ON ESTIMATION AND PREDICTION FOR THE XLINDLEY DISTRIBUTION BASED ON RECORD DATA

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Abstract

This paper investigates the estimation of the unknown parameter in the XLindley distribution using record values and inter-record times, both in classical and Bayesian frameworks. It also delves into Bayesian prediction of a future record value. We also study the problem of estimation and prediction for the XLindley distribution based on lower records alone. A simulation study, as well as an analysis of a real data example, are conducted for comparison and illustration. The numerical findings underline that including the inter-record times in the study may enhance the performance of the estimators and predictors.

Keywords: XLindley distribution, lower record values, inter-record times, Bayesian estimation and prediction.

1. Introduction

The XLindley distribution was first proposed by [8] as an effective new distribution in modeling lifetime data. Suppose that X is a random variable following the one-parameter XLindley distribution. The probability density function (PDF) and cumulative distribution function (CDF) of X are given by

$$f(x;\theta) = \frac{\theta^2}{(1+\theta)^2}(2+\theta+x)\mathrm{e}^{-\theta x}, \tag{1}$$

$$F(x;\theta) = 1 - \left(1 + \frac{\theta x}{(1+\theta)^2}\right)\mathrm{e}^{-\theta x}. \tag{2}$$

respectively.

We write $X \sim XL(\theta)$ if the PDF of X is given by (1). The XLindley distribution enjoys an increasing hazard rate function. Chouia and Zeghdoudi [8] demonstrated that the XLindley distribution can fit better than some other one-parameter distributions such as the exponential, xgamma and Lindley distributions. Due to the flexibility of the XLindley model, several inferential researches have been accomplished by authors since its inception, for example, Alotaibi et al. [2] addressed the estimation problem for the XLindley distribution using an adaptive Type-II progressively hybrid censored data, Nassar et al. [31] investigated the reliability estimation of the XLindley constant-stress partially accelerated life tests using progressively censored samples and Alotaibi et al. [3] worked on the reliability estimation under normal operating conditions

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for progressively Type-II XLindley censored data. Moreover, Metiri et al. [29] focused on the characterization of XLindley distribution using the relation between the truncated moment and failure rate function or reverse failure rate function.

Suppose that $\{X_n, n = 1, 2, \cdots\}$ is a sequence of identical and independent random variables. Let $\{X_n, n = 1, 2, \cdots\}$ be a sequence of identically distributed and independent random variables. If an observation X_j is less than all its preceding observations, then it is termed a lower record value. Similarly, upper record values can be defined based on the comparisons with preceding observations in the sequence. The sequence of lower record values along with the inter-record times can be denoted by $(\boldsymbol{R}, \boldsymbol{T}) = \{R_1, T_1, R_2, T_2, \cdots, R_{m-1}, T_{m-1}, R_m\}$ where R_i represents the i-th record value and T_i is the i-th inter-record time, which is the number of observations needed after occurrence of R_i to obtain a new record value R_{i+1}. Record data play a crucial role in various practical scenarios, see for example [5]. Record values and the related subjects have been studied by many authors; see for example [1, 11, 12, 28]. For instance, Samaniego and Whitaker [38] explored the estimation problem of the mean parameter of the exponential distribution using records and inter-record times. Doostparast [9] delved into the Bayesian and non-Bayesian estimation of the two parameters of the exponential distribution based on records and inter-record times. In a similar study, Doostparast et al. [10] investigated the Bayesian estimation of the parameters of the Pareto distribution utilizing records and inter-record times. Kzlaslan and Nadar [21] estimated the parameter of the proportional reversed hazard rate model based on records and inter-record times. Nadar and Kzlaslan [30] discussed inferential methods for the Burr type XII distribution using record values and inter-record times. Additionally, Kzlaslan and Nadar [22, 23] centered their research on inferential procedures for the generalized exponential and Kumaraswamy distributions based on record values and inter-record time statistics, respectively. Amini and MirMostafaee [4] examined interval prediction of future order statistics from the exponential distribution based on records given the inter-record times. Pak and Dey [32] developed inferential procedures for the estimation of parameters and prediction of future record values for the power Lindley model using lower record values and inter-record times. Kumar et al. [24] directed their attention towards the estimation and prediction for the unit-Gompertz distribution based on records and inter-record times. Bastan and MirMostafaee [6] explored inferential problems for the Poisson-exponential distribution based on record values and inter-record times. Khoshkhoo Amiri and MirMostafaee [19] studied estimation and prediction issues for the xgamma distribution based on lower records and inter-record times. Most recently, Khoshkhoo Amiri and MirMostafaee [20] addressed the estimation and prediction problems for the Chen distribution, utilizing lower records and inter-record times.

In this paper, we intend to discuss estimation and prediction for the XLindley distribution based on lower records and inter-record times, as well as based on lower records alone. In what follows, first, we obtain maximum likelihood (ML) estimates and asymptotic confidence intervals (ACIs) for the parameter of the XLindley distribution in Section 2. In Section 3, we go through the Bayesian estimation method and find the Bayes estimates of the parameter under a symmetric loss function and an asymmetric loss function. The Bayes estimates do not seem to be expressible in closed forms, so we become inclined to use an approximation method such as the Metropolis-Hastings algorithm. Section 4 is devoted to the Bayesian prediction of a lower future record value. A simulation study and a real data example are given in Section 5. The numerical outcomes highlight the effect of incorporating inter-record times in the study on the performance of estimators and predictors. The paper is concluded with several remarks in Section 6.

2. Maximum Likelihood Estimation

In this section, we proceed to obtain the ML estimates, as well as ACIs, for the unknown parameter θ for the XLindley model based on record data. The record data are obtained through an inverse sampling scheme, where the units are sequentially observed until the mth record occurs. Additionally, for ease of computation, the mth inter-record time is assumed to be one.

2.1. ML Estimation Based on Records and Inter-Record Times

In this subsection, our attention shifts towards the ML estimate and an ACI for the parameter. Let $\boldsymbol{r} = (r_1, \cdots, r_m)$ and $\boldsymbol{t} = (t_1, \cdots, t_{m-1})$ be the observed sets of $\boldsymbol{R} = \{R_1, \cdots, R_m\}$ and $\boldsymbol{T} = \{T_1, \cdots, T_{m-1}\}$ respectively coming from $XL(\theta)$ distribution. Then, the likelihood function of θ, given the observed lower records and inter-record times, becomes

$$L(\theta; \boldsymbol{r}, \boldsymbol{t}) = \prod_{i=1}^{m} f(r_i)[1 - F(r_i)]^{t_i - 1} = \left(\frac{\theta}{1+\theta}\right)^{2m} e^{-\theta \sum_{i=1}^{m} r_i} \prod_{i=1}^{m} \left[(2 + \theta + r_i)[\xi(r_i, \theta)]^{t_i - 1}\right], \quad (3)$$

where

$$\xi(x, \theta) = \left(1 + \frac{\theta x}{(1+\theta)^2}\right) e^{-\theta x}, \qquad \theta > 0. \quad (4)$$

It is important to note that t_m is set to one for the sake of simplifying the equations. Therefore, the resulting log-likelihood function can be expressed as

$$l(\theta; \boldsymbol{r}, \boldsymbol{t}) = 2m \ln \theta - 2m \ln(1+\theta) - \theta \sum_{i=1}^{m} r_i + \sum_{i=1}^{m} \ln(2 + \theta + r_i) + \sum_{i=1}^{m} (t_i - 1) \ln \xi(r_i, \theta).$$

Upon taking the partial derivative of the log-likelihood function with respect to (w.r.t.) θ and setting it equal to zero, we get

$$\frac{\partial l(\theta; \boldsymbol{r}, \boldsymbol{t})}{\partial \theta} = \frac{2m}{\theta(1+\theta)} - \sum_{i=1}^{m} r_i + \sum_{i=1}^{m} \frac{1}{2 + \theta + r_i} + \sum_{i=1}^{m} (t_i - 1) \frac{\psi(r_i, \theta)}{\xi(r_i, \theta)} = 0,$$

where

$$\psi(x, \theta) = \frac{\partial \xi(x, \theta)}{\partial \theta} = -x e^{-\theta x} \left(1 + \frac{\theta x}{(1+\theta)^2} + \frac{\theta - 1}{(1+\theta)^3}\right), \quad \theta > 0. \quad (5)$$

The ML estimate of θ may be determined through solving the above equation. However, it appears that there is no explicit form for the equation presented above, which necessitates the use of a numerical method. Subsequently, our focus shifts to constructing an ACI for the parameter θ. In this context, Fisher's information is defined as follows $I(\theta) = -E\left(\frac{\partial^2 l \ln f_\theta(\boldsymbol{R}, \boldsymbol{T})}{\partial \theta^2}\right)$, if the integral exists, where $f_\theta(\boldsymbol{r}, \boldsymbol{t})$ denotes the joint probability function of $R_1, T_1, R_2, T_2, \cdots, R_{m-1}, T_{m-1}, R_m$.

The second partial derivative of the log-likelihood function w.r.t. θ is given by

$$\frac{\partial^2 l(\theta; \boldsymbol{r}, \boldsymbol{t})}{\partial \theta^2} = -\frac{2m(1+2\theta)}{[\theta(1+\theta)]^2} - \sum_{i=1}^{m} \frac{1}{(2 + \theta + r_i)^2} + \sum_{i=1}^{m} (t_i - 1) \left(\frac{\psi'(r_i, \theta)\xi(r_i, \theta) - [\psi(r_i, \theta)]^2}{[\xi(r_i, \theta)]^2}\right),$$

where

$$\psi'(x, \theta) = \frac{\partial \psi(x, \theta)}{\partial \theta} = x e^{-\theta x} \left(x + \frac{\theta x^2}{(1+\theta)^2} + \frac{2x(\theta - 1)}{(1+\theta)^3} - \frac{2(2-\theta)}{(1+\theta)^4}\right), \quad \theta > 0. \quad (6)$$

Let $\hat{\theta}_{ML}$ denote the ML estimator (MLE) of θ. Then, the $100(1-\alpha)\%$ modified asymptotic two-sided equi-tailed confidence interval (MATE CI) for θ can be given by (see for example [25])

$$\left(\max\left\{0, \hat{\theta}_{ML} - \frac{z_{\frac{\alpha}{2}}}{\sqrt{\tilde{I}(\hat{\theta}_{ML})}}\right\}, \hat{\theta}_{ML} + \frac{z_{\frac{\alpha}{2}}}{\sqrt{\tilde{I}(\hat{\theta}_{ML})}}\right),$$

where z_γ represents the γ-th upper quantile of the standard normal distribution and

$$\tilde{I}(\hat{\theta}_{ML}) = -\frac{\partial^2 l(\theta | \boldsymbol{R}, \boldsymbol{T})}{\partial \theta^2}\bigg|_{\theta = \hat{\theta}_{ML}}.$$

2.2. ML Estimation Based on Record Values

The likelihood function of θ given the lower records $\boldsymbol{r}$ (without considering the inter-record times) is given by

$$L^*(\theta;\boldsymbol{r}) = f(r_m)\prod_{i=1}^{m-1}\frac{f(r_i)}{F(r_i)} = (\frac{\theta}{1+\theta})^{2m}\mathrm{e}^{-\theta\sum_{i=1}^{m} r_i}\frac{\prod_{i=1}^{m}(2+\theta+r_i)}{\prod_{i=1}^{m-1}(1-\xi(r_i,\theta))}, \tag{7}$$

where $\xi(x,\theta)$ is defined in (4).

The corresponding log-likelihood function of θ is then given by

$$l^*(\theta;\boldsymbol{r}) = 2m\ln\theta - 2m\ln(1+\theta) - \theta\sum_{i=1}^{m} r_i + \sum_{i=1}^{m}\ln(2+\theta+r_i) - \sum_{i=1}^{m-1}\ln[1-\xi(r_i,\theta)]. \tag{8}$$

Taking the first partial derivative of the log-likelihood (8) w.r.t. θ and equating it with zero, we have

$$\frac{\partial l^*(\theta;\boldsymbol{r})}{\partial\theta} = \frac{2m}{\theta(1+\theta)} - \sum_{i=1}^{m} r_i + \sum_{i=1}^{m}\frac{1}{2+\theta+r_i} + \sum_{i=1}^{m-1}\frac{\psi(r_i,\theta)}{1-\xi(r_i,\theta)} = 0,$$

where $\psi(x,\theta)$ is defined in (5).

So the ML estimate may be obtained by solving the above equation with the help of a numerical technique.

The second partial derivate of (8) w.r.t. θ is obtained to be

$$\frac{\partial^2 l^*(\theta;\boldsymbol{r})}{\partial\theta^2} = -\frac{2m(1+2\theta)}{[\theta(1+\theta)]^2} - \sum_{i=1}^{m}\frac{1}{(2+\theta+r_i)^2} + \sum_{i=1}^{m-1}\left(\frac{\psi'(r_i,\theta)[1-\xi(r_i,\theta)] + [\psi(r_i,\theta)]^2}{[1-\xi(r_i,\theta)]^2}\right),$$

where $\psi'(x,\theta)$ is defined in (6).

Let $\hat{\theta}^*_{ML}$ denote the MLE of θ based on lower records. Following the same approach described in the previous subsection, the $100(1-\alpha)\%$ MATE CI for θ can be given by

$$\left(\max\left\{0, \hat{\theta}^*_{ML} - \frac{z_{\frac{\alpha}{2}}}{\sqrt{\tilde{I}^*(\hat{\theta}^*_{ML})}}\right\}, \hat{\theta}_{ML} + \frac{z_{\frac{\alpha}{2}}}{\sqrt{\tilde{I}^*(\hat{\theta}^*_{ML})}}\right),$$

where

$$\tilde{I}^*(\hat{\theta}^*_{ML}) = -\frac{\partial^2 l^*(\theta|\boldsymbol{R},\boldsymbol{T})}{\partial\theta^2}\bigg|_{\theta=\hat{\theta}^*_{ML}}.$$

3. Bayesian Estimation

In the context of Bayesian estimation, the experimenter's information can be conveyed through a probability distribution for the parameter, referred to as the prior distribution. Due to the constraint that the parameter of the XLindley distribution must be positive, we use the popular gamma prior for θ, whose PDF is given by

$$\Pi(\theta) = \frac{b^a\theta^{a-1}\mathrm{e}^{-b\theta}}{\Gamma(a)}, \tag{9}$$

where, the positive hyperparameters a and b can be set based on the prior information available to the experimenter. In what follows, we focus on the Bayesian estimation of θ based on records and inter-record times and based on records alone.

3.1. Bayesian Estimation Based on Records and Inter-Record Times

Using (3) and the prior (9), we can derive the posterior density of θ given $\boldsymbol{r}$ and $\boldsymbol{t}$ as follows

$$\Pi(\theta|\boldsymbol{r},\boldsymbol{t}) = \frac{\theta^{2m+a-1}}{D(1+\theta)^{2m}} e^{-\theta(b+\sum_{i=1}^{m} r_i)} \prod_{i=1}^{m} \left[(2+\theta+r_i)[\xi(r_i,\theta)]^{t_i-1}\right],$$

where $\xi(x,\theta)$ is defined in (4) and

$$D = \int_0^\infty \frac{\theta^{2m+a-1}}{(1+\theta)^{2m}} e^{-\theta(b+\sum_{i=1}^{m} r_i)} \prod_{i=1}^{m} \left[(2+\theta+r_i)[\xi(r_i,\theta)]^{t_i-1}\right] d\theta.$$

The squared error loss function (SELF) is widely used in Bayesian analyses. However, the SELF may not be appropriate for many real-world scenarios due to its equal weighting of overestimation and underestimation. An alternative asymmetric loss function is the linear-exponential loss function (LELF), proposed by [42], which is given by

$$L_{LE}(\theta,\hat{\theta}) = b[\exp\{c(\hat{\theta}-\theta)\} - c(\hat{\theta}-\theta) - 1], \qquad b > 0, \qquad c \neq 0,$$

where $\hat{\theta}$ denotes an estimator of θ.

Without loss of generality, we assume $b = 1$. The appropriate determination of c involves considering both its sign and magnitude. When $c > 0$, then overestimation is more serious than underestimation and vice versa, see [43] for more details. The Bayes estimates of θ under the SELF and LELF become

$$\hat{\theta}_{SE} = \int_0^\infty \theta \Pi(\theta|\boldsymbol{r},\boldsymbol{t}) d\theta = \frac{1}{D} \int_0^\infty \frac{\theta^{2m+a}}{(1+\theta)^{2m}} e^{-\theta(b+\sum_{i=1}^{m} r_i)} \prod_{i=1}^{m} \left[(2+\theta+r_i)[\xi(r_i,\theta)]^{t_i-1}\right] d\theta,$$

and

$$\begin{aligned}\hat{\theta}_{LE} &= -\frac{1}{c} \ln M(-c|\boldsymbol{r},\boldsymbol{t}) = -\frac{1}{c} \ln[\int_0^\infty e^{-c\theta} \Pi(\theta|\boldsymbol{r},\boldsymbol{t}) d\theta] \\ &= -\frac{1}{c} \ln\left(\frac{1}{D} \int_0^\infty \frac{\theta^{2m+a-1}}{(1+\theta)^{2m}} e^{-\theta(c+b+\sum_{i=1}^{m} r_i)} \prod_{i=1}^{m} \left[(2+\theta+r_i)[\xi(r_i,\theta)]^{t_i-1}\right] d\theta\right),\end{aligned}$$

respectively, provided that the integrals exist.

It appears that the above Bayes estimates of θ may not be expressible in closed forms. Therefore, we resort to an approximation method, called the Metropolis-Hastings (M-H) algorithm [27, 15]. An M-H algorithm suitable for our scenario can be outlined as follows.

Algorithm 1

Step1. Start with an initial guess $\theta_0 = \hat{\theta}_{ML}$ and set $t = 1$.

Step2. Given θ_{t-1}, generate θ^* from a truncated-normal distribution, $N(\theta_{t-1}, \sigma^2)I_{\{\theta>0\}}$. Then, assign $\theta_t = \theta^*$ with the following probability

$$P = \min\left\{\frac{\Pi(\theta^*|\boldsymbol{r},\boldsymbol{t})q(\theta_{t-1}|\theta^*)}{\Pi(\theta_{t-1}|\boldsymbol{r},\boldsymbol{t})q(\theta^*|\theta_{t-1})}, 1\right\},$$

where $q(x|b)$ represents the density of $N(b,\sigma^2)I_{\{x>0\}}$, otherwise set $\theta_t = \theta_{t-1}$.

Step3. Set $t = t+1$ and repeat Step 2, T times, where T is a considerably large number. So, $\{\theta_{M+1}, \theta_{M+2}, \cdots, \theta_T\}$ constitutes the generated sample, where M denotes the burn-in period.

The approximate Bayes point estimates of θ under the SELF and LELF are then given by

$$\hat{\theta}_{SM} = \frac{1}{M^*} \sum_{t=M+1}^{T} \theta_t, \qquad \text{and} \qquad \hat{\theta}_{LM} = -\frac{1}{c} \ln\left(\frac{1}{M^*} \sum_{t=M+1}^{T} e^{-c\theta_t}\right),$$

respectively, with $M^* = T - M$. In Section 5, we have taken $\sigma^2 = 1$.

Let $\theta_{(1)} \cdots \theta_{(M^*)}$ denote the ordered values of $\theta_{M+1}, \cdots, \theta_T$. Define the intervals $L_j(M^*) = [\theta_{(j)}, \theta_{(j+\lceil(1-\alpha)M^*\rceil)}]$ for $j = 1, 2, \cdots, M^* - \lceil(1-\alpha)M^*\rceil$. Consequently, the $100(1-\alpha)\%$ Chen and Shao short width credible interval (CSSW CrI) for θ can be represented as $L_q(M^*)$, where q is determined such that [7]

$$\theta_{(q+[(1-\alpha)M^*])} - \theta_{(q)} = \min_{1 \leq j \leq M^* - [(1-\alpha)M^*]} \theta_{(j+[(1-\alpha)M^*])} - \theta_{(j)}.$$

3.2. Bayesian Estimation Based on Record Values

Using (7) and the prior (9), the posterior density of θ given $\boldsymbol{r}$ is derived to be

$$\Pi^*(\theta|\boldsymbol{r}) = \frac{\theta^{2m+a-1}}{D^*(1+\theta)^{2m}} \mathrm{e}^{-\theta(b+\sum_{i=1}^m r_i)} \frac{\prod_{i=1}^m (2+\theta+r_i)}{\prod_{i=1}^{m-1}[1-\xi(r_i,\theta)]},$$

where

$$D^* = \int_0^\infty \frac{\theta^{2m+a-1}}{(1+\theta)^{2m}} \mathrm{e}^{-\theta(b+\sum_{i=1}^m r_i)} \frac{\prod_{i=1}^m (2+\theta+r_i)}{\prod_{i=1}^{m-1}[1-\xi(r_i,\theta)]} \mathrm{d}\theta.$$

The Bayes estimates of θ under the SELF and LELF become

$$\hat{\theta}^*_{SE} = \frac{1}{D^*} \int_0^\infty \frac{\theta^{2m+a}}{(1+\theta)^{2m}} \mathrm{e}^{-\theta(b+\sum_{i=1}^m r_i)} \frac{\prod_{i=1}^m (2+\theta+r_i)}{\prod_{i=1}^{m-1}[1-\xi(r_i,\theta)]} \mathrm{d}\theta,$$

and

$$\hat{\theta}^*_{LE} = -\frac{1}{c} \ln \left(\frac{1}{D^*} \int_0^\infty \frac{\theta^{2m+a-1}}{(1+\theta)^{2m}} \mathrm{e}^{-\theta(c+b+\sum_{i=1}^m r_i)} \frac{\prod_{i=1}^m (2+\theta+r_i)}{\prod_{i=1}^{m-1}[1-\xi(r_i,\theta)]} \mathrm{d}\theta \right),$$

respectively, provided that the related integrals exist.

It appears that the above Bayes estimates of θ may not have closed-form expressions. So we may use the M-H algorithm (similar to that described in Algorithm 1) to approximate these Bayes estimates, see Subsection 3.1. We can also obtain the $100(1-\alpha)\%$ CSSW CrI for θ using a similar approach detailed in Subsection 3.1.

4. Bayesian Prediction

Let $R_1, T_1, R_2, T_2, \cdots, R_{m-1}, T_{m-1}, R_m$ be the first m of lower record values and their corresponding inter-record times from $XL(\theta)$. Let further $\boldsymbol{r} = (r_1, \cdots, r_m)$ and $\boldsymbol{t} = (t_1, \cdots, t_{m-1})$ be the observed sets of $\boldsymbol{R} = \{R_1, \cdots, R_m\}$ and $\boldsymbol{T} = \{T_1, \cdots, T_{m-1}\}$. We intend to predict the s-th unobserved lower record value, R_s, where $s > m$. Using the Morkovian property of records, the conditional PDF of R_s given $\boldsymbol{R} = \boldsymbol{r}$ and $\boldsymbol{T} = \boldsymbol{t}$, denoted by $f(r_s|\theta, \boldsymbol{r}, \boldsymbol{t})$ is identical to the conditional PDF of R_s given $R_m = r_m$, denoted by $f(r_s|\theta, r_m)$ (see for example [5, 20]). So, we have

$$\begin{aligned} f(r_s|\theta, \boldsymbol{r}, \boldsymbol{t}) \equiv f(r_s|\theta, r_m) &= \frac{f(r_s;\theta)[Q(r_s,\theta) - Q(r_m,\theta)]^{s-m-1}}{F(r_m,\theta)\Gamma(s-m)} \\ &= [Q(r_s,\theta) - Q(r_m,\theta)]^{s-m-1} \frac{\left(\frac{\theta}{1+\theta}\right)^2 (\theta+2+r_s)}{[1-\xi(r_m,\theta)]\Gamma(s-m)} \mathrm{e}^{-\theta r_s}, \end{aligned} \tag{10}$$

where $0 < r_s < r_m$, $Q(x,\theta) = -\ln(F(x;\theta))$ and $\xi(x,\theta)$ is defined in (4).

The Bayes predictive density of R_s given the lower records and inter-record times is obtained to be

$$h(r_s|\boldsymbol{r}, \boldsymbol{t}) = \int_0^\infty f(r_s|\theta, r_m) \Pi(\theta|\boldsymbol{r}, \boldsymbol{t}) \mathrm{d}\theta.$$

It can be easily seen that the associated posterior predictive density may not be obtained analytically. Thus, we estimate $h(r_s|,\boldsymbol{r},\boldsymbol{t})$ by means of a sample generated using the M-H algorithm. Let $\{\theta_v, v=1,\cdots,M^*\}$ be the generated sample using Algorithm 1, where $M^*=T-M$. Then, an estimate of $h(r_s|\boldsymbol{r},\boldsymbol{t})$ is given by

$$\tilde{h}(r_s|\boldsymbol{r},\boldsymbol{t}) = \frac{1}{M^*}\sum_{v=1}^{M^*} f(r_s|\theta_v, r_m).$$

The approximate predictions of R_s under the SELF and LELF (provided that they exist) can be obtained as

$$\tilde{R}_s^{SEM} = \int_0^{r_m} r_s\tilde{h}(r_s|\boldsymbol{r},\boldsymbol{t})\mathrm{d}r_s = \frac{1}{M^*}\sum_{v=1}^{M^*}\int_0^{r_m} r_s f(r_s|\theta_v, r_m)\mathrm{d}r_s, \tag{11}$$

and

$$\tilde{R}_s^{LEM} = \frac{-1}{c}\ln\left[\int_0^{r_m} \mathrm{e}^{-cr_s}\tilde{h}(r_s|\boldsymbol{r},\boldsymbol{t})\mathrm{d}r_s\right] = \frac{-1}{c}\ln[\frac{1}{M^*}\sum_{v=1}^{M^*}\int_0^{r_m}\mathrm{e}^{-cr_s} f(r_s|\theta_v, r_m)\mathrm{d}r_s], \tag{12}$$

respectively.

A $100(1-\alpha)\%$ two-sided Bayesian prediction interval for R_s is given by $(L(\boldsymbol{r},\boldsymbol{t}), U(\boldsymbol{r},\boldsymbol{t}))$, where $L(\boldsymbol{r},\boldsymbol{t})$ and $U(\boldsymbol{r},\boldsymbol{t})$ satisfy the following equations at the same time

$$\int_0^{L(\boldsymbol{r},\boldsymbol{t})} h(r_s \mid \boldsymbol{r},\boldsymbol{t})\mathrm{d}r_s = \frac{\alpha}{2}, \qquad \text{and} \qquad \int_0^{U(\boldsymbol{r},\boldsymbol{t})} h(r_s \mid \boldsymbol{r},\boldsymbol{t})\mathrm{d}r_s = 1-\frac{\alpha}{2}.$$

A $100(1-\alpha)\%$ approximate two-sided Bayesian prediction interval (ATB PI) for R_s is given by (L, U), where L and U satisfy the following equations at the same time

$$\frac{1}{M^*}\sum_{v=M+1}^{T}\int_0^{L} f(r_s \mid \theta_v, r_m)\mathrm{d}r_s = \frac{\alpha}{2}, \qquad \text{and} \qquad \frac{1}{M^*}\sum_{v=M+1}^{T}\int_0^{U} f(r_s \mid \theta_v, r_m)\mathrm{d}r_s = 1-\frac{\alpha}{2}.$$

4.1. Special Case: $s = m+1$

For the special case, when $s=m+1$, then $Y=R_{m+1}$ given $R_m=r_m$ follows the truncated XLindley distribution on interval $(0, r_m)$. So, we have

$$\begin{aligned} f(r_{m+1}|\theta,\boldsymbol{r},\boldsymbol{t}) \equiv f(r_{m+1}|\theta, r_m) &= \frac{f(r_{m+1};\theta)}{F(r_m,\theta)} \\ &= \frac{\left(\frac{\theta}{1+\theta}\right)^2(\theta+2+r_{m+1})}{1-\xi(r_m,\theta)}\,\mathrm{e}^{-\theta r_{m+1}}, \quad 0<r_{m+1}<r_m. \end{aligned} \tag{13}$$

Moreover, we have following two relations

$$\begin{aligned} \int_0^{r_m} r_{m+1} f(r_{m+1}|\theta, r_m)\mathrm{d}r_{m+1} &= -\frac{(\theta+2)(\theta r_{m+1}+1)+r_{m+1}(\theta r_{m+1}+2)+2/\theta}{(1+\theta)^2[1-\xi(r_m,\theta)]}\mathrm{e}^{-\theta r_{m+1}}\Bigg]_0^{r_m} \\ &= \frac{\theta+2+2/\theta-[(\theta+2)(\theta r_m+1)+r_m(\theta r_m+2)+2/\theta]\mathrm{e}^{-\theta r_m}}{(1+\theta)^2[1-\xi(r_m,\theta)]}, \end{aligned}$$

$$\begin{aligned} \int_0^{r_m} \mathrm{e}^{-cr_{m+1}} f(r_{m+1}|\theta, r_m)\mathrm{d}r_{m+1} &= -\frac{\theta^2[1+(\theta+c)(\theta+2+r_{m+1})]}{(\theta+c)^2(1+\theta)^2[1-\xi(r_m,\theta)]}\mathrm{e}^{-(\theta+c)r_{m+1}}\Bigg]_0^{r_m} \\ &= \frac{\theta^2\{1+(\theta+c)(\theta+2)-[1+(\theta+c)(\theta+2+r_m)]\mathrm{e}^{-(\theta+c)r_m}\}}{(\theta+c)^2(1+\theta)^2[1-\xi(r_m,\theta)]}. \end{aligned}$$

Therefore, from (11) and (12), the approximate predictions of R_s under the SELF and LELF can be obtained as

$$\begin{aligned}\tilde{R}_{m+1}^{SEM} &= \frac{1}{M^*}\sum_{v=1}^{M^*}\int_0^{r_m} r_{m+1} f(r_{m+1}|\theta_v, r_m)\mathrm{d}r_{m+1} \\ &= \frac{1}{M^*}\sum_{v=1}^{M^*}\frac{\theta_v + 2 + 2/\theta_v - [(\theta_v+2)(\theta_v r_m + 1) + r_m(\theta_v r_m + 2) + 2/\theta_v]\mathrm{e}^{-\theta_v r_m}}{(1+\theta_v)^2[1-\xi(r_m,\theta_v)]},\end{aligned}$$

and

$$\begin{aligned}\tilde{R}_{m+1}^{LEM} &= \frac{-1}{c}\ln\left[\frac{1}{M^*}\sum_{v=1}^{M^*}\int_0^{r_m}\mathrm{e}^{-cr_s} f(r_s|\theta_v, r_m)\mathrm{d}r_s\right] \\ &= \frac{-1}{c}\ln\left[\frac{1}{M^*}\sum_{v=1}^{M^*}\frac{\theta_v^2\{1+(\theta_v+c)(\theta_v+2) - [1+(\theta_v+c)(\theta_v+2+r_m)]\mathrm{e}^{-(\theta_v+c)r_m}\}}{(\theta_v+c)^2(1+\theta_v)^2[1-\xi(r_m,\theta_v)]}\right]\end{aligned}$$

respectively.

Additionally, A $100(1-\alpha)\%$ ATB PI for R_{m+1} is given by (L, U), where L and U satisfy the following nonlinear equations

$$\frac{1}{M^*}\sum_{v=M+1}^{T}\frac{1-\xi(L,\theta_v)}{1-\xi(r_m,\theta_v)} = \frac{\alpha}{2}, \qquad \text{and} \qquad \frac{1}{M^*}\sum_{v=M+1}^{T}\frac{1-\xi(U,\theta_v)}{1-\xi(r_m,\theta_v)} = 1-\frac{\alpha}{2},$$

where $\xi(x,\theta)$ is defined in (4).

Remark 1. Using the Morkovian property of records, the conditional PDF of R_s given $\boldsymbol{R} = \boldsymbol{r}$ is identical to the conditional PDF of R_s given $R_m = r_m$ (see for example [5, 20]). Therefore, the approximate Bayesian point predictions and a $100(1-\alpha)\%$ ATB PI for R_s based on record values can be obtained using a similar procedure described above, with this difference that the M-H sample, $\{\theta_v, v = 1, \cdots, M^*\}$, must be generated based on only records.

5. Numerical Illustration

This section involves a simulation study, as well as a real data analysis.

5.1. A Simulation Study

Here, we conduct a Monte Carlo simulation to evaluate the accuracy of the point and interval estimators and approximate predictors that are mentioned in this paper. In this simulation study, we set the number of replications to $N^* = 1000$. For each replication, we generate $(m+1)$ records and their associated inter-record times from $XL(\theta)$. We consider the values of m to be $m = 3, 4, 5$ and the values of the parameter to be $\theta = 0.5, 1$ and 2. In the context of the Bayesian estimation, we use the approximate non-informative prior with $a = b = 0.1$. A few replications for which the predictions became negative were removed from the simulation.

We obtain the ML estimates and the approximate Bayes estimates based on the first m records and their corresponding $(m-1)$ record times and based on the first m records alone. Furthermore, we use Geweke's test [13], Raftery and Lewiss diagnostic [36, 37] and Heidelberger and Welch's convergence diagnostic [18] to assess the convergence of the generated M-H Markov chains. It is worth noting that Heidelberger and Welch [18] made use of or referenced the findings of [39, 16, 17, 40, 41]. In some cases, we have taken every second sampled value (and adjusted the number of sampled values accordingly) to ensure a convergent M-H Markov chain. All the final chains have sizes equal to 10000. Figure 1 shows the M-H Markov chains (the figure is for $m = 4$ and $\theta = 1$), from which the convergence of the M-H algorithm may be confirmed.

The performance of the different estimators is compared based on their estimated biases (biases for short) and estimated risks (ERs). Additionally, we evaluate the interval estimators

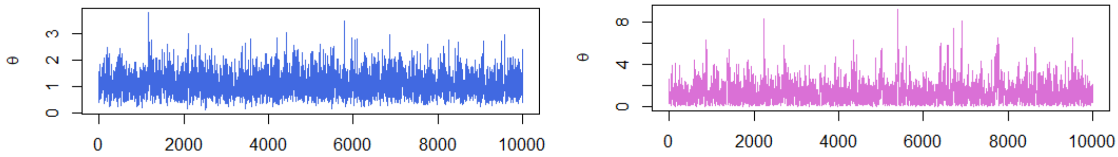

Figure 1: *Plots of Markov chains for θ, the left panel is for the case based on records and inter-records times, whereas the right panel is for the case based on lower records alone ($m = 4$ and $\theta = 1$).*

and predictors using the average width (AW) and coverage probability (CP) criteria. If $\widehat{\theta}$ is an estimator of θ and $\widehat{\theta}_i$ is the corresponding estimate obtained in the i-th replication, then the bias and ERs of $\widehat{\theta}$ w.r.t. the SELF and LELF are given by

$$Bias(\widehat{\theta}) = \frac{1}{N^*}\sum_{i=1}^{N^*}(\widehat{\theta}_i - \theta), \quad (14)$$

$$ER_S(\widehat{\theta}) = \frac{1}{N^*}\sum_{i=1}^{N^*}(\widehat{\theta}_i - \theta)^2, \quad (15)$$

and

$$ER_L(\widehat{\theta}) = \frac{1}{N^*}\sum_{i=1}^{N^*}\left(\exp[c(\widehat{\theta}_i - \theta)] - c(\widehat{\theta}_i - \theta) - 1\right), \quad (16)$$

respectively.

The point and interval predictions for the $(m+1)$-th record value, namely R_{m+1}, are also calculated. In terms of prediction assessment, we consider the estimated bias (bias for short) and the estimated prediction risks (EPRs) w.r.t. to the SELF and LELF for the point predictors, which are defined similarly to (14), (15) and (16), respectively. The simulation results are given in Table 1 for point estimation, Table 2 for point prediction and Table 3 for interval estimation and prediction. The results for point estimation and prediction in Tables 1 and 2 are provided for $m = 4$ and 5 for the sake of brevity, whereas the results presented in Table 3 are provided for $m = 3, 4$ and 5.

Based on Tables 1-3, we draw the following conclusions:

- The point estimators based on records and inter-record times outperform the corresponding point estimators based on record alone in terms of bias and ER in the most cases. Additionally, the biases and EPRs of the approximate point predictors based on records and inter-record times are smaller than those of approximate point predictors based on records alone in the most cases, as well.
- The ERs of the point estimators for $\theta = 1$ and 2 decrease w.r.t. to m in the most cases, whereas the EPRs of the point predictors decrease w.r.t. m for all selected values of θ without any exception.
- The AWs of the 95% approximate interval estimators and predictors based on records and inter-record times are less than those of the 95% approximate interval estimators and predictors based on records alone (except for one case for which they are equal up to 5 decimals).
- The CPs of the 95% approximate interval estimators and predictors are all equal to or close to the nominal value 0.95, as expected.

Table 1: *The biases and ERs of the point estimators of θ based on records and inter-record times (first row) and based on records alone (second row).*

	$m = 4$				$m = 5$			
$\theta = 0.5$	bias	ER_S	ER_L $c = 0.5$	ER_L $c = -0.5$	bias	ER_S	ER_L $c = 0.5$	ER_L $c = -0.5$
MLE	0.0868	0.0971	0.0156	0.0101	0.0669	0.0483	0.0066	0.0056
	1.6656	99.041	> 100	0.6617	0.1215	> 100	> 100	0.8858
Bayes (SELF)	0.0943	0.0991	0.0156	0.0104	0.0729	0.0510	0.0070	0.0059
	0.6746	2.7953	1.7739	0.1732	0.6977	3.0185	2.6072	0.1802
Bayes (LELF)	0.0795	0.0821	0.0123	0.0089	0.0634	0.0457	0.0062	0.0053
$c = 0.5$	0.3751	0.7616	0.1494	0.0684	0.3854	0.7803	0.1546	0.0697
Bayes (LELF)	0.1112	0.1246	0.0212	0.0126	0.0831	0.0573	0.0079	0.0066
$c = -0.5$	1.7028	25.912	> 100	0.6366	1.7503	26.783	> 100	0.6545
$\theta = 1$								
MLE	0.2293	0.5745	0.2497	0.0466	0.1564	0.3371	0.1164	0.0300
	7.4661	> 100	> 100	3.4686	4.8365	> 100	> 100	2.1844
Bayes (SELF)	0.2394	0.5045	0.1371	0.0443	0.1688	0.3186	0.0806	0.0297
	1.2180	6.3531	43.106	0.3665	1.0927	5.8714	9.0584	0.3326
Bayes (LELF)	0.1652	0.3243	0.0602	0.0317	0.1186	0.2277	0.0412	0.0230
$c = 0.5$	0.4764	1.1666	0.2182	0.1077	0.4107	1.0520	0.1980	0.0973
Bayes (LELF)	0.3505	1.2209	14.838	0.0718	0.2316	0.5204	0.5563	0.0406
$c = -0.5$	3.9390	75.039	> 100	1.6089	3.4741	67.636	> 100	0.0481
$\theta = 2$								
MLE	0.5934	2.9392	2.0688	0.1996	0.4055	1.7905	0.9308	0.1353
	9.4667	> 100	> 100	4.4454	9.9441	> 100	> 100	4.6788
Bayes (SELF)	0.5169	2.1067	0.7134	0.1628	0.3695	1.4141	0.4473	0.1166
	1.5136	9.0327	7.3213	0.5359	1.5155	8.9470	6.8274	0.5322
Bayes (LELF)	0.2013	0.9833	0.1922	0.0943	0.1448	0.7758	0.1505	0.0759
$c = 0.5$	0.1222	1.0090	0.1527	0.1139	0.1291	1.0102	0.1549	0.1128
Bayes (LELF)	1.0939	6.8180	59.371	0.3483	0.7305	3.6409	17.917	0.2150
$c = -0.5$	6.5314	> 100	> 100	2.7891	6.6037	> 100	> 100	2.8190

Table 2: *The biases and EPRs of the approximate Bayes point predictors of θ based on records and inter-record times (first row) and based on records alone (second row)..*

	$m=4$				$m=5$			
$\theta=0.5$	bias	ER_S	ER_L $c=0.5$	ER_L $c=-0.5$	bias	ER_S	ER_L $c=0.5$	ER_L $c=-0.5$
SELF	0.002842	0.017065	0.002167	0.002128	0.000548	0.006948	0.000912	0.000843
	0.005136	0.017106	0.002211	0.002098	0.001393	0.007154	0.000958	0.000854
LELF	-0.001371	0.017407	0.002159	0.002222	-0.000844	0.006759	0.000865	0.000837
$c=0.5$	0.000895	0.017163	0.002166	0.002155	-0.000003	0.006893	0.000899	0.000841
LELF	0.007078	0.017195	0.002235	0.002097	0.001943	0.007233	0.000975	0.000858
$c=-0.5$	0.009345	0.017514	0.002317	0.002102	0.002780	0.007493	0.001031	0.000874
$\theta=1$								
SELF	0.002136	0.002491	0.000312	0.000312	0.001369	0.000653	0.000082	0.000081
	0.002754	0.002453	0.000309	0.000305	0.001667	0.000663	0.000084	0.000082
LELF	0.001516	0.002487	0.000310	0.000313	0.001114	0.000645	0.000081	0.000080
$c=0.5$	0.002130	0.002437	0.000305	0.000305	0.001411	0.000651	0.000082	0.000081
LELF	0.002760	0.002504	0.000315	0.000312	0.001625	0.000664	0.000084	0.000082
$c=-0.5$	0.003379	0.000248	0.000313	0.000307	0.001924	0.000678	0.000086	0.000084
$\theta=2$								
SELF	0.000476	0.000378	0.000047	0.000047	-0.000272	0.000105	0.000013	0.000013
	0.000689	0.000397	0.000050	0.000049	-0.000208	0.000106	0.000013	0.000013
LELF	0.000381	0.000374	0.000047	0.000047	-0.000302	0.000105	0.000013	0.000013
$c=0.5$	0.000594	0.000392	0.000049	0.000049	-0.000239	0.000106	0.000013	0.000013
LELF	0.000572	0.000382	0.000048	0.000048	-0.000242	0.000105	0.000013	0.000013
$c=-0.5$	0.000785	0.000401	0.000050	0.000050	-0.000177	0.000106	0.000013	0.000013

Table 3: *The AWs and CPs of 95% approximate interval estimators and predictors based on records and inter-record times (first row) and based on records alone (second row).*

	$m=3$		$m=4$		$m=5$	
$\theta=0.5$	AW	CP	AW	CP	AW	CP
MATE CI	1.09579	0.962	0.83556	0.963	0.70605	0.956
	6.92765	0.964	5.89956	0.959	7.21595	0.960
CSSW CrI	1.03103	0.955	0.80036	0.952	0.68460	0.951
	2.98580	0.957	2.93030	0.952	3.00196	0.957
ATB PI	0.47067	0.945	0.23426	0.938	0.11708	0.948
	0.47077	0.942	0.23428	0.938	0.11708	0.949
$\theta=1$						
MATE CI	2.55093	0.953	1.95860	0.954	1.61990	0.951
	11.8789	0.966	24.2695	0.977	16.4109	0.965
CSSW CrI	2.29057	0.954	1.82186	0.955	1.54003	0.946
	5.46901	0.962	5.77409	0.975	5.43254	0.960
ATB PI	0.18213	0.955	0.08794	0.946	0.04852	0.950
	0.18224	0.955	0.08797	0.946	0.04853	0.950
$\theta=2$						
MATE CI	5.81158	0.946	4.58167	0.956	3.75708	0.956
	29.2559	0.962	32.4821	0.955	33.8740	0.967
CSSW CrI	4.66783	0.952	4.00381	0.958	3.39805	0.953
	8.67913	0.959	9.29240	0.952	9.29188	0.962
ATB PI	0.08007	0.935	0.03415	0.954	0.01698	0.950
	0.08012	0.937	0.03416	0.952	0.01699	0.950

5.2. Real Data Example

Here, we consider the following data on the amount of rainfall (in inches) recorded at the Los Angeles Civic Center in February from 1999 to 2018; visit the website of Los Angeles Almanac: www.laalmanac.com/weather/we08aa.php.

```
0.56, 5.54,  8.87, 0.29,  4.64, 4.89, 11.02, 2.37, 0.92,  1.64,
3.57, 4.27,  3.29, 0.16,  0.20, 3.58, 0.83,  0.79, 4.17,  0.03.
```

We have used the Kolmogorov-Smirnov (K-S) test to check if the XLindley model fits the data. The K-S test statistic confirms that the XLindley distribution is quite suitable for fitting the above data (p-value greater than 0.5). We have extracted the lower records and the corresponding inter-record times as follows:

i	1	2	3	4
r_i	0.56	0.29	0.16	0.03
k_i	3	10	6	1

Here, we have used the approximate non-informative prior with $a = b = 0.1$. We have computed the ML and approximate Bayes point estimates, along with the 95% approximate interval estimates of the parameter for the XLindley distribution. Additionally, we have derived the point predictions and 95% ATB PIs for the next future record, namely R_5. The numerical results of this example are given in Table 4, where Case I denotes the case based on records and inter-record times, whereas Case II denotes the case based on records alone. Our findings suggest that the subsequent lowest rainfall amount (after 2018) is expected to be around 0.015 inches, which is the predicted 5-th lower record value since 1999.

Table 4: *The numerical results of the real data example.*

Estimation	MLE	SELF	LELF ($c = 0.5$)	LELF ($c = -0.5$)	95% MATE CI	95% CSSW CrI
Case I	0.9535	0.9729	0.9405	1.0087	(0.2470, 1.6601)	(0.3781, 1.7523)
Case II	1.8809	1.8679	1.4782	2.9436	(0, 4.9336)	(0.0400, 4.6741)
Prediction		**SELF**	**LELF ($c = 0.5$)**	**LELF ($c = -0.5$)**	**95% ATB PI**	
Case I		0.01495	0.01493	0.01497	(0.00074, 0.02924)	
Case II		0.01488	0.01486	0.01490	(0.00073, 0.02923)	

6. Concluding Remarks

Recently, the XLindley distribution has been introduced by [8] aiming at proposing a flexible distribution for lifetime phenomena. In our study, first, we obtained the ML estimates of the XLindley parameter based on record values and inter-record times, as well as solely based on records. Then, we considered the Bayesian estimation of the parameter, and we employed both symmetric and asymmetric loss functions. The Bayesian point estimates involve integrals that seem to lack closed forms, so we have utilized the M-H method to evaluate them. Our study extended to predicting future records, especially the immediate subsequent lower record value as a special case has been explored in detail. A simulation study has been conducted to evaluate the point and interval estimators of the unknown parameter of the XLindley distribution along with the approximate point and interval predictors of a future lower record value. The simulation study revealed the impact of including the inter-record times on the performance of the estimators and predictors. Furthermore, a real data set containing the rainfall data was analyzed, where a lower record value could serve as an indicator of an impending drought. The predicted values of the 5-th lower record have been obtained in the example. Summing up, the results of this paper are anticipated to offer practical utility in the estimation and prediction in real phenomena. All the computations of the paper were carried out using the statistical software R [35], and the packages coda [33, 34], nleqslv [14] and truncnorm [26] therein.

Data Availability Statement

The data set used in this paper is provided in the manuscript.

Declaration of Conflicting Interests

The Authors declare that there is no conflict of interest.

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