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# A COMPREHENSIVE STUDY OF LENGTH-BIASED TRANSMUTED DISTRIBUTION

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**Abstract**

*In this study, we explore a new probability distribution termed as the Length-Biased Transmuted Mukherjee-Islam (LBTMI) distribution. This exploration enhances the conventional Transmuted Mukherjee-Islam distribution by integrating a weighted transformation approach. The paper examines the probability density function and the corresponding cumulative distribution function associated with the LBTMI distribution. A comprehensive examination of the unique structural properties of the proposed model is carried out, including the survival function, conditional survival function, hazard function, cumulative hazard function, mean residual life, moments, moment generating function (MGF), characteristic function (CF), cumulant generating function (CGF), likelihood ratio test, ordered statistics, entropy measures, and Bonferroni and Lorenz curves. To ensure precise estimation of model parameters, the study employs the maximum likelihood estimation method, contributing significantly to the advancement of statistical modelling in this domain.*

**Key words:** Transmuted Mukherjee-Islam distribution, Weighted transformation, Reliability analysis, Maximum likelihood estimator, Ordered statistics

## 1. Introduction

One fundamental aim of statistics is to develop precise predictive models for real-world phenomena. However, due to the complex nature of these phenomena, conventional modelling approaches may prove inadequate. As a result, an array of probability distributions has been devised to address these challenges by using various transformation approaches. In our study, we introduce a novel extension of Transmuted Mukherjee-Islam distribution termed as LBTMI distribution. This distribution is crafted through the Fisher's [6] weighted transformation technique, initially introduced in 1934, and further elucidated by Rao [17] in 1965. This method allows us to construct weighted models of observations based on predefined weighted functions. The weighted distribution reduces to length biased distribution when the weight function considers only the length of the units. The concept of length biased sampling was first introduced by Cox [4] and Zelen [26]. Scholars and researchers have extensively delved into weighted probability models and their wide-ranging applications across diverse domains. Modi and Gill [13] discussed the length-biased weighted Maxwell distribution, while Sanat [23] derived the beta-length biased Pareto distribution. Reyad et al. [22] examined the length-biased weighted Frechet distribution, elucidating its properties and practical applications. Rather and Subramanian [20] introduced a method for characterizing and estimating the length-biased weighted generalized uniform distribution. Mudasir and Ahmad [14] provided an in-depth discussion on the characterization and estimation of the length-biased Nakagami distribution, and Khan et al. [7] discussed the weighted modified Weibull distribution. In subsequent years, Rather and Subramanian [19] explored the length-biased Erlang truncated exponential distribution, highlighting its practical applications. Mathew and Chesneau [12] studied the Marshall-Olkin length-biased Maxwell distribution. In recent developments, Rather and Ozel [18] introduced a new length-biased power Lindley distribution with applications. Al-Omari and Alanzi [1] presented the inverse length-biased Maxwell distribution and conducted statistical inference with an illustrative application. Mustafa and Khan [15] developed the length-biased powered inverse Rayleigh distribution with practical applications.

The Transmuted Mukherjee-Islam distribution had explored by Rather and Subramanian [21] using the quadratic rank transmutation map studied first by Shaw and Buckley [24] in 2007. Loai M. A. Al-Zou'bi [2] also obtained various properties of Transmuted Mukherjee-Islam distribution and its applications. The probability density function of a random variable say $Z$ following Transmuted Mukherjee- Islam distribution with parameters say $(\varepsilon, \nu, \omega)$ is given by

$$f(z;\varepsilon,\nu,\omega) = \frac{\varepsilon}{\nu^{\varepsilon}} z^{\varepsilon-1}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right) \; ;0 < z < \nu, \varepsilon > 0, \nu > 0, -1 \le \omega \le 1 \tag{1}$$

And the corresponding cumulative distribution function is

$$F_Z(z) = \left(\frac{z}{\nu}\right)^{\varepsilon}\left(1+\omega-\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right) \tag{2}$$

Several researchers have explored the quadratic transmutation mapping approach, introducing new members to this family across various baseline distributions. These include the transmuted extreme value distribution by Aryal and Tsokos [3], the transmuted Frechet distribution by Mahmoud and Mandouh [10], the transmuted generalized linear exponential distribution by Elbatal et al. [5], the transmuted additive Weibull distribution by Mansour et al. [11], the transmuted Gompertz distribution by Khan et al. [8], and the transmuted generalized inverse Weibull distribution by Khan et al. [9]. Additionally, Subramanian and Rather [25] examined the weighted version of the exponentiated Mukherjee-Islam distribution, deriving its statistical properties. Furthermore, Otiniano et al. [16] delved into the transmuted generalized extreme value distribution.

## 2. Probability density function (PDF) and cumulative distribution function (CDF)

Using the weighted transformation approach, the PDF $y_w(z)$ of a non-negative random variable $Z$ is given by

$$y_w(z) = \frac{w(z)\,y(z)}{E\big(w(z)\big)}; \qquad z > 0$$

Where $w(z)$ be a non-negative weight function and

$$E\big(w(z)\big) = \int_{-\infty}^{\infty} w(z)\,y(z)\,dz < \infty.$$

Note that different choices of the weight function $w(z)$ give different weighted distributions. Consequently for weight function $w(z) = z$, the resulting distribution is called length-biased distribution. Let us assume that the PDF of the random variable $Z$ to be Transmuted Mukherjee-Islam distribution, so the PDF of Length-Biased Mukherjee-Islam(LBTMI) distribution is given by

$$g(z;\varepsilon,\nu,\omega) = \frac{z\,f(z;\varepsilon,\nu,\omega)}{E(z)} \tag{3}$$

Now

$$E(z)=\int_0^{\nu} z\frac{\varepsilon}{\nu^{\varepsilon}}z^{\varepsilon-1}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)dz \tag{4}$$

$$E(z)=\frac{\varepsilon}{\nu^{\varepsilon}}\left((1+\omega)\int_0^{\nu} z^{\varepsilon}dz-\frac{2\omega}{(\nu)^{\varepsilon}}\int_0^{\nu} z^{2\varepsilon}dz\right) \tag{5}$$

After simplification we get

$$E(z^{s})=\frac{\varepsilon\,\nu\big((1-\omega)+2\varepsilon\big)}{(\varepsilon+1)(2\varepsilon+1)} \tag{6}$$

Using (1) and (6) in (3) we get

$$g(z;\varepsilon,\nu,\omega)=\frac{z\frac{\varepsilon}{\nu^{\varepsilon}}z^{\varepsilon-1}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\frac{\varepsilon\,\nu\big((1-\omega)+2\varepsilon\big)}{(\varepsilon+1)(2\varepsilon+1)}} \tag{7}$$

$$g(z;\varepsilon,\nu,\omega)=\frac{(\varepsilon+1)(2\varepsilon+1)z^{\varepsilon}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{\varepsilon+1}\big((1-\omega)+2\varepsilon\big)} \tag{8}$$

The corresponding CDF of LBTMI distribution is given by

$$G_Z(z)=\int_0^{z}\left(\frac{(\varepsilon+1)(2\varepsilon+1)z^{\varepsilon}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{\varepsilon+1}\big((1-\omega)+2\varepsilon\big)}\right)dz \tag{9}$$

$$G_Z(z)=\frac{(\varepsilon+1)(2\varepsilon+1)}{\nu^{\varepsilon+1}\big((1-\omega)+2\varepsilon\big)}\left((1+\omega)\int_0^{z} z^{\varepsilon}dz-\frac{2\omega}{(\nu)^{\varepsilon}}\int_0^{z} z^{2\varepsilon}dz\right) \tag{10}$$

After simplification we get

$$G_Z(z)=\frac{(2\varepsilon+1)(1+\omega)(\nu)^{\varepsilon}z^{\varepsilon+1}-2\omega(\varepsilon+1)z^{2\varepsilon+1}}{\nu^{2\varepsilon+1}\big((1-\omega)+2\varepsilon\big)} \tag{11}$$

## 3. Reliability Analysis

### 3.1 Survival function

The survival function of LBTMI distribution is given by

$$R_T(t)=P_r(T>t)$$

$$R_T(t)=1-P_r(T\le t)$$

$$R_T(t)=1-\frac{(2\varepsilon+1)(1+\omega)(\nu)^{\varepsilon}t^{\varepsilon+1}-2\omega(\varepsilon+1)t^{2\varepsilon+1}}{\nu^{2\varepsilon+1}\big((1-\omega)+2\varepsilon\big)} \tag{12}$$

$$R_T(t) = \frac{\nu^{2\varepsilon+1}\big((1-\omega)+2\varepsilon\big)-(2\varepsilon+1)(1+\omega)(\nu)^{\varepsilon}t^{\varepsilon+1}+2\omega(\varepsilon+1)t^{2\varepsilon+1}}{\nu^{2\varepsilon+1}\big((1-\omega)+2\varepsilon\big)} \tag{13}$$

After simplification we get

$$R_T(t) = \frac{\nu^{2\varepsilon+1}\big((1-\omega)+2\varepsilon\big)-(2\varepsilon+1)(1+\omega)(\nu)^{\varepsilon}t^{\varepsilon+1}+2\omega(\varepsilon+1)t^{2\varepsilon+1}}{\nu^{2\varepsilon+1}\big((1-\omega)+2\varepsilon\big)} \tag{14}$$

After simplification we get

$$R_T(t) = \frac{\nu^{2\varepsilon+1}\big((1-\omega)+2\varepsilon\big)-t^{\varepsilon+1}\big((2\varepsilon+1)(1+\omega)(\nu)^{\varepsilon}-2\omega(\varepsilon+1)t\big)}{\nu^{2\varepsilon+1}\big((1-\omega)+2\varepsilon\big)} \tag{15}$$

## 3.2 Conditional survival function

In case of LBTMI distribution the conditional survival function is given by

$$R_T(t\,|\,t_0) = P_r(T > t_0 + t\,|\,T > t_0)$$

$$R_T(t\,|\,t_0) = \frac{P_r(T > t_0 + t)}{P_r(T > t_0)}$$

$$R_T(t\,|\,t_0) = \frac{R_T(t_0 + t)}{R_T(t_0)}$$

$$R_T(t\,|\,t_0) = \frac{\dfrac{\nu^{2\varepsilon+1}\big((1-\omega)+2\varepsilon\big)-(t_0+t)^{\varepsilon+1}\big((2\varepsilon+1)(1+\omega)(\nu)^{\varepsilon}-2\omega(\varepsilon+1)(t_0+t)\big)}{\nu^{2\varepsilon+1}\big((1-\omega)+2\varepsilon\big)}}{\dfrac{\nu^{2\varepsilon+1}\big((1-\omega)+2\varepsilon\big)-t^{\varepsilon+1}\big((2\varepsilon+1)(1+\omega)(\nu)^{\varepsilon}-2\omega(\varepsilon+1)t\big)}{\nu^{2\varepsilon+1}\big((1-\omega)+2\varepsilon\big)}} \tag{16}$$

$$R_T(t\,|\,t_0) = \frac{\nu^{2\varepsilon+1}\big((1-\omega)+2\varepsilon\big)-(t_0+t)^{\varepsilon+1}\big((2\varepsilon+1)(1+\omega)(\nu)^{\varepsilon}-2\omega(\varepsilon+1)(t_0+t)\big)}{\nu^{2\varepsilon+1}\big((1-\omega)+2\varepsilon\big)-t^{\varepsilon+1}\big((2\varepsilon+1)(1+\omega)(\nu)^{\varepsilon}-2\omega(\varepsilon+1)t\big)} \tag{17}$$

## 3.3 Hazard function

The hazard function of LBTMI distribution is given by

$$H_T(t) = \frac{g(t;\varepsilon,\nu,\omega)}{1-G_T(t)}$$

$$H_T(t) = \frac{\dfrac{(\varepsilon+1)(2\varepsilon+1)t^{\varepsilon}\left(1+\omega-2\omega\left(\dfrac{t}{\nu}\right)^{\varepsilon}\right)}{\nu^{\varepsilon+1}\big((1-\omega)+2\varepsilon\big)}}{1-\dfrac{(2\varepsilon+1)(1+\omega)(\nu)^{\varepsilon}t^{\varepsilon+1}-2\omega(\varepsilon+1)t^{2\varepsilon+1}}{\nu^{2\varepsilon+1}\big((1-\omega)+2\varepsilon\big)}}$$

After simplification we have

$$H_T(t) = \frac{(\varepsilon+1)(2\varepsilon+1)(\nu)^{\varepsilon}t^{\varepsilon}\left(1+\omega-2\omega\left(\dfrac{t}{\nu}\right)^{\varepsilon}\right)}{\nu^{2\varepsilon+1}\big((1-\omega)+2\varepsilon\big)-t^{\varepsilon+1}\big((2\varepsilon+1)(1+\omega)(\nu)^{\varepsilon}-2\omega(\varepsilon+1)t^{\varepsilon}\big)} \tag{18}$$

## 3.4 Cumulative hazard function

The cumulative hazard function of LBTMI distribution is given by

$$ {}_C H_T(t) = -\ln\left(R_T(t)\right) $$

$$ {}_C H_T(t) = -\ln\left( \frac{\nu^{2\varepsilon+1}\left((1-\omega)+2\varepsilon\right) - t^{\varepsilon+1}\left((2\varepsilon+1)(1+\omega)(\nu)^{\varepsilon} - 2\omega(\varepsilon+1)t\right)}{\nu^{2\varepsilon+1}\left((1-\omega)+2\varepsilon\right)} \right) \tag{19} $$

Similarly, the Conditional Cumulative hazard function of LBTMI distribution is given by

$$ {}_C H_T(t \mid t_0) = -\ln\left(R_T(t \mid t_0)\right) $$

$$ {}_C H_T(t \mid t_0) = -\ln\left( \frac{\nu^{2\varepsilon+1}\left((1-\omega)+2\varepsilon\right) - (t_0+t)^{\varepsilon+1}\left((2\varepsilon+1)(1+\omega)(\nu)^{\varepsilon} - 2\omega(\varepsilon+1)(t_0+t)\right)}{\nu^{2\varepsilon+1}\left((1-\omega)+2\varepsilon\right) - t^{\varepsilon+1}\left((2\varepsilon+1)(1+\omega)(\nu)^{\varepsilon} - 2\omega(\varepsilon+1)t\right)} \right) $$

## 3.5 Reverse Hazard function

The reverse hazard function of LBTMI distribution is given by

$$ H_r(t) = \frac{g(t;\varepsilon,\nu,\omega)}{G_T(t)} $$

$$ H_r(t) = \frac{\dfrac{(\varepsilon+1)(2\varepsilon+1)t^{\varepsilon}\left(1+\omega-2\omega\left(\dfrac{t}{\nu}\right)^{\varepsilon}\right)}{\nu^{\varepsilon+1}\left((1-\omega)+2\varepsilon\right)}}{\dfrac{(2\varepsilon+1)(1+\omega)(\nu)^{\varepsilon}t^{\varepsilon+1} - 2\omega(\varepsilon+1)t^{2\varepsilon+1}}{\nu^{2\varepsilon+1}\left((1-\omega)+2\varepsilon\right)}} $$

After simplification we get

$$ H_r(t) = \frac{(\varepsilon+1)(2\varepsilon+1)(\nu)^{\varepsilon}\, t^{\varepsilon}\left(1+\omega-2\omega\left(\dfrac{t}{\nu}\right)^{\varepsilon}\right)}{(2\varepsilon+1)(1+\omega)(\nu)^{\varepsilon}t^{\varepsilon+1} - 2\omega(\varepsilon+1)t^{2\varepsilon+1}} \tag{20} $$

## 3.6 Mills Ratio

The Mills ratio of LBTMI distribution is given by

$$ Mills\ \ ratio = \frac{1}{H_r(t)} $$

$$ Mills\ \ ratio = \frac{(2\varepsilon+1)(1+\omega)(\nu)^{\varepsilon}t^{\varepsilon+1} - 2\omega(\varepsilon+1)t^{2\varepsilon+1}}{(\varepsilon+1)(2\varepsilon+1)(\nu)^{\varepsilon}\, t^{\varepsilon}\left(1+\omega-2\omega\left(\dfrac{t}{\nu}\right)^{\varepsilon}\right)} \tag{21} $$

## 3.7 Mean residual life

The mean residual life (MRL) in case of LBTMI distribution is given by

$$ MRL = \frac{1}{1-G_Z(z)}\int_z^{\nu} t\, g(t;\varepsilon,\nu,\omega)\,dt - z $$

$$MRL = \frac{1}{1-G_Z(z)}\int_z^{\nu} t \frac{(\varepsilon+1)(2\varepsilon+1)t^{\varepsilon}\left(1+\omega-2\omega\left(\frac{t}{\nu}\right)^{\varepsilon}\right)}{\nu^{\varepsilon+1}\left((1-\omega)+2\varepsilon\right)} dt - z \tag{22}$$

$$MRL = \frac{1}{1-G_Z(z)}\int_z^{\nu} \frac{(\varepsilon+1)(2\varepsilon+1)t^{\varepsilon+1}\left(1+\omega-2\omega\left(\frac{t}{\nu}\right)^{\varepsilon}\right)}{\nu^{\varepsilon+1}\left((1-\omega)+2\varepsilon\right)} dt - z \tag{23}$$

$$MRL = \frac{(\varepsilon+1)(2\varepsilon+1)}{\left(1-G_Z(z)\right)\nu^{\varepsilon+1}\left((1-\omega)+2\varepsilon\right)}\left((1+\omega)\int_z^{\nu} t^{\varepsilon+1}dt - \frac{2\omega}{(\nu)^{\varepsilon}}\int_z^{\nu} t^{2\varepsilon+1}dt\right) - z \tag{24}$$

After simplification we get

$$MRL = \frac{(\varepsilon+1)(2\varepsilon+1)}{\left(1-G_Z(z)\right)\nu^{2\varepsilon+1}\left((1-\omega)+2\varepsilon\right)(\varepsilon+2)(2\varepsilon+2)}\left\{\begin{array}{l}(\nu)^{2\varepsilon+2}\left((1-\omega)(\varepsilon+2)+(1+\omega)\varepsilon\right)+\\ 2\omega(\varepsilon+2)z^{2\varepsilon+2}-(\nu)^{\varepsilon}(1+\omega)(2\varepsilon+2)z^{\varepsilon+2}\end{array}\right\} - z$$

## 4. Moments

The rth raw moment about origin of LBTMI distributionis defined as

$$\mu_r' = \int_0^{\nu} z^r\, g(z;\varepsilon,\nu,\omega)dz$$

$$\mu_r' = \int_0^{\nu} z^r \frac{(\varepsilon+1)(2\varepsilon+1)z^{\varepsilon}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{\varepsilon+1}\left((1-\omega)+2\varepsilon\right)} dz$$

$$\mu_r' = \frac{(\varepsilon+1)(2\varepsilon+1)}{\nu^{\varepsilon+1}\left((1-\omega)+2\varepsilon\right)}\left((1+\omega)\int_0^{\nu} z^{\varepsilon+r}dz - \frac{2\omega}{(\nu)^{\varepsilon}}\int_0^{\nu} z^{2\varepsilon+r}dz\right) \tag{25}$$

$$\mu_r' = \frac{(1+\varepsilon)(1+2\varepsilon)}{\nu^{1+\varepsilon}\left((1-\omega)+2\varepsilon\right)}\left((1+\omega)\frac{(\nu)^{\varepsilon+r+1}}{\varepsilon+r+1} - \frac{2\omega(\nu)^{2\varepsilon+r+1}}{(\nu)^{\varepsilon}(2\varepsilon+r+1)}\right) \tag{26}$$

After simplification we get

$$\mu_r' = \frac{(\varepsilon+1)(2\varepsilon+1)(\nu)^r\left((1-\omega)(\varepsilon+r+1)+(1+\omega)\varepsilon\right)}{\left((1-\omega)+2\varepsilon\right)(\varepsilon+r+1)(2\varepsilon+r+1)} \tag{27}$$

Putting $r = 1, 2, 3, 4$ in (26) we get

$$\mu_1' = \frac{(\varepsilon+1)(2\varepsilon+1)(\nu)\left((1-\omega)(\varepsilon+2)+(1+\omega)\varepsilon\right)}{\left((1-\omega)+2\varepsilon\right)(\varepsilon+2)(2\varepsilon+2)} \tag{28}$$

$$\mu_2' = \frac{(\varepsilon+1)(2\varepsilon+1)(\nu)^2\left((1-\omega)(\varepsilon+3)+(1+\omega)\varepsilon\right)}{\left((1-\omega)+2\varepsilon\right)(\varepsilon+3)(2\varepsilon+3)} \tag{29}$$

$$\mu_3' = \frac{(\varepsilon+1)(2\varepsilon+1)(\nu)^3\left((1-\omega)(\varepsilon+4)+(1+\omega)\varepsilon\right)}{\left((1-\omega)+2\varepsilon\right)(\varepsilon+4)(2\varepsilon+4)} \tag{30}$$

$$\mu_4' = \frac{(\varepsilon+1)(2\varepsilon+1)(\nu)^4\left((1-\omega)(\varepsilon+5)+(1+\omega)\varepsilon\right)}{\left((1-\omega)+2\varepsilon\right)(\varepsilon+5)(2\varepsilon+5)} \tag{31}$$

The variance and coefficient of variance $(C.V)$ respectively are given by

$$\sigma^2 = \mu_2' - (\mu_1')^2$$

And

$$C.V = \frac{\sigma}{\mu_1'}; \qquad \text{where } \sigma = \sqrt{\mu_2' - (\mu_1')^2}$$

## 5. Harmonic mean

The harmonic mean of LBTMI distributionis defined as

$$Harmonic\ \ mean = E\left(\frac{1}{Z}\right)$$

$$Harmonic\, mean = \int_0^{\nu} \frac{1}{z} \frac{(\varepsilon+1)(2\varepsilon+1)z^{\varepsilon}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{\varepsilon+1}\big((1-\omega)+2\varepsilon\big)} dz$$

$$Harmonic\, mean = \int_0^{\nu} \frac{(\varepsilon+1)(2\varepsilon+1)z^{\varepsilon-1}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{\varepsilon+1}\big((1-\omega)+2\varepsilon\big)} dz \tag{32}$$

$$Harmonic\, mean = \frac{(\varepsilon+1)(2\varepsilon+1)}{\nu^{\varepsilon+1}\big((1-\omega)+2\varepsilon\big)}\left((1+\omega)\int_0^{\nu} z^{\varepsilon-1}dz - \frac{2\omega}{(\nu)^{\varepsilon}}\int_0^{\nu} z^{2\varepsilon-1}dz\right) \tag{33}$$

After simplification we get

$$Harmonic\, mean = \frac{(\varepsilon+1)(2\varepsilon+1)\big((1-\omega)\varepsilon+(1+\omega)\big)}{\nu\big((1-\omega)+2\varepsilon\big)(2\varepsilon^2)} \tag{34}$$

## 6. MGF, CF, and CGF

The moment generating function of LBTMI distribution is

$$M_Z(t) = E(e^{tz})$$

$$M_Z(t) = \int_0^{\nu} e^{tz} \frac{(\varepsilon+1)(2\varepsilon+1)z^{\varepsilon}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{\varepsilon+1}\big((1-\omega)+2\varepsilon\big)} dz$$

$$M_Z(t) = \int_0^{\nu}\sum_{k=0}^{\infty}\frac{(tz)^k}{k!} \frac{(\varepsilon+1)(2\varepsilon+1)z^{\varepsilon}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{\varepsilon+1}\big((1-\omega)+2\varepsilon\big)} dz \tag{35}$$

$$M_Z(t) = \sum_{k=0}^{\infty}\frac{(t)^k}{k!}\int_0^{\infty} z^k g(z;\varepsilon,\nu,\omega)dz$$

$$M_Z(t)=\sum_{k=0}^{\infty}\frac{(t)^k}{k!}\mu_k'$$

$$M_Z(t)=\sum_{k=0}^{\infty}\frac{(t)^k}{k!}\frac{(\varepsilon+1)(2\varepsilon+1)(\nu)^k\left((1-\omega)(\varepsilon+k+1)+(1+\omega)\,\varepsilon\right)}{\left((1-\omega)+2\varepsilon\right)(\varepsilon+k+1)(2\varepsilon+k+1)} \tag{36}$$

The characteristics function of LBTMI distribution is

$$\phi_Z(t)=E(e^{itz})$$

$$\phi_Z(t)=\int_0^{\nu}e^{\imath tz}\frac{(\varepsilon+1)(2\varepsilon+1)z^{\varepsilon}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{\varepsilon+1}\left((1-\omega)+2\varepsilon\right)}dz$$

$$\phi_Z(t)=\int_0^{\nu}\sum_{k=0}^{\infty}\frac{(\imath tz)^k}{k!}\frac{(\varepsilon+1)(2\varepsilon+1)z^{\varepsilon}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{\varepsilon+1}\left((1-\omega)+2\varepsilon\right)}dz \tag{37}$$

$$\phi_Z(t)=\sum_{k=0}^{\infty}\frac{(\imath)^k(t)^k}{k!}\int_0^{\infty}z^k g(z;\varepsilon,\nu,\omega)dz \tag{38}$$

$$\phi_Z(t)=\sum_{k=0}^{\infty}\frac{(\imath t)^k}{k!}\mu_k'$$

$$\phi_Z(t)=\sum_{k=0}^{\infty}\frac{(\imath t)^k}{k!}\frac{(\varepsilon+1)(2\varepsilon+1)(\nu)^k\left((1-\omega)(\varepsilon+k+1)+(1+\omega)\,\varepsilon\right)}{\left((1-\omega)+2\varepsilon\right)(\varepsilon+k+1)(2\varepsilon+k+1)} \tag{39}$$

The cumulant generating function of LBTMI distribution is

$$\kappa_Z(t)=\log\left(M_Z(t)\right)$$

$$\kappa_Z(t)=\log\left(\sum_{k=0}^{\infty}\frac{(t)^k}{k!}\frac{(\varepsilon+1)(2\varepsilon+1)(\nu)^k\left((1-\omega)(\varepsilon+k+1)+(1+\omega)\,\varepsilon\right)}{\left((1-\omega)+2\varepsilon\right)(\varepsilon+k+1)(2\varepsilon+k+1)}\right) \tag{40}$$

## 7. Estimation of Parameters

Let $z_1, z_2, z_3, \ldots, z_n$ be a random sample of size $n$ from LBTMI distribution. Then the likelihood function is defined as the joint density of the random sample, which is given as

$$L(\varepsilon,\nu,\omega)=\prod_{l=1}^{n}g\left(z_l;\varepsilon,\nu,\omega\right)=\frac{(\varepsilon+1)(2\varepsilon+1)z^{\varepsilon}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{\varepsilon+1}\left((1-\omega)+2\varepsilon\right)} \tag{41}$$

$$L(\varepsilon,\nu,\omega)=\prod_{l=1}^{n}\frac{(\varepsilon+1)(2\varepsilon+1)z_l^{\varepsilon 1}\left(1+\omega-2\omega\left(\frac{z_l}{\nu}\right)^{\varepsilon}\right)}{\nu^{\varepsilon+1}\left((1-\omega)+2\varepsilon\right)} \tag{42}$$

$$L(\varepsilon,\nu,\omega)=\frac{(\varepsilon+1)^n\,(2\varepsilon+1)^n}{\nu^{n(\varepsilon+1)}\left((1-\omega)+2\varepsilon\right)^n}\left(\prod_{l=1}^{n}z_l^{\varepsilon}\right)\left(\prod_{l=1}^{n}\left(1+\omega-2\omega\left(\frac{z_l}{\nu}\right)^{\varepsilon}\right)\right) \tag{43}$$

Taking logarithm on both sides we get

$$\log L(\varepsilon,\nu,\omega)=n\log(\varepsilon+1)+n\log(2\varepsilon+1)-n(\varepsilon+1)\log(\nu)-n\log\left((1-\omega)+2\varepsilon\right)$$
$$+(\varepsilon)\sum_{l=1}^{n}\log z_l+\sum_{l=1}^{n}\log\left(1+\omega-\frac{2\omega}{(\nu)^{\varepsilon}}(z_l)^{\varepsilon}\right) \tag{44}$$

Differentiating equation (43) partially with respect to $\varepsilon$ and equating to zero we get

$$\frac{n}{\varepsilon+1}+\frac{2n}{2\varepsilon+1}-n\log(\nu)-\frac{2n}{(1-\omega)+2\varepsilon}+\sum_{l=1}^{n}\log z_l-\sum_{l=1}^{n}\frac{2\omega}{\left(1+\omega-\frac{2\omega}{(\nu)^{\varepsilon}}(z_l)^{\varepsilon}\right)}\left(\frac{z}{\nu}\right)^{\varepsilon}\log\left(\frac{z}{\nu}\right)=0 \tag{45}$$

Differentiating equation (43) partially with respect to $\nu$ and equating to zero we get

$$\sum_{l=1}^{n}\frac{2\,\omega\,(z_l)\,\varepsilon}{\left(1+\omega-\frac{2\omega}{(\nu)^{\varepsilon}}(z_l)^{\varepsilon}\right)(\nu)^{\varepsilon+1}}-\frac{n(\varepsilon+1)}{\nu}=0 \tag{46}$$

Differentiating equation (43) partially with respect to $\omega$ and equating to zero we get

$$\frac{n}{(1-\omega)+2\varepsilon}+\sum_{l=1}^{n}\frac{1-2\left(\frac{z_l}{\nu}\right)^{\varepsilon}}{\left(1+\omega-\frac{2\omega}{(\nu)^{\varepsilon}}(z_l)^{\varepsilon}\right)}=0 \tag{47}$$

Simultaneously solving equation (44), (45), and (46), gives the maximum likelihood estimators of parameters involved in the given distribution. However, direct evaluation of the aforementioned system of nonlinear equations is unfeasible. To obtain maximum likelihood estimates for the distribution parameters, it is necessary to employ iterative methods such as the Newton-Raphson method, Mathematica, or the Secant method to solve this system effectively.

## 8. Distribution of ordered statistics

Suppose we draw a random sample $z_1, z_2, z_3, ..., z_n$ of size $n$ from LBTMI distribution. Then the ordered statistics corresponding to the given sample is $Z_{(1)}, Z_{(2)}, Z_{(3)}, ..., Z_{(n)}$ such that $Z_{(1)} \le Z_{(2)} \le Z_{(3)} \le ... \le Z_{(n)}$, Where

$$Z_{(1)}=\min(z_1, z_2, z_3, ..., z_n)$$

$$\text{and}\qquad Z_{(n)}=\max(z_1, z_2, z_3, ..., z_n)$$

The PDF of $k^{th}$ ordered statistics from LBTMI distribution is given by

$$g_{Z_{(k)}}(z)=\frac{n!}{(k-1)!(n-k)!}g(z;\varepsilon,\nu,\omega)\left(G_Z(z)\right)^{k-1}\left(1-G_Z(z)\right)^{n-k}$$

$$g_{Z_{(k)}}(z)=\frac{n!}{(k-1)!(n-k)!}\frac{(\varepsilon+1)(2\varepsilon+1)z^{\varepsilon}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{\varepsilon+1}((1-\omega)+2\varepsilon)}\left(\frac{(2\varepsilon+1)(1+\omega)(\nu)^{\varepsilon}z^{\varepsilon+1}-2\omega(\varepsilon+1)z^{2\varepsilon+1}}{\nu^{2\varepsilon+1}((1-\omega)+2\varepsilon)}\right)^{k-1} \times\left(1-\frac{(2\varepsilon+1)(1+\omega)(\nu)^{\varepsilon}z^{\varepsilon+1}-2\omega(\varepsilon+1)z^{2\varepsilon+1}}{\nu^{2\varepsilon+1}((1-\omega)+2\varepsilon)}\right)^{n-k} \tag{48}$$

$$g_{Z_{(k)}}(z)=\frac{n!}{(k-1)!(n-k)!}\frac{(\varepsilon+1)(2\varepsilon+1)z^{\varepsilon}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{\varepsilon+1}((1-\omega)+2\varepsilon)}\left(\frac{(2\varepsilon+1)(1+\omega)(\nu)^{\varepsilon}z^{\varepsilon+1}-2\omega(\varepsilon+1)z^{2\varepsilon+1}}{\nu^{2\varepsilon+1}((1-\omega)+2\varepsilon)}\right)^{k-1} \times\left(\frac{\nu^{2\varepsilon+1}((1-\omega)+2\varepsilon)-z^{\varepsilon+1}\left((2\varepsilon+1)(1+\omega)(\nu)^{\varepsilon}-2\omega(\varepsilon+1)z^{\varepsilon}\right)}{\nu^{2\varepsilon+1}((1-\omega)+2\varepsilon)}\right)^{n-k} \tag{49}$$

And the corresponding CDF of $k^{th}$ ordered statistics is

$$G_{Z_{(k)}}(z)=\sum_{j=k}^{n}({}^{n}C_{j})\left(G_Z(z)\right)^{j}\left(1-G_Z(z)\right)^{n-j}$$

$$G_{Z_{(k)}}(z)=\sum_{j=k}^{n}\left(\begin{array}{l}({}^{n}C_{j})\left(\frac{(2\varepsilon+1)(1+\omega)(\nu)^{\varepsilon}z^{\varepsilon+1}-2\omega(\varepsilon+1)z^{2\varepsilon+1}}{\nu^{2\varepsilon+1}((1-\omega)+2\varepsilon)}\right)^{j}\\ \times\left(\frac{\nu^{2\varepsilon+1}((1-\omega)+2\varepsilon)-z^{\varepsilon+1}\left((2\varepsilon+1)(1+\omega)(\nu)^{\varepsilon}-2\omega(\varepsilon+1)z^{\varepsilon}\right)}{\nu^{2\varepsilon+1}((1-\omega)+2\varepsilon)}\right)^{n-j}\end{array}\right) \tag{50}$$

On substituting $k=1,n$ in equation (48) we get the PDF of smallest and highest ordered statistics respectively and are given as

$$g_{Z_{(1)}}(z)=n\frac{(\varepsilon+1)(2\varepsilon+1)z^{\varepsilon}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{\varepsilon+1}((1-\omega)+2\varepsilon)}\times\left(\frac{\nu^{2\varepsilon+1}((1-\omega)+2\varepsilon)-z^{\varepsilon+1}\left((2\varepsilon+1)(1+\omega)(\nu)^{\varepsilon}-2\omega(\varepsilon+1)z^{\varepsilon}\right)}{\nu^{2\varepsilon+1}((1-\omega)+2\varepsilon)}\right)^{n-1} \tag{51}$$

And

$$g_{Z_{(n)}}(z)=\frac{(\varepsilon+1)(2\varepsilon+1)z^{\varepsilon}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{\varepsilon+1}((1-\omega)+2\varepsilon)}\left(\frac{(2\varepsilon+1)(1+\omega)(\nu)^{\varepsilon}z^{\varepsilon+1}-2\omega(\varepsilon+1)z^{2\varepsilon+1}}{\nu^{2\varepsilon+1}((1-\omega)+2\varepsilon)}\right)^{n-1} \tag{52}$$

Their corresponding CDFs are obtained on substituting $k=1,n$ in equation (49) and are given by

$$G_{Z_{(1)}}(z)=1-\left(1-\left(\frac{(2\varepsilon+1)(1+\omega)(\nu)^{\varepsilon}z^{\varepsilon+1}-2\omega(\varepsilon+1)z^{2\varepsilon+1}}{\nu^{2\varepsilon+1}((1-\omega)+2\varepsilon)}\right)\right)^{n} \tag{53}$$

And

$$G_{Z_{(n)}}(z)=\left(\frac{(2\varepsilon+1)(1+\omega)(\nu)^{\varepsilon}z^{\varepsilon+1}-2\omega(\varepsilon+1)z^{2\varepsilon+1}}{\nu^{2\varepsilon+1}((1-\omega)+2\varepsilon)}\right)^{n} \tag{54}$$

## 9. Likelihood ratio test

The likelihood ratio test is a statistical technique designed to explore the adequacy of fit between two models. Specifically, in the realm of probability distributions, it is utilized to check the suitability of two distinct distributions in explaining observed data and its purpose is to ascertain whether the inclusion of additional parameters in a statistical model substantially enhances its ability to accurately represent the data or not. Suppose $z_1, z_2, z_3, ..., z_n$ be a random sample of size $n$ from LBTMI distribution.. To test the hypothesis

$$H_0 : g(z) = f(z;\varepsilon,\nu,\omega,) \qquad \text{against} \qquad H_1 : g(z) = g(z;\varepsilon,\nu,\omega)$$

The likelihood ratio test is defined as

$$\ell = \prod_{k=1}^{n} \frac{g(z_k;\varepsilon,\nu,\omega)}{f(z_k;\varepsilon,\nu,\omega)}$$

$$\ell = \prod_{k=1}^{n} \frac{\dfrac{(\varepsilon+1)(2\varepsilon+1)z_k^{\varepsilon}\left(1+\omega-2\omega\left(\dfrac{z_k}{\nu}\right)^{\varepsilon}\right)}{\nu^{\varepsilon+1}\big((1-\omega)+2\varepsilon\big)}}{\dfrac{\varepsilon}{\nu^{\varepsilon}} z_k^{\varepsilon-1}\left(1+\omega-2\omega\left(\dfrac{z_k}{\nu}\right)^{\varepsilon}\right)} \tag{55}$$

$$\ell = \left(\frac{(\varepsilon+1)(2\varepsilon+1)}{\varepsilon\nu\big((1-\omega)+2\varepsilon\big)}\right)^n \prod_{k=1}^{n} z_k \tag{56}$$

So, we reject null hypothesis at $\alpha$ level of significance if $\ell > \mathrm{K}^*$ such that $P(\ell > \mathrm{K}^*) = \alpha$, where $\mathrm{K}^*$ is the critical value at $\alpha$ level of significance of the given test statistics. That is,

$$\left(\frac{(\varepsilon+1)(2\varepsilon+1)}{\varepsilon\nu\big((1-\omega)+2\varepsilon\big)}\right)^n \prod_{k=1}^{n} z_k > \mathrm{K}^* \tag{57}$$

$$\prod_{k=1}^{n} z_k > \mathrm{K}^* \left(\frac{\varepsilon\nu\big((1-\omega)+2\varepsilon\big)}{(\varepsilon+1)(2\varepsilon+1)}\right)^n \tag{58}$$

For large sample size $n$, $-2\log(\ell)$ is distributed as Chi-square distribution with one degree of freedom. Also p-value is calculated from the chi-square distribution. On the basis of p-value, we reject the null hypothesis when the p-value is less than level of significance.

## 10. Entropy measures

### 10.1 Renyi entropy and Tsallis entropy

By definition, the Renyi entropy is given by

$$R(\tau) = \frac{1}{1-\tau}\log\left(\int_0^{\nu}\left(g\left(z_k;\varepsilon,\nu,\omega\right)\right)^{\tau}dz\right) \tag{59}$$

$$R(\tau) = \frac{1}{1-\tau}\log\left(\int_0^{\nu}\left(\frac{(\varepsilon+1)(2\varepsilon+1)z^{\varepsilon}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{\varepsilon+1}\left((1-\omega)+2\varepsilon\right)}\right)^{\tau}dz\right) \tag{60}$$

$$R(\tau) = \frac{1}{1-\tau}\log\left(\left(\frac{(\varepsilon+1)(2\varepsilon+1)}{\nu^{\varepsilon+1}\left((1-\omega)+2\varepsilon\right)}\right)^{\tau}\int_0^{\nu}\left(z^{\tau\varepsilon}\sum_{k=0}^{\tau}({}^{\tau}C_k)(-1)^k(1+\omega)^{\tau-k}\left(2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)^k\right)dz\right)$$

$$R(\tau) = \frac{1}{1-\tau}\log\left(\left(\frac{(\varepsilon+1)(2\varepsilon+1)}{\nu^{\varepsilon+1}\left((1-\omega)+2\varepsilon\right)}\right)^{\tau}\sum_{k=0}^{\tau}({}^{\tau}C_k)(-1)^k(1+\omega)^{\tau-k}\left(\frac{2\omega}{(\nu)^{\varepsilon}}\right)^k\int_0^{\nu}z^{\varepsilon(k+\tau)}dz\right) \tag{61}$$

After simplification we get

$$R(\tau) = \frac{1}{1-\tau}\log\left(\frac{\left((\varepsilon+1)(2\varepsilon+1)\right)^{\tau}(\nu)^{1-\tau}}{\left((1-\omega)+2\varepsilon\right)^{\tau}}\sum_{k=0}^{\tau}\frac{({}^{\tau}C_k)(-1)^k(1+\omega)^{\tau-k}(2\omega)^k}{\varepsilon(k+\tau)+1}\right) \tag{62}$$

Similarly, the Tsallis entropy associated with the given distribution is given by

$$T_s(\xi) = \frac{1}{\xi-1}\left(1-\int_0^{\nu}\left(g\left(z_k;\varepsilon,\nu,\omega\right)\right)^{\xi}dz\right)$$

$$T_s(\xi) = \frac{1}{\xi-1}\left(1-\frac{\left((\varepsilon+1)(2\varepsilon+1)\right)^{\xi}(\nu)^{1-\xi}}{\left((1-\omega)+2\varepsilon\right)^{\xi}}\sum_{k=0}^{\xi}\frac{({}^{\xi}C_k)(-1)^k(1+\omega)^{\xi-k}(2\omega)^k}{\varepsilon(k+\xi)+1}\right) \tag{63}$$

## 11. Bonferroni and Lorenz curves

The Bonferroni curve of the given distribution is given by

$$\Psi(\zeta) = \frac{1}{\zeta\,\mu_1'}\int_0^{\varphi} z\,g\left(z;\varepsilon,\nu,\omega\right)dz$$

Where $\mu_1' = \frac{(\varepsilon+1)(2\varepsilon+1)(\nu)\left((1-\omega)(\varepsilon+2)+(1+\omega)\,\varepsilon\right)}{\left((1-\omega)+2\varepsilon\right)(\varepsilon+2)(2\varepsilon+2)}$ and $\varphi = F^{-1}(\zeta)$

$$\Psi(\zeta) = \frac{1}{\zeta\,\mu_1'}\int_0^{\varphi} z\,\frac{(\varepsilon+1)(2\varepsilon+1)z^{\varepsilon}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{\varepsilon+1}\left((1-\omega)+2\varepsilon\right)} \tag{64}$$

$$\Psi(\zeta) = \frac{1}{\zeta\,\mu_1'}\frac{(\varepsilon+1)(2\varepsilon+1)}{\nu^{\varepsilon+1}\left((1-\omega)+2\varepsilon\right)}\left((1+\omega)\int_0^{\varphi}z^{\varepsilon+1}dz-\frac{2\omega}{(\nu)^{\varepsilon}}\int_0^{\varphi}z^{2\varepsilon+1}dz\right) \tag{65}$$

$$\Psi(\zeta) = \frac{1}{\zeta\,\mu_1'}\frac{(\varepsilon+1)(2\varepsilon+1)}{\nu^{\varepsilon+1}\left((1-\omega)+2\varepsilon\right)}\left((1+\omega)\left(\frac{(\varphi)^{\varepsilon+2}}{\varepsilon+2}\right)-\frac{2\omega}{(\nu)^{\varepsilon}}\left(\frac{(\varphi)^{2\varepsilon+2}}{2\varepsilon+2}\right)\right) \tag{66}$$

After simplification we get

$$\Psi(\zeta) = \frac{(\varepsilon+1)(2\varepsilon+1)(\varphi)^{\varepsilon+2}\left((1+\omega)(2\varepsilon+2)(\nu)^{\varepsilon}-2\omega(\varepsilon+2)(\varphi)^{\varepsilon}\right)}{\zeta\,\mu_1'\nu^{2\varepsilon+1}\left((1-\omega)+2\varepsilon\right)(\varepsilon+2)(2\varepsilon+2)} \tag{67}$$

Also, the Lorenz curve of the given distribution is given by

$$\Phi(\zeta)=\zeta\,\Psi(\zeta)$$

$$\Phi(\zeta)=\zeta\left(\frac{(\varepsilon+1)(2\varepsilon+1)(\varphi)^{\varepsilon+2}\left((1+\omega)(2\varepsilon+2)(\nu)^{\varepsilon}-2\omega(\varepsilon+2)(\varphi)^{\varepsilon}\right)}{\zeta\;\mu_1'\nu^{2\varepsilon+1}\left((1-\omega)+2\varepsilon\right)(\varepsilon+2)(2\varepsilon+2)}\right) \tag{68}$$

$$\Phi(\zeta)=\frac{(\varepsilon+1)(2\varepsilon+1)(\varphi)^{\varepsilon+2}\left((1+\omega)(2\varepsilon+2)(\nu)^{\varepsilon}-2\omega(\varepsilon+2)(\varphi)^{\varepsilon}\right)}{\mu_1'\nu^{2\varepsilon+1}\left((1-\omega)+2\varepsilon\right)(\varepsilon+2)(2\varepsilon+2)} \tag{69}$$

## 12. Conclusion

In this paper, we have introduced a novel extension of the Transmuted Mukherjee-Islam distribution. This extension incorporates a weighted transformation approach and the existing three-parameter Transmuted Mukherjee-Islam distribution and generates four parametric innovative model known as the Length-Biased Transmuted Mukherjee-Islam distribution. We conduct a thorough analysis of the Length-Biased Transmuted Mukherjee-Islam distribution, investigating its mathematical formulation and statistical properties in detail. Parameter estimation for this new distribution is performed using maximum likelihood estimation techniques. Additionally, to assess the goodness of fit between these two models, we employ the likelihood ratio test.

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