THE LENGTH-BIASED WEIGHTED WILSON HILFERTY DISTRIBUTION AND ITS APPLICATIONS

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Abstract

In this article, we propose a new length-biased weighted form of Wilson Hilferty distribution named as Length-Biased Weighted Wilson Hilferty Distribution. The various Statistical properties of the proposed distribution like, reliability function, hazard rate function, reverse hazard rate function, moment generating function, quantile function, the coefficient of variation etc. are considered to understand its nature. Furthermore, we have used the method of maximum likelihood for estimation of the parameters of proposed distribution. Also, we obtain the Shannon's entropy, stochastic ordering, Lorenz and Bonferroni curves. The performance of the proposed distribution is compared with competitive distributions using two real data sets.

Keywords*:* Wilson Hilferty distribution, length-biased weighted Wilson Hilferty distribution, hazard function, reversed hazard function, maximum likelihood estimation.

1. Introduction

The weighted distribution arises when the observations are recorded from random process, the probability of recorded observations are not equal, and instead they are recorded according to some weighted function. The concept of weighted distributions was first given by [2]. Subsequently, [3] introduced a general form for model specification and data interpretation problems and identified that many situations can be modelled by weighted distributions.
Let T is as non-negative random variable with the probability density function (pdf) $f(t)$, then the weighted distribution is given by

$$g^{w}(t) = \frac{w(t)\,f(t)}{\Omega} \tag{1}$$

on the support of T, where $w(t) > 0$ and $\Omega = \int w(t)f(t)dt$ is considered as a normalizing factor that forces $g^w(t)$ to integrate to unity. When we replace $w(t) = t$ ($i.e.$ the length of units) in equation (1), we get a special case of the weighted distribution called length biased distribution see, [5]. A $r.v.\,T$, is said to have a length biased weighted distribution if its pdf is defined as

$$g^L(t) = \frac{w_L(t)\, w_j(t)\, f(t)}{\Omega} \tag{2}$$

Where, $\Omega = \int w_L(t)\, w_j(t)\, f(t)dt$ and $w_j(t) > 0$, provided thet $w_L(t) = t$.

Weighted distribution in general and length biased distributions are specifically very useful and convenient for the analysis of life time data. Weighted distributions are commonly used in study related to reliability, biomedicine, ecology, analysis of family data, branching process and various other fields of research, see [6], [8], [10] and [12].

Many length-biased weighted distributions with applications in different fields have been presented in the literature, see [9] discussed the characterization of inverse Gaussian and gamma distribution through their length biased distributions, [13] introduced a new class of weighted exponential distribution, [14], [15] discussed the length-biased weighted generalized Rayleigh distribution and also studied the length biased weighted Weibull distribution for rainfall data in India, a weighted Lindley distribution for survival data is given by [16], the length-biased lognormal distribution with application in the analysis of oil field exploration data is discussed by [17], [18] proposed the length-biased weighted exponential and Rayleigh distribution and its properties, different methods of estimation of parameters are applied for weighted exponential distribution by [19] which introduced by [13], [20] presented the length-biased weighted Lomax distribution and application with cancer data, [21] proposed inverted weighted exponential distribution and its properties, [24] discussed the length-biased exponential distribution for Bayesian reliability estimation, [23] proposed the length-biased weighted Lindley distribution, [25] proposed weighted exponentiated inverted exponential distribution and its properties, time and failure censoring schemes for Marshall Olkin alpha power extended Weibull distribution is presented by [26]. Recently, [28] proposed power weighted Sujatha distribution and application to survival times of patients head and neck cancer data, [27] proposed a weighted intervened exponential distribution as a lifetime model.

[1] introduced a Wison Hilferty distribution. For some recent developments of the Wilson Hilferty destitution the readers may, see [22]. Its probability density function (pdf), and cumulative distribution function (cdf), respectively as

$$\varphi(t) = \frac{3}{\Gamma(\alpha)} t^{3\alpha-1} \left(\frac{\alpha}{\beta}\right)^{\alpha} \exp\left\{-\frac{\alpha}{\beta} t^3\right\}; \qquad t,\ \alpha,\ \beta > 0 \tag{3}$$

$$\Phi(t) = \frac{\gamma\left(\alpha,\ \frac{\alpha}{\beta} t^3\right)}{\Gamma(\alpha)}; \qquad t,\ \alpha,\ \beta > 0 \tag{4}$$

where, α and β are the shape and scale parameters, respectively, $\Gamma(\alpha) = \int_0^\infty t^{\theta-1} e^{-t} dt$ and $\gamma(x, y) = \int_0^x w^{y-1} e^{-w} dw$ the complete and lower incomplete gamma functions, respectively.

In this paper, we propose a new length-biased distribution, called length-biased weighted Wilson-Hilferty distribution. Rest of the paper is structured as follows, in Section 2, the proposed distribution is introduced and its properties and reliability characteristics are discussed. In Section 3, the method of maximum likelihood is discussed for estimating the model parameters. Stochastic ordering and entropy are discussed in Section 4. Bonferroni and Lorenz curves and random number generation & Quantiles are discussed in Section 5 and 6, respectively. The applications of two real data sets are presented in Section 7. Finally, the conclusion is summarized in section 8.

2. Length-Biased Weighted Wilson Hilferty Distribution, its Properties and Reliability Characteristics

In this section, we develop the Length-Biased Weighted Wilson Hilferty distribution. For this proposed new distribution, we present the pdf, cdf, reliability function, hazard function, moments, skewness and discuss some properties.

Consider the weight function as $w_j(t) = \frac{n}{\alpha} t^k, k > 0, w_L(t) = t$. Hence using considered weight function, unit length and equation (3) into the equation (2), we obtain the density of Length-biased weighted Wilson Hilferty distribution (LBWWHD) of the form

$$f(t) = \frac{3\left(\frac{\alpha}{\beta}\right)^{\alpha+\frac{k+1}{3}} t^{3\alpha+k} \exp\left\{-\frac{\alpha}{\beta} t^3\right\}}{\Gamma\left(\alpha+\frac{k+1}{3}\right)}; \quad t,\ \alpha,\ \beta,\ k > 0 \tag{5}$$

Where, α, β and k are the shape, scale and weighted parameters, respectively.

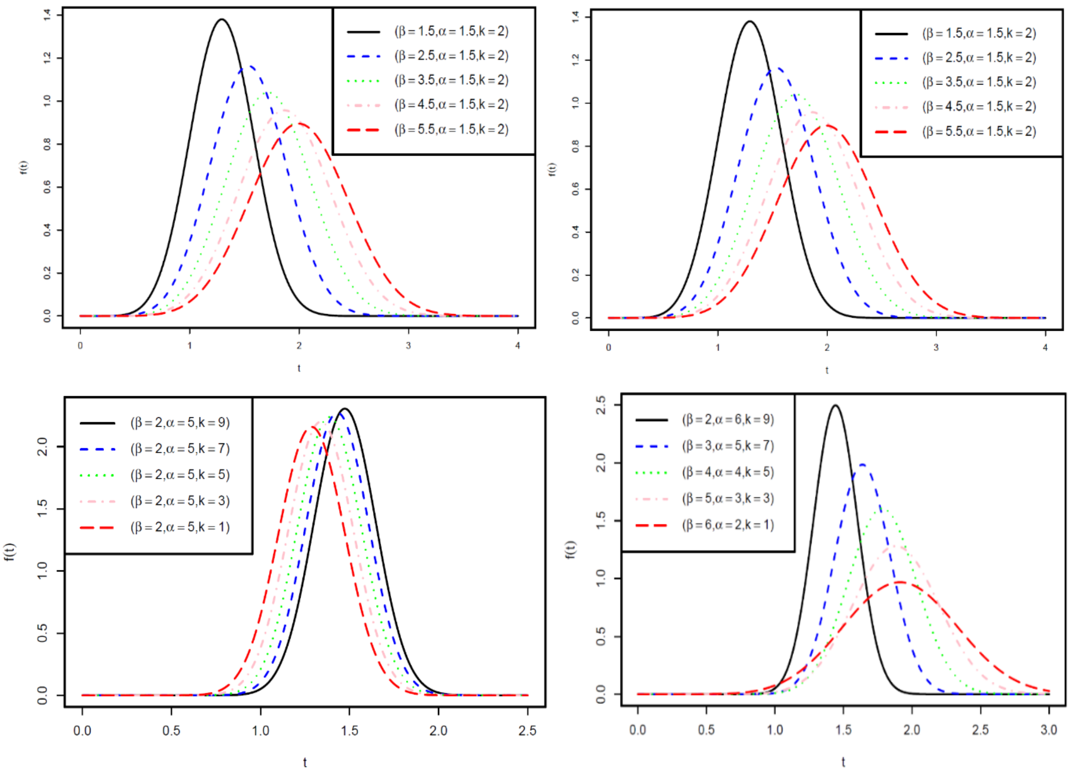


Figure 1: *pdf plots of LBWWHD*

Figure 1, clearly shows that LBWWH distribution is positively skewed.

The cdf of LBWWHD is given by

$$F(t) = \frac{\gamma\left(\alpha+\frac{k+1}{3},\ \frac{\alpha}{\beta} t^3\right)}{\Gamma\left(\alpha+\frac{k+1}{3}\right)}; \quad t,\ \alpha,\ \beta,\ k > 0 \tag{6}$$

where, $\int_0^x t^{s-1} e^{-t} dt = \gamma(s, x)$ is a lower incomplete gamma function.

The reliability function $R(t)$ is given by

$$R(t) = 1 - \frac{\gamma\left(\alpha + \frac{k+1}{3},\ \frac{\alpha}{\beta}\ t^3\right)}{\Gamma\left(\alpha + \frac{k+1}{3}\right)}$$

On using some basic concept of an upper incomplete gamma integral's, it reduces to

$$R(t) = \frac{\Gamma\left(\alpha+\frac{k+1}{3},\ \frac{\alpha}{\beta}t^3\right)}{\Gamma\left(\alpha+\frac{k+1}{3}\right)};\quad t,\ \alpha,\ \beta,\ k > 0 \tag{7}$$

where, $\int_x^\infty t^{s-1}\, e^{-t} dt = \Gamma(s,x)$ is an upper incomplete gamma function.

Table 1: Reliability function $R(t)$ of LBWWHD for $\alpha = 2\ and\ \beta = 3$

t	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
0.5	0.9999827	0.9999965	0.9999993	0.9999999	1.0000000
0.6	0.9999268	0.9999822	0.9999958	0.999999	0.9999998
0.7	0.9997537	0.9999303	0.9999809	0.9999949	0.9999987
0.8	0.9993024	0.9997745	0.9999294	0.9999785	0.9999936
0.9	0.9982712	0.9993722	0.999779	0.9999244	0.9999748
1.0	0.9961542	0.9984503	0.9993945	0.99977	0.9999149

Table 2: *Reliability function $R(t)$ of LBWWHD for $k = 2\ and\ \beta = 3$*

t	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$	$\alpha = 5$
0.5	0.9999044	0.9999965	0.9999999	1.0000000	1.0000000
0.6	0.9997166	0.9999822	0.9999989	0.9999999	1.0000000
0.7	0.999292	0.9999303	0.9999935	0.9999994	0.9999999
0.8	0.9984419	0.9997745	0.9999691	0.9999958	0.9999994
0.9	0.9968914	0.9993722	0.9998803	0.9999773	0.9999957
1.0	0.9942659	0.9984503	0.9996054	0.9999000	0.9999746

Table 3: *Reliability function $R(t)$ of LBWWHD for $k = 2\ and\ \alpha = 3$*

t	$\beta = 1$	$\beta = 2$	$\beta = 3$	$\beta = 4$	$\beta = 5$
0.5	0.9993884	0.999997	0.9999999	1.0000000	1.0000000
0.6	0.9955998	0.9999748	0.9999989	0.9999999	1.0000000
0.7	0.979182	0.9998513	0.9999935	0.9999993	0.9999999
0.8	0.9297582	0.9993323	0.9999691	0.9999967	0.9999994
0.9	0.8219026	0.9975858	0.9998803	0.999987	0.9999977
1.0	0.6472319	0.9927078	0.9996054	0.9999557	0.9999921

From Table 1, 2 & 3, we conclude that, for the different values of α, β and k the reliability of the distribution decreases with increase in the value of t.

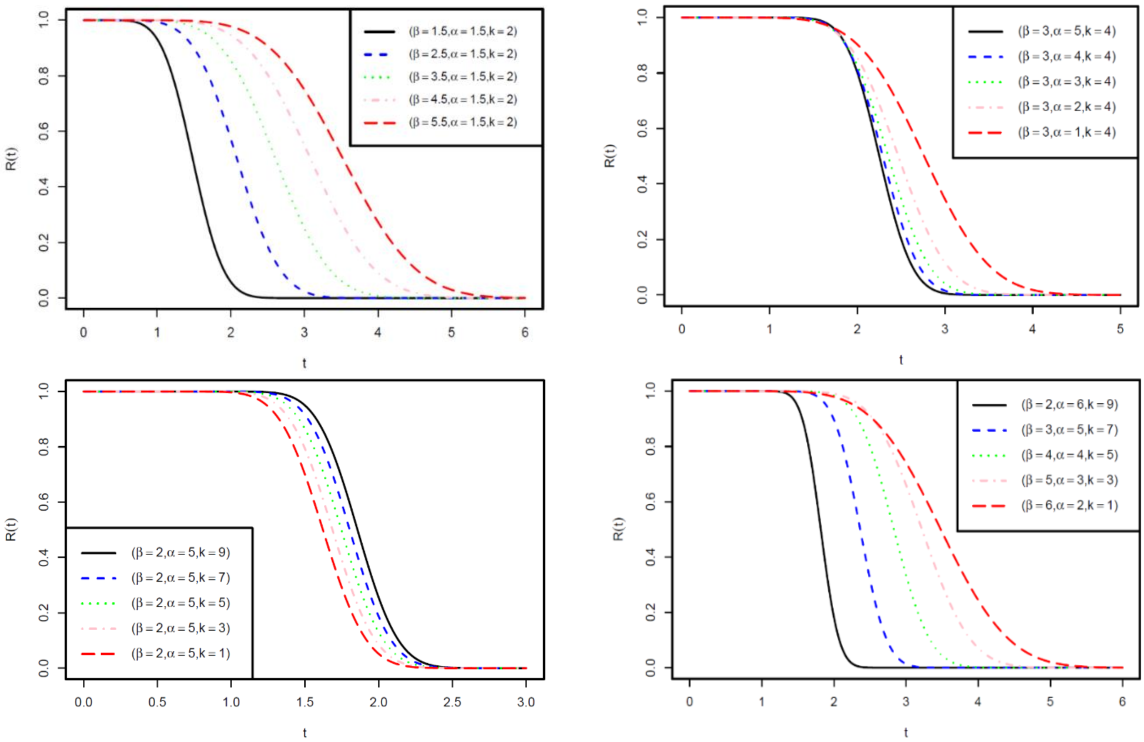


Figure 2: *Reliability plots of LBWWHD*

Figure 2, shows the reliability behavior of the LBWWHD for varying values of shape parameter α, scale parameter β and weighted parameter k. Reliability function behaves like decreasing function.

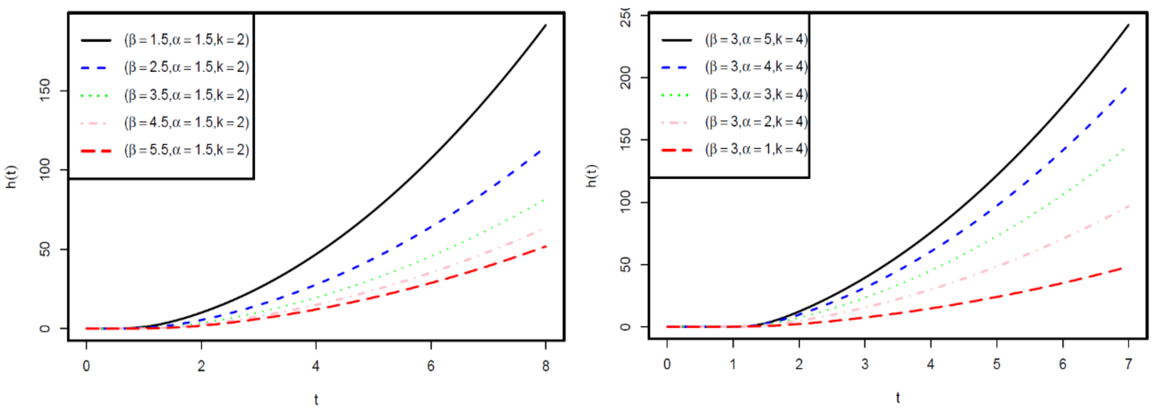
The hazard function is defined as

$$h(t) = \frac{f(t)}{R(t)}$$

$$= \frac{3\left(\frac{\alpha}{\beta}\right)^{\alpha+\frac{k+1}{3}} t^{3\alpha+k} \exp\left\{-\frac{\alpha}{\beta}t^3\right\} \Gamma\left(\alpha+\frac{k+1}{3}\right)}{\Gamma\left(\alpha+\frac{k+1}{3}\right) \Gamma\left(\alpha+\frac{k+1}{3},\ \frac{\alpha}{\beta}t^3\right)}$$

On simplifying, we get

$$h(t) = \frac{3\left(\frac{\alpha}{\beta}\right)^{\alpha+\frac{k+1}{3}} t^{3\alpha+k} \exp\left\{-\frac{\alpha}{\beta}t^3\right\}}{\Gamma\left(\alpha+\frac{k+1}{3},\ \frac{\alpha}{\beta}t^3\right)}; \quad t,\ \alpha,\ \beta,\ k > 0 \tag{8}$$



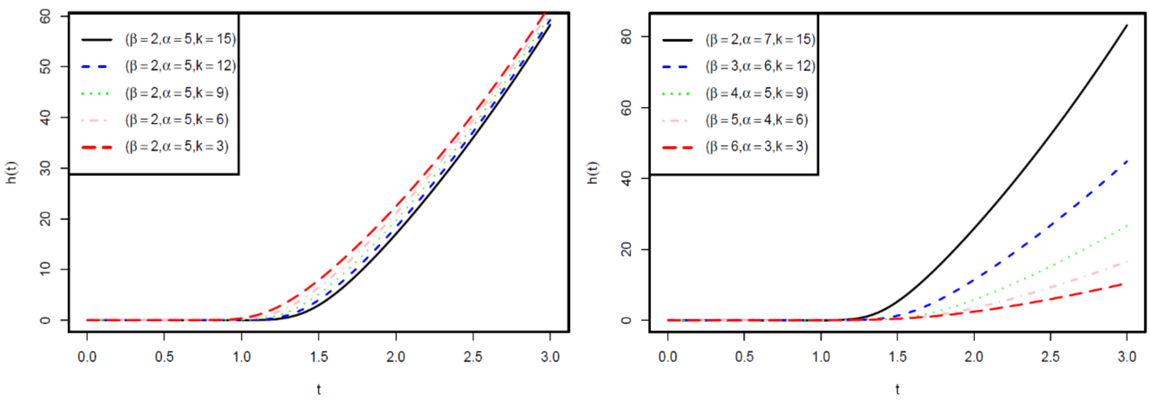


Figure 3: *Hazard rate plots of LBWWHD*

Figure 3, shows the behaviour of hazard function for distinct values of α, β and k. Clearly, it shows that the hazard function of LBWWH behaves increasing hazard rate.
The reverse hazard rate is defined as

$$R_h(t) = \frac{f(t)}{F(t)}$$

$$R_h(t) = \frac{3\left(\frac{\alpha}{\beta}\right)^{\alpha+\frac{k+1}{3}} t^{3\alpha+k} \exp\left\{-\frac{\alpha}{\beta}\, t^3\right\} \Gamma\left(\alpha+\frac{k+1}{3}\right)}{\Gamma\left(\alpha+\frac{k+1}{3}\right) \gamma\left(\alpha+\frac{k+1}{3},\ \frac{\alpha}{\beta}\, t^3\right)}$$

On simplifying, we get

$$R_h(t) = \frac{3\left(\frac{\alpha}{\beta}\right)^{\alpha+\frac{k+1}{3}} t^{3\alpha+k} \exp\left\{-\frac{\alpha}{\beta} t^3\right\}}{\gamma\left(\alpha+\frac{k+1}{3},\ \frac{\alpha}{\beta} t^3\right)};\quad t,\ \alpha,\ \beta,\ k > 0$$

Theorem 2.1. For $r = 0, 1, 2, 3, \ldots$ r^{th} moment of random variable T is given by

$$\mu_r' = E(T^r) = \left(\frac{\beta}{\alpha}\right)^{r/3} \frac{\Gamma\left(\alpha+\frac{k+r+1}{3}\right)}{\Gamma\left(\alpha+\frac{k+1}{3}\right)} \tag{9}$$

Proof. It T is a random variable with pdf $f(t)$ from equation (5), then the rth moment is

$$E(T^r) = \mu_r' = \int_0^\infty t^r \frac{3\left(\frac{\alpha}{\beta}\right)^{\left(\alpha+\frac{k+1}{3}\right)} t^{3\alpha+k} \exp\left\{-\frac{\alpha}{\beta}\, t^3\right\}}{\Gamma\left(\alpha+\frac{k+1}{3}\right)} dt$$

$$= \frac{3\left(\frac{\alpha}{\beta}\right)^{\left(\alpha+\frac{k+1}{3}\right)}}{\Gamma\left(\alpha+\frac{k+1}{3}\right)} \int_0^\infty t^{(3\alpha+k+r)} \exp\left\{-\frac{\alpha}{\beta}\, t^3\right\} dt \tag{10}$$

Theorem follows on taking $y = \left(\frac{\alpha}{\beta}\, t^3\right)$, and using the gamma function in equation (10).

Lemma 2.1. If a random variable T follows Length-biased weighted Wilson Hilferty distribution then on substituting $r = 1, 2$ in equation (10), we obtain the mean and variance, respectively.

$$E(T) = \left(\frac{\beta}{\alpha}\right)^{1/3} \frac{\Gamma\left(\alpha+\frac{k+2}{3}\right)}{\Gamma\left(\alpha+\frac{k+1}{3}\right)}$$

$$E(T^2) = \left(\frac{\beta}{\alpha}\right)^{2/3} \frac{\Gamma\left(\alpha+1+\frac{k}{3}\right)}{\Gamma\left(\alpha+\frac{k+1}{3}\right)}$$

and,

$$\text{Variance } (T) = \left(\frac{\beta}{\alpha}\right)^{2/3} \frac{\Gamma\left(\alpha+1+\frac{k}{3}\right)}{\Gamma\left(\alpha+\frac{k+1}{3}\right)} - \left\{\left(\frac{\beta}{\alpha}\right)^{1/3} \frac{\Gamma\left(\alpha+\frac{k+2}{3}\right)}{\Gamma\left(\alpha+\frac{k+1}{3}\right)}\right\}^2$$

$$= \left(\frac{\beta}{\alpha}\right)^{2/3} \left[\frac{\Gamma\left(\alpha+1+\frac{k}{3}\right)}{\Gamma\left(\alpha+\frac{k+1}{3}\right)} - \left\{\frac{\Gamma\left(\alpha+\frac{k+2}{3}\right)}{\Gamma\left(\alpha+\frac{k+1}{3}\right)}\right\}^2\right]$$

Lemma 2.2. If a random variable T follows Length-biased Weighted Wilson Hilferty distribution then the coefficient of variation (C.V) is given by

$$\frac{\left[\Gamma\left(\alpha+1+\frac{k}{3}\right)\Gamma\left(\alpha+\frac{k+1}{3}\right)-\left\{\Gamma\left(\alpha+\frac{k+2}{3}\right)\right\}^2\right]^{1/2}}{\Gamma\left(\alpha+\frac{k+2}{3}\right)} \tag{11}$$

Proof. Coefficient of variation is given by,

$$C.V. = \frac{\sqrt{var(T)}}{E(T)} = \frac{\left(\frac{\beta}{\alpha}\right)^{1/3}\left[\frac{\Gamma\left(\alpha+1+\frac{k}{3}\right)}{\Gamma\left(\alpha+\frac{k+1}{3}\right)} - \left\{\frac{\Gamma\left(\alpha+\frac{k+2}{3}\right)}{\Gamma\left(\alpha+\frac{k+1}{3}\right)}\right\}^2\right]^{1/2}}{\left(\frac{\beta}{\alpha}\right)^{1/3}\frac{\Gamma\left(\alpha+\frac{k+2}{3}\right)}{\Gamma\left(\alpha+\frac{k+1}{3}\right)}}$$

$$= \frac{\left[\Gamma\left(\alpha+1+\frac{k}{3}\right)\Gamma\left(\alpha+\frac{k+1}{3}\right) - \left\{\Gamma\left(\alpha+\frac{k+2}{3}\right)\right\}^2\right]^{1/2}}{\Gamma\left(\alpha+\frac{k+2}{3}\right)}$$

Lemma 2.2, follows on using Lemma 2.1.

Table 4: *Coefficients of LBWWHD for $\beta = 3$*

α	k	*Mean*	*Variance*	*CV*	*Skewness*	*Kurtosis*
2	1	1.521644	0.10416361	0.212102093	0.03819147	2.895791
	2	1.590099	0.10032135	0.199192288	0.03175884	2.908649
	3	1.65319	0.09697548	0.188368465	0.02692476	2.918847
	4	1.71185	0.09402637	0.179126206	0.0231893	2.927098
	5	1.766776	0.09139999	0.17111638	0.02023536	2.933893
4	1	1.482214	0.05474587	0.157857336	0.01590133	2.944386
	2	1.519149	0.05351908	0.152283885	0.01427804	2.94851
	3	1.554379	0.05239432	0.147260131	0.01291154	2.95208
	4	1.588086	0.05135774	0.14270163	0.01174866	2.955199
	5	1.620426	0.05039801	0.138540736	0.01074952	2.957945

Table 5: *Coefficients of LBWWHD for $\alpha = 3$*

β	k	*Mean*	*Variance*	*CV*	*Skewness*	*Kurtosis*
2	1	1.306386	0.05475973	0.179126208	0.0231893	2.927098
	2	1.348303	0.05323016	0.171116302	0.02023536	2.933893
	3	1.387782	0.05185586	0.164088321	0.01785365	2.939574
	4	1.425148	0.05061155	0.157857366	0.01590133	2.944386
	5	1.460661	0.04947741	0.152283919	0.01427804	2.94851
4	1	1.645943	0.08692566	0.179126239	0.0231893	2.927098
	2	1.698755	0.08449762	0.171116343	0.02023536	2.933893
	3	1.748496	0.08231604	0.164088291	0.01785365	2.939574
	4	1.795574	0.08034083	0.157857365	0.01590133	2.944386
	5	1.840318	0.07854049	0.152283878	0.01427804	2.94851

According to the Table 4 & 5, skewness decreases and kurtosis increases whenever the values of k increases.

Lemma 2.3. If a random variable T follows Length-biased Weighted Wilson Hilferty distribution then harmonic mean (H) is given by

$$\frac{1}{H} = \left(\frac{\alpha}{\beta}\right)^{1/3} \frac{\Gamma\left(\alpha+\frac{k}{3}\right)}{\Gamma\left(\alpha+\frac{k+1}{3}\right)} \tag{12}$$

Proof. The harmonic mean (H) is defined as

$$\frac{1}{H} = E\left(\frac{1}{T}\right)$$
$$= \int_0^\infty \frac{1}{t} f(t)dt \tag{13}$$

Using equation(5), we get

$$\frac{1}{H} = \int_0^\infty \frac{1}{t} \frac{3\left(\frac{\alpha}{\beta}\right)^{\alpha+\frac{k+1}{3}} t^{3\alpha+k} \exp\left\{-\frac{\alpha}{\beta}\, t^3\right\}}{\Gamma\left(\alpha+\frac{k+1}{3}\right)} dt$$

$$= \frac{3\left(\frac{\alpha}{\beta}\right)^{\alpha+\frac{k+1}{3}}}{\Gamma\left(\alpha+\frac{k+1}{3}\right)} \int_0^\infty t^{3\alpha+k-1} \exp\left\{-\frac{\alpha}{\beta}\, t^3\right\} dt$$

Lemma 2.3, follows on using the transformation $y = \left(\frac{\alpha}{\beta}\right) t^3$, and the gamma function.

Lemma 2.4. If a random variable T follows Length-biased Weighted Wilson Hilferty distribution then moment generating function (MGF) and characteristic function (CF) of T are respectively, given by

$$M_T(x) = \sum_{r=0}^{\infty} \frac{x^r}{r!} \left(\frac{\beta}{\alpha}\right)^{r/3} \frac{\Gamma\left(\alpha+\frac{k+r+1}{3}\right)}{\Gamma\left(\alpha+\frac{k+1}{3}\right)} \tag{14}$$

$$\emptyset_T(x) = \sum_{r=0}^{\infty} \frac{(ix)^r}{r!} \left(\frac{\beta}{\alpha}\right)^{r/3} \frac{\Gamma\left(\alpha+\frac{k+r+1}{3}\right)}{\Gamma\left(\alpha+\frac{k+1}{3}\right)} \tag{15}$$

Proof: On using equation (5) and Taylor's series expansion the Lemma 2.4, follows.

3. Parameter Estimation

In this section, we estimate the parameters of the LBWWHD by using the maximum likelihood technique. Let $T_1, T_2 \dots T_n$ be the random sample of size n follows the LBWWHD(α, β, k), then the likelihood function given as

$$L(t) = \frac{3^n \left(\frac{\alpha}{\beta}\right)^{n\left(\alpha+\frac{k+1}{3}\right)}}{\left(\Gamma\left(\alpha+\frac{k+1}{3}\right)\right)^n} \prod_{i=1}^{n} t^{3\alpha+k} \exp\left\{-\frac{\alpha}{\beta}\, t^3\right\}$$

The log-likelihood function can be written as

$$\log L(t) = n\log 3 + n\left(\alpha+\tfrac{k+1}{3}\right)\log\alpha - n\left(\alpha+\tfrac{k+1}{3}\right)\log\beta \\ -n\log\Gamma\left(\alpha+\tfrac{k+1}{3}\right) - \tfrac{\alpha}{\beta}\sum {t_i}^3 + (3\alpha+k)\sum \log t_i \qquad (16)$$

Differentiating equations (16) partially with respect to α, β and k then equate to zero, we get normal equations on the following form

$$\frac{\partial \log L(t)}{\partial \beta} = 0 \;\Rightarrow\; \hat{\beta} = \frac{\alpha \sum {t_i}^3}{n\left(\alpha+\frac{k+1}{3}\right)} \qquad (17)$$

$$\frac{\partial \log L(t)}{\partial \alpha} = 0 \;\Rightarrow\; n\log\alpha + \frac{n}{\alpha}\left(\alpha+\tfrac{k+1}{3}\right) - n\log\beta - \frac{1}{\beta}\sum {t_i}^3 - n\,\psi\left(\Gamma\left(\alpha+\tfrac{k+1}{3}\right)\right) \\ +3\sum \log t_i = 0 \qquad (18)$$

$$\frac{\partial \log L(t)}{\partial k} = 0 \;\Rightarrow\; \frac{n}{3}\log\alpha - \frac{n}{3}\log\beta - \frac{n}{3}\,\psi\left(\Gamma\left(\alpha+\tfrac{k+1}{3}\right)\right) + \sum \log t_i = 0 \qquad (19)$$

where, $\psi(z) = \frac{d}{dz}\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is a logarithmic derivative of gamma function. As it seems, from equations (17), (18) and (19), the analytical solution of α, β and k are not available. Consequently, we have to use to non-linear estimation of the parameters using iterative method.

4. Stochastic Ordering and Entropy

Let X and Y be two independent random variables follows LBWWHD with shape parameter α, weighted parameter k and the scale parameters β_1 and β_2, respectively.

When $f_X(t)$ and $f_Y(t)$ be the density functions of X and Y, then X less than Y in likelihood order ($X \le_{lr} Y$) if $\frac{f_Y(t)}{f_X(t)}$ is an increasing function of t. Here,

$$\frac{f_Y(t)}{f_X(t)} = \frac{3\left(\frac{\alpha}{\beta_2}\right)^{\alpha+\frac{k+1}{3}} t^{3\alpha+k}\exp\left\{-\frac{\alpha}{\beta_2}\,t^3\right\}}{\Gamma\left(\alpha+\frac{k+1}{3}\right)} \frac{\Gamma\left(\alpha+\frac{k+1}{3}\right)}{3\left(\frac{\alpha}{\beta_1}\right)^{\alpha+\frac{k+1}{3}} t^{3\alpha+k}\exp\left\{-\frac{\alpha}{\beta_1}\,t^3\right\}}$$

$$\frac{f_Y(t)}{f_X(t)} = \frac{\left(\frac{\alpha}{\beta_2}\right)^{\alpha+\frac{k+1}{3}} t^{3\alpha+k}\exp\left\{-\frac{\alpha}{\beta_2}\,t^3\right\}}{\left(\frac{\alpha}{\beta_1}\right)^{\alpha+\frac{k+1}{3}} t^{3\alpha+k}\exp\left\{-\frac{\alpha}{\beta_1}\,t^3\right\}}$$

$$\frac{f_Y(t)}{f_X(t)} = \left(\frac{\beta_1}{\beta_2}\right)^{\alpha+\frac{k+1}{3}} \exp\left\{\alpha\left(\frac{1}{\beta_1}-\frac{1}{\beta_2}\right)t^3\right\} \qquad (20)$$

Differentiating equation (20), with respect to t, we get

$$\frac{d}{dt}\left(\frac{f_Y(t)}{f_X(t)}\right) = 3\alpha\, t^2 \left(\frac{\beta_1}{\beta_2}\right)^{\alpha+\frac{k+1}{3}} \left(\frac{1}{\beta_1} - \frac{1}{\beta_2}\right) \exp\left\{\alpha\left(\frac{1}{\beta_1} - \frac{1}{\beta_2}\right) t^3\right\} \geq 0$$

hence, $\frac{d}{dt}\left(\frac{f_Y(t)}{f_X(t)}\right) \geq 0$ when, $\left(\frac{1}{\beta_1} - \frac{1}{\beta_2}\right) \geq 0$ i.e. $\beta_2 \geq \beta_1$.

Therefore, $X \leq_{lr} Y$ for, $\beta_2 \geq \beta_1$. If $X \leq_{lr} Y$, then the following ordering shall also holds for the LBWWHD ([11]) ,

$$\begin{array}{c} X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{mrl} Y \\ \Downarrow \\ X \leq_{st} Y \end{array}$$

for $\beta_2 \geq \beta_1$.

The idea of entropy has significant importance in several academic disciplines, including probability and statistics, physics, communication theory, and economics. Entropy is a measure that quantifies the level of variety, uncertainty, or unpredictability shown by a given system. The entropy of a random variable T may be defined as a quantitative measure of the level of uncertainty or variation associated with it.
The Shannon's entropy defined by

$$S(x) = -E[\log f(t)]$$

Using equation (5), we get

$$\begin{aligned} S(x) &= -E\left[\log\left(\frac{3\left(\frac{\alpha}{\beta}\right)^{\alpha+\frac{k+1}{3}} t^{3\alpha+k} \exp\left\{-\frac{\alpha}{\beta}\, t^3\right\}}{\Gamma\left(\alpha+\frac{k+1}{3}\right)}\right)\right] \\ &= -\log\left(\frac{3\left(\frac{\alpha}{\beta}\right)^{\alpha+\frac{k+1}{3}}}{\Gamma\left(\alpha+\frac{k+1}{3}\right)}\right) - (3\alpha+k)E(\log(t)) + \frac{\alpha}{\beta}E(t^3) \end{aligned} \quad (21)$$

By solving the value of $E(\log(t))$ and $E(t^3)$ and put in equation (21), we get

$$S(x) = -\log\left(\frac{3\left(\frac{\alpha}{\beta}\right)^{\alpha+\frac{k+1}{3}}}{\Gamma\left(\alpha+\frac{k+1}{3}\right)}\right) - (3\alpha+k)\left(\log\left(\frac{\beta}{\alpha}\right) + \Psi\left(\alpha+\frac{k+1}{3}\right)\right) + \frac{\Gamma\left(\alpha+\frac{k+4}{3}\right)}{\Gamma\left(\alpha+\frac{k+1}{3}\right)}$$

5. Bonferroni and Lorenz Curves

Let's assume that the random variable T is a non-negative with a continuous and twice differentiable cumulative distribution function. The Bonferroni curve of the random variable T is defined as

$$B(p) = \frac{1}{\mu p}\int_0^q t f(t)dt$$

where, $p = F(t)$, $q = F^{-1}(p)$ and $\mu = E(t)$

$$B(p) = \frac{\Gamma\left(\alpha+\frac{k+1}{3}\right)}{p\left(\frac{\beta}{\alpha}\right)^{1/3}\Gamma\left(\alpha+\frac{k+2}{3}\right)}\int_0^q t\,\frac{3\left(\frac{\alpha}{\beta}\right)^{\alpha+\frac{k+1}{3}} t^{3\alpha+k}\exp\left\{-\frac{\alpha}{\beta}t^3\right\}}{\Gamma\left(\alpha+\frac{k+1}{3}\right)}dt \tag{22}$$

$$= \frac{3\left(\frac{\alpha}{\beta}\right)^{\alpha+\frac{k+1}{3}}}{p\left(\frac{\beta}{\alpha}\right)^{1/3}\Gamma\left(\alpha+\frac{k+2}{3}\right)}\int_0^q t^{3\alpha+k+1}\exp\left\{-\frac{\alpha}{\beta}\,t^3\right\}dt$$

Substituting, $y = \left(\frac{\alpha}{\beta}\right)t^3$ in equation (22), we get Bonferroni curve

$$B(p) = \frac{\gamma\left(\alpha+\frac{k+2}{3},\ \frac{\alpha}{\beta}q^3\right)}{p\,\Gamma\left(\alpha+\frac{k+2}{3}\right)} \tag{23}$$

Lorenz curve is defined as

$$L(p) = \frac{1}{\mu}\int_0^q tf(t)dt = pB(p)$$

Using equation (23), we get

$$L(p) = \frac{\gamma\left(\alpha+\frac{k+2}{3},\ \frac{\alpha}{\beta}q^3\right)}{\Gamma\left(\alpha+\frac{k+2}{3}\right)} \tag{24}$$

6. Random Number Generation and Quantiles

Random numbers of LBWWH can be easily generate by using the following function

$$t = \left[\left(\frac{\beta}{\alpha}\right)Q^{-1}\left(\alpha+\frac{k+1}{3},\ 1-U\right)\right]^{1/2}$$

where, $U \sim U(0,1)$ and $Q^{-1}(a,z)$ is inverse of regularized incomplete gamma function, where regularized incomplete gamma function is defined as $Q(a,z) = \frac{\Gamma(a,z)}{\Gamma(a)}$.
Quantiles are given by

$$t_q = \left[\left(\frac{\beta}{\alpha}\right)Q^{-1}\left(\alpha+\frac{k+1}{3},\ 1-q\right)\right]^{1/2} \tag{25}$$

By putting $q = 0.5$ in equation (25), we get the median of LBWWHD

$$t_{0.5} = \left[\left(\frac{\beta}{\alpha}\right)Q^{-1}\left(\alpha+\frac{k+1}{3},\ 0.5\right)\right]^{\frac{1}{2}}$$

7. Applications

In this section, we have considered two real data sets to check the suitability of the proposed distribution. Further, we have compared the distribution with the Length-Biased weighted Lindley distribution (LBWLD), Length-Biased Susheela distribution ($LBSD_1$) and length-Biased Suja distribution ($LBSD_2$), for suitability of proposed distribution. For this, we have used Akaike information criterion (AIC), Bayesian information criteria (BIC), Akaike Information Criterion Corrected (AICC) and Hannan-Quinn Information Criterion (HQIC), respectively. The AIC, BIC, AICC and HQIC are defined as:

$$\text{AIC} = 2K - 2\log L,\ \text{BIC} = K\log n - 2\log L,$$

$$\text{AICC} = \text{AIC} + \frac{2K(K+1)}{n-K-1}, \text{ and } \text{HQIC} = 2K\log(\log(n)) - 2\log L.$$

where n is the sample size, K is the number of parameters, and L denotes the likelihood function. Any probability model having smaller value of AIC, BIC and $-LogL$ being the best model to fit the data set.

Dataset 1: The first dataset is taken from [7] which represent the tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20mm.

Table 6: *MLE, AIC and BIC for gauge lengths of 20 mm*

Distribution	α	β	k	Log L	AIC	BIC	AICC	HQIC
LBWWHD	0.540	3.014	6.259	−48.895	103.790	110.492	104.148	106.449
LBWLD	--	--	--	−87.8984	179.796	184.265	179.973	181.569
LBSD1	--	--	--	−92.3037	188.607	193.075	188.78	190.380
LBSD2	--	--	--	−134.128	186.607	188.841	186.66	187.493

According to the results in Table 6, the LBWWHD has the smallest values of these statistics, followed by LBWLD, $LBSD_1$ and $LBSD_2$. Therefore, the suggested distribution is the best choice for the tensile strength data. The fitted $pdfs$ and empirical $cdfs$ plots of the four models are sketched in Figure 4. Therefore, we assert that the LBWWHD fitting successfully the empirical plots of the data set.

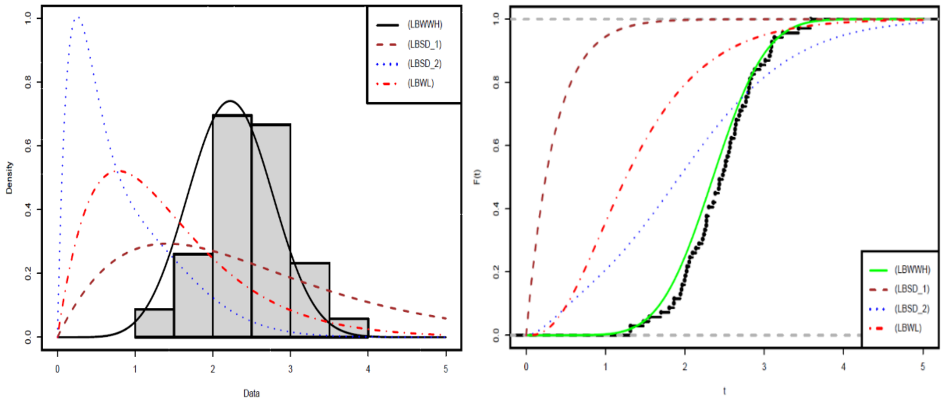


Figure 4: *Estimated densities and cdf plot of the models based on the real dataset 1.*

Dataset 2- The second dataset used by [7], which represent the tensile strength, measured in GPa, of 63 carbon fibers tested under tension at gauge lengths of 10mm.

Table 7: *MLE, AIC and BIC for gauge lengths of 10 mm*

Distribution	α	β	k	Log L	AIC	BIC	AICC	HQIC
LBWWHD	0.1587	1.730	7.402	−58.7320	123.464	129.893	123.857	125.992
LBWLD	--	--	--	−93.4265	190.85	195.133	191.046	192.538
$LBSD_1$	--	--	--	−97.9972	199.994	204.280	200.188	201.680
$LBSD_2$	--	--	--	−65.9256	133.851	135.995	133.914	134.694

For the dataset 2, we infer from the Table 7, the LBWWHD has the lowest values of AIC, BIC, AICC and HQIC followed by LBWLD, $LBSD_1$ and $LBSD_2$. Therefore, we conclude that LBWWHD is the most suitable choice for this dataset among the considered distributions. The fitted *pdfs* and empirical cdf plots of the four models are presented in Figure 5 and see that the LBWWHD fitting successfully the empirical plots of the data set.

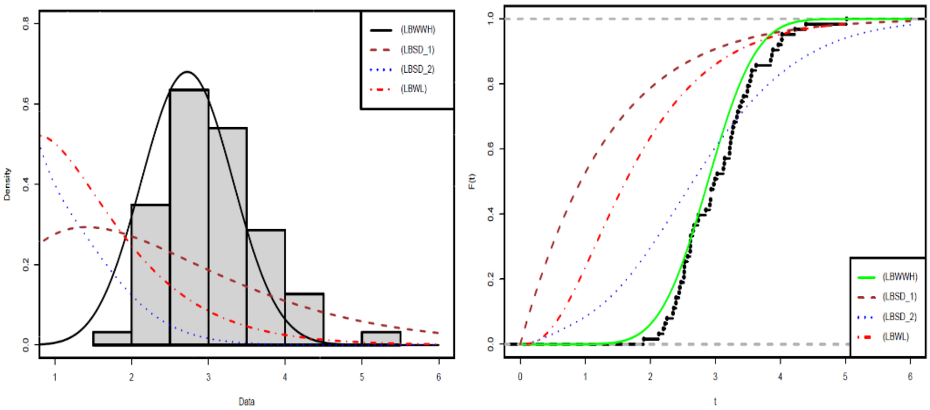


Figure 5: *Estimated densities and* cdf *plot of the models based on the real dataset 2.*

8. Conclusion

In the present article a new distribution known as the Length-biased Weighted Wilson Hilferty distribution, has been proposed. The distribution in discussion is characterized by three parameters called shape, scale and weighted parameter. Through the use of certain formulae, the properties and characteristics of this distribution such as its moments, failure rate, reliability function *etc.*, comprehensively examined and, the parameters estimation and stochastic comparison is also done. The examination and subsequent comparison of the criteria for AIC, BIC, AICC and HQIC have been conducted in relation to the Length-biased weighted Lindley distribution, Length-biased Sushila distribution and Length-biased Suja distribution. The actual lifetime of two sets of data has been successfully modelled and the resulting fit has been determined to be satisfactory.

Author Contributions: All authors have equal contribution. All authors reviewed the results and approved the final version of the manuscript.

Funding: This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors

Conflict of interest: The Authors declares that there is no conflict of interest.

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