A NEW BAYESIAN CONTROL CHART FOR PROCESS MEAN USING EMPIRICAL BAYES ESTIMATES

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Abstract

This article develops a new control chart for the mean using empirical Bayes estimates. We assume that the quality characteristic of the proposed control chart follows a normal distribution with unknown mean and variance. Both the parameters have known prior probability distributions. In practice, the parameters of priors are unknown and are estimated using the empirical Bayes approach. For the performance assessment of the new control chart, the Average Run Length (ARL) procedure is used while the process is in control and out of control. A real-life example is also considered to evaluate the performance of the proposed control chart.

Keywords: Average Run Length, Empirical Bayes, Mean Chart, Posterior, Statistical Process Control.

1. Introduction

Statistical Process Control (SPC) is a popular methodology for monitoring and assessing the quality of a manufacturing process. The main objective of SPC is to minimize the process variability. A control chart is the main technique SPC uses to measure whether a manufacturing process is in control. Dr. Walter Shewhart first proposed the control chart technique in the 1920s. If the quality characteristic under study is quantifiable, we use variable control charts like $\bar{X}$, R, and S charts, etc. For these control charts, it is assumed that the quality characteristic follows a normal distribution. Over the years, researchers have developed control charts for means by considering different aspects. [5] proposed a $\bar{X}$ chart when the quality characteristic follows a skewed distribution. [9] introduced a new $\bar{X}$ chart by considering variable sample size and sampling intervals, which can detect the shift in the process mean in less time than a traditional $\bar{X}$ chart. [8] gave an idea of the Max chart by combining the $\bar{X}$ chart and S chart. [18] proposed a new control chart for mean based on variable and attribute inspections.

The Bayesian approach has recently become very popular among researchers for constructing control charts. Using empirical Bayes, [11] developed a multivariate process control chart. [19] compared the effectiveness of different mean charts under the Bayesian approach. [13] have constructed a new control chart for the coefficient of variation using prior information when the mean is variable, and the variance is the function of the mean. [17] designed a two-sided $\bar{X}$ control chart for mean. [4] developed a new control chart for mean using posterior distribution. [10] measured the performance of a Bayesian Control Chart using empirical Bayes based on

Weibull data. [2] proposed a mean control chart using a uniform prior. [12] used Empirical Bayes methods based on loss functions for a sequential sampling plan. [6] used the Bayesian model for constructing predictive control charts. [1] designed a Bayesian Shewhart-type control chart for the Maxwell distributed process.

The entire article is arranged in the following way. Section 2 discusses the Shewhart $\bar{X}$ chart. In section 3, a discussion is made on the posterior mean control chart. Section 4 briefly describes the empirical Bayes method. In section 5, we explain the construction of the new control chart for mean using empirical Bayes estimates. In section 6, the performance of the proposed control chart is evaluated concerning the Average Run Length values. In section 7, a real-life dataset is taken to analyze the performance of the proposed control chart for mean. In the last section (section 8), concluding remarks are given.

2. X-bar Control Chart

Let $X_1, X_2, \ldots, X_n$ be n observations of a quality characteristic X following a normal distribution with mean, μ and variance, σ^2 of a manufacturing process. Then, according to W. Shewhart, the 3-sigma control limits of $\bar{X}$ chart are

$$UCL = \mu + 3\frac{\sigma}{\sqrt{n}}$$
$$LCL = \mu - 3\frac{\sigma}{\sqrt{n}}$$

3. Posterior Control Chart for Mean

[4] proposed a new posterior $\bar{X}$ control chart for process mean. Suppose $X_1, X_2, \ldots, X_n$ be n observations of a quality characteristic X. It is assumed that X_i's are independently and identically distributed normal variables with mean μ and variance σ^2(known). Here, the process average μ has normal prior with known parameters.Then X_i's $\sim N(\mu, \sigma^2)$ and $\mu \sim N(\theta, \lambda^2)$, where θ and λ are known. So, the posterior mean $\alpha_0 = \bar{x}\zeta_0 + \theta(1-\zeta_0)$ and the posterior variance is $\rho_0 = \frac{n}{\sigma^2\zeta_0}$ where $\zeta_0 = \frac{n\lambda^2}{n\lambda^2+\sigma^2}$. Hence, the three-sigma control limits of the posterior control chart for the mean are

$$UCL = \bar{x}\zeta_0 + \theta(1-\zeta_0) + 3\frac{\sigma}{\sqrt{n}}\sqrt{\zeta_0}$$
$$CL = \bar{x}\zeta_0 + \theta(1-\zeta_0)$$
$$LCL = \bar{x}\zeta_0 + \theta(1-\zeta_0) - 3\frac{\sigma}{\sqrt{n}}\sqrt{\zeta_0}$$

4. Empirical Bayes method

In the Bayesian method, the probability distribution function's unknown parameters are considered the random variables. Suppose $X_1, X_2, \ldots, X_n$ are n observations from f(θ). Here, the parameter θ has some prior information. , θ has the prior distribution $\pi(\theta|\omega)$, where ω is the hyperparameter. The Bayes' theorem states that the posterior distribution of θ can be expressed as proportional multiplication of the likelihood L(θ) and the prior distribution $\pi(\theta|\omega)$. Symbolically, $h(\theta|x) = \frac{L(\theta)\pi(\theta|\omega)}{\int L(\theta)\pi(\theta|\omega)} \propto L(\theta)\pi(\theta|\omega)$

The Bayesian method is different from the frequentist method. In the parametric empirical Bayes method, the prior distribution $\pi(\theta|\omega)$ takes parametric form, where the prior distribution parameters are unknown. [7] estimates the prior parameters using the observed data. These parameters could be estimated using the empirical Bayes procedure (see [14] and [3]). Given the observations, the joint likelihood distributions have been compared with the joint prior distributions. The joint likelihood distributions are just the multiplication of the likelihood distribution of X, and joint prior distributions are the multiplication of the prior distributions of the parameters. We can estimate the prior parameters by comparing them individually with their corresponding likelihood functions.

5. Control Chart for Mean using Empirical Bayes

Using the empirical Bayes method, we propose a new control chart for the mean. Suppose X is a quality characteristic of a manufacturing process and is assumed to follow a normal distribution with mean, μ and variance, σ^2. The location parameter, μ, has a normal prior with unknown parameters, and σ follows an inverse gamma distribution with unknown parameters. Let X has n observations $X_1, X_2, \ldots, X_n$, such that $X_i \sim N(\mu, \sigma^2)$. here $\mu \sim N(\mu_0, \sigma^2)$ and $\sigma \sim InverseGamma(\alpha, \beta)$.

$$\begin{aligned}
g(x|\mu,\sigma^2) &= \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2} \quad -\infty < \mu < \infty, \sigma^2 > 0 \\
g(\mu|\sigma^2, x) &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(\mu-\mu_0)} \quad -\infty < \mu_0 < \infty \\
g(\sigma^2|x) &= \frac{\beta^\alpha}{\Gamma\alpha}(\sigma^2)^{-\alpha-1} e^{\frac{\beta}{\sigma^2}} \quad \alpha > 0, \beta > 0
\end{aligned}$$

Hence, the posterior distribution of (μ, σ^2) is given by,

$$g(\mu,\sigma^2|x) = \frac{g(x|\mu,\sigma^2)g(\mu|\sigma^2)}{\int_0^\infty \int_{-\infty}^\infty g(x|\mu,\sigma^2)g(\mu|\sigma^2)d\mu d\sigma^2}$$

So, posterior mean $E(\mu|x) = \frac{\sum x_i + \mu_0}{n+1}$. The empirical Bayes estimate of μ_0 is $\bar{X}$.
So, $E(\hat{\mu}|x) = \frac{(n+1)\bar{x}}{n+1} = \bar{x}$.
Now,

$$\begin{aligned}
E(\sigma^2|x) &= \int_0^\infty \sigma g(\mu,\sigma^2|x)d\sigma^2 \\
&= \int_0^\infty \sigma g(\sigma^2|x)g(\mu|\sigma^2,x)d\sigma^2 \\
&= \int_0^\infty \sigma g(\sigma^2|x)g(\mu|\sigma^2,x)d\sigma^2 \\
&= \frac{\Gamma(\frac{n}{2}+\alpha)}{\Gamma(\frac{n+1}{2}+\alpha)} \frac{w_1^{\frac{n+1}{2}+\alpha}}{w_1^{\frac{n}{2}+\alpha}} \\
&= \frac{\Gamma(\frac{n}{2}+\alpha)}{\Gamma(\frac{n+1}{2}+\alpha)}\sqrt{w_1} \\
&= \frac{2^{n+2\alpha-1}\Gamma(\frac{n}{2}+\alpha)}{\sqrt{\pi}\Gamma(n+2\alpha)}\sqrt{w_1}
\end{aligned}$$

here $w_1 = \sum x_i^2 + 2\beta + \mu_0^2 - \frac{x_0^2}{n+1}$.
The empirical Bayes procedure will be used to estimate the parameters. So the estimated values of the parameters of the likelihood function of σ^2 are $\hat{\alpha} = (n-3)/2$ and $\hat{\beta} = \sum_{i=1}^n (X-\bar{X})^2/2$ (see [15]). Therefore

$$\begin{aligned}
E(\hat{\sigma}|x) &= \frac{\Gamma(\frac{n}{2}+\frac{n-3}{2})}{\Gamma(\frac{n+1}{2}+\frac{n-3}{2})}\sqrt{2\sum_{i=1}^n (x_i-\bar{x})^2} \\
&= \frac{\Gamma(\frac{2n-3}{2})}{\Gamma(\frac{2n-2}{2})}\sqrt{2\sum_{i=1}^n (x_i-\bar{x})^2} \\
&= \frac{\Gamma(\frac{2n-3}{2})}{\Gamma(\frac{2n-2}{2}+\frac{1}{2})}\sqrt{2\sum_{i=1}^n (x_i-\bar{x})^2}
\end{aligned}$$

Hence, the control limits of the proposed control chart for the mean are

$$
\begin{aligned}
UCL &= \bar{x} + L\frac{\Gamma(\frac{2n-3}{2})}{\Gamma(\frac{2n-2}{2}+\frac{1}{2})}\sqrt{2\sum_{i=1}^{n}(x_i-\bar{x})^2} \\
CL &= \bar{x} \\
LCL &= \bar{x} - L\frac{\Gamma(\frac{2n-3}{2})}{\Gamma(\frac{2n-2}{2}+\frac{1}{2})}\sqrt{2\sum_{i=1}^{n}(x_i-\bar{x})^2}
\end{aligned}
$$

Here, L is the control chart coefficient.

6. Evaluation of Performance and Comparisons

This section uses Monte Carlo Simulation to compute the proposed control chart's Average Run Length (ARL) for mean using empirical Bayes. We consider different sample values of n and compare the computed in-control-ARL (ARL_0) and out-of-control ARL (ARL_1) with the existing posterior mean control chart and Shewhart Control Chart. The decision is based on the value of ARL_1. The control chart with a smaller ARL_1 value is more efficient in detecting a shift in the process mean than other control charts. We consider the shift in process mean as $\mu^* = \mu + c\sigma$.

Algorithm for construction of UCL and LCL is as follows.

- **Step 1:** Select a random sample of size n, say, $x_1, x_2, \ldots, x_n$ from $N(\mu, \sigma^2)$ distribution. Here we assume that μ has a normal prior and σ has an Inverse Gamma prior with unknown parameters.
- **Step 2:** Estimate the posterior distribution parameters using the empirical Bayes procedure.
- **Step 3:** For given values of n and fixed in-control Average Run Length(ARL), say r_0, find the control chart coefficient L.
- **Step 4:** Find UCL and LCL for each i, $i = 1, 2, \ldots, n$. The process is in control if all the values of x_i fall within the UCL and LCL of the proposed mean chart.
- **Step 5:** Next, we find the ARL_0 value for a particular choice of the process mean μ.
- **Step 6:** We shift the process mean μ to a certain amount, say c and compute ARL_1 by repeating steps 1 to steps 5.

Here, we fixed the ARL_0 at 370. The ARL values of the proposed control chart are given in Table 1 - Table 3.

We can see the proposed control chart for mean using empirical Bayes estimators has the least ARL among all the control charts under consideration. The ARL_1 of the proposed chart decreases quickly for a small shift in the process mean. As we increase the sample size, the ARL_1 values of the control chart decrease. Therefore, we can conclude that the new control chart for using empirical Bayes estimators is more efficient than the posterior control chart and Shewhart $\bar{X}$ control chart.

7. Illustrative Example

In recent trends, SPC researchers use both simulated and real-life data to evaluate the performance of a control chart. In this study, we have considered a real dataset from [16] to evaluate the performance of the new control chart for mean using empirical Bayes estimates. The data set is given in the appendix section. Here, we have filled out height data in ten subgroups of size 10. It is assumed that the control chart statistic, fill height, follows normal distribution where the parameters μ and σ have unknown prior distribution. Using the empirical Bayes procedure, the

Table 1: *Comparison of average run lengths of Empirical Bayes control chart for Mean with Posterior Mean Control Chart and Shewhart $\bar{X}$ Control Chart for n = 10 and ARL = 370*

Shift	Empirical Bayes Mean Chart	Posterior Mean Chart	Shewhart $\bar{X}$ Chart
	L = 3	L = 3	L = 3
0.0	370.398	370.398	370.398
0.05	281.397	312.467	328.011
0.1	183.248	221.991	249.167
0.15	128.813	144.631	181.701
0.2	86.003	94.297	123.981
0.25	58.238	68.997	87.457
0.3	34.184	39.832	59.301
0.4	9.327	18.115	28.034
0.7	2.265	3.168	4.387
0.9	1.183	1.719	2.814

Table 2: *Comparison of average run lengths of Empirical Bayes control chart for Mean with Posterior Mean Control Chart and Shewhart $\bar{X}$ Control Chart for n = 20 and ARL = 370*

Shift	Empirical Bayes Mean Chart	Posterior Mean Chart	Shewhart $\bar{X}$ Chart
	L = 3	L = 3	L = 3
0.0	370.398	370.398	370.398
0.05	236.234	289.754	300.373
0.1	131.197	168.103	185.559
0.15	67.469	91.476	106.358
0.2	34.482	50.893	61.539
0.25	21.107	29.555	36.807
0.3	12.893	17.985	22.885
0.4	5.221	7.663	9.959
0.7	1.934	2.988	3.824
0.9	1.021	1.504	2.357

Table 3: *Comparison of average run lengths of Empirical Bayes control chart for Mean with Posterior Mean Control Chart and Shewhart $\bar{X}$ Control Chart for n = 30 and ARL = 370*

Shift	Empirical Bayes Mean Chart	Posterior Mean Chart	Shewhart $\bar{X}$ Chart
	L = 3	L = 3	L = 3
0.0	370.398	370.398	370.398
0.05	183.609	262.736	271.659
0.1	96.064	130.865	142.164
0.15	34.627	63.37	71.433
0.2	24.922	32.409	37.614
0.25	9.088	17.731	20.860
0.3	3.167	10.384	12.343
0.4	2.081	4.338	5.163
0.7	1.355	2.805	3.532
0.9	1.008	1.244	1.841

prior parameters are estimated. The UCL and LCL of the new control chart for mean based on empirical Bayes for the data set are 0.7288446 and -0.6888446, respectively. From figure 2, we can see that the proposed control chart based on empirical Bayes can detect an out-of-control observation more precisely than the posterior control chart for mean and Shewhart $\bar{X}$ chart. In figure 4, the Average Run Lengths of the proposed chart and other control charts are drawn.

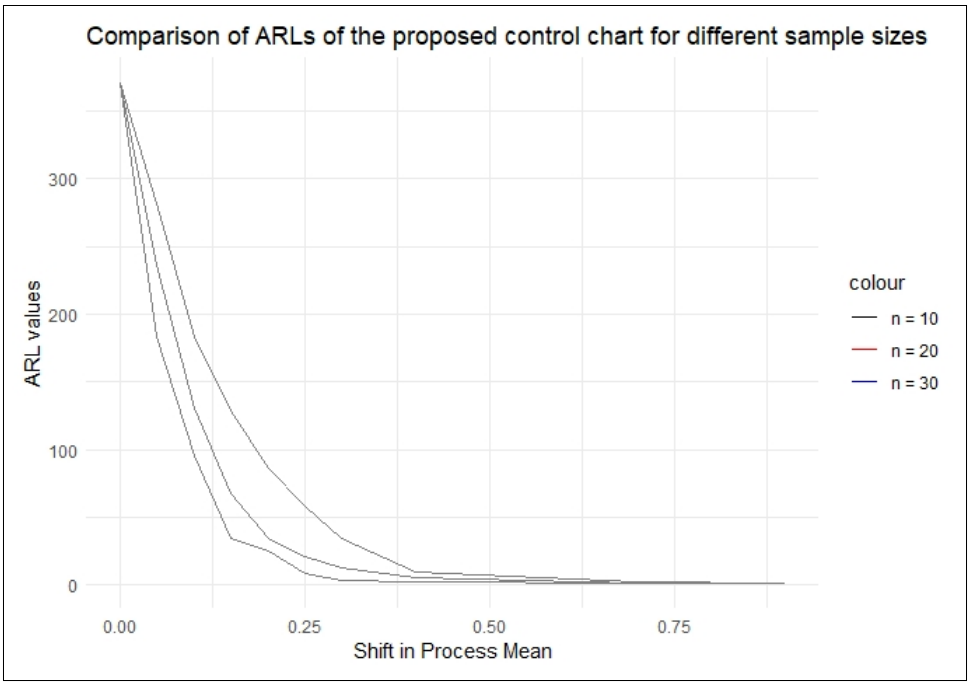

Figure 1: *Comparison of ARLs of Empirical Bayes Mean Control Chart for different sample values*

Table 4: *Comparison of average run lengths of Empirical Bayes control chart for Mean with Posterior Mean Control Chart and Shewhart $\bar{X}$ Control Chart*

Shift	Empirical Bayes Mean Chart	Posterior Mean Chart	Shewhart $\bar{X}$ Chart
	L = 3	L = 3	L = 3
0.0	370.398	370.398	370.398
0.05	297.351	322.097	334.916
0.1	180.398	227.721	257.719
0.15	101.842	147.533	182.071
0.2	58.247	94.044	124.894
0.25	34.533	60.687	85.584
0.3	21.331	40.032	59.301
0.4	9.217	18.786	29.912
0.7	1.867	3.538	5.911
0.9	1.198	1.829	2.822

From table 4 and figure 2, we can conclude that the proposed mean chart using the empirical Bayes estimator has smaller ARL values than the posterior mean control chart as well as Shewhart $\bar{X}$ control chart when there occurs a shift. In figure 1, we can see that the width of control limits for the proposed control chart is narrower than the posterior mean control chart and Shewhart $\bar{X}$ chart. This implies that the new mean chart based on empirical Bayes estimate can detect an 'out of control state' of a process mean earlier.

8. Conclusion

This article proposes a new control chart for mean using the empirical Bayes approach. For the mean, we compare the performance of the new control chart with that of the existing control charts. We used ARL values to measure the performance of the control charts for the mean. It is observed that the proposed control chart can detect the smaller shift in the process mean quickly than the posterior mean control chart and Shewhart $\bar{X}$ control chart. It was also noted that the

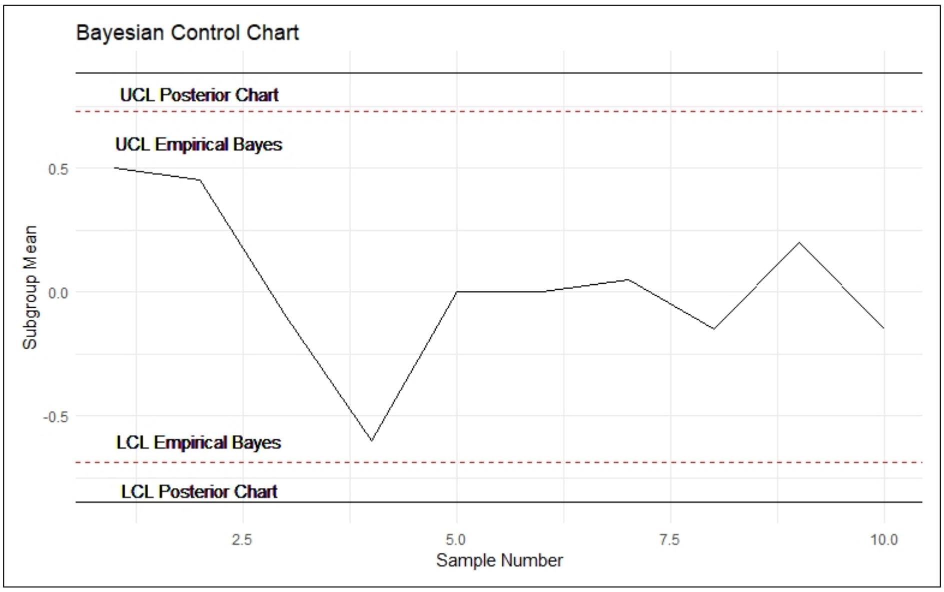

Figure 2: *Empirical Bayes control chart with that of Posterior Control Chart*

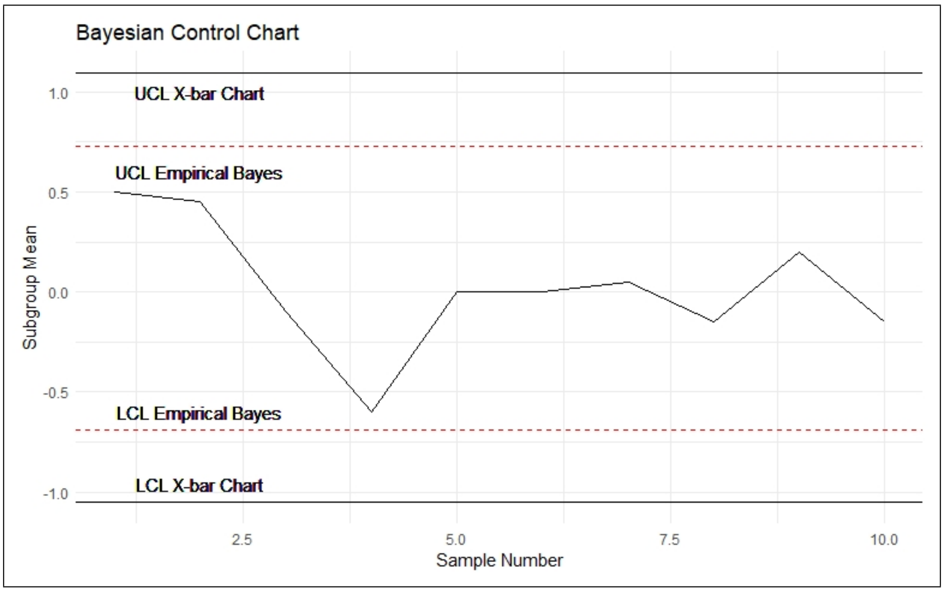

Figure 3: *Empirical Bayes Control Chart and Shewhart $\bar{X}$ control chart*

proposed control chart performed better for larger sample sizes.

Declarations

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Ethical Approval: This article does not contain any studies with human participants performed by authors.
Code availability: Codes are available on request.

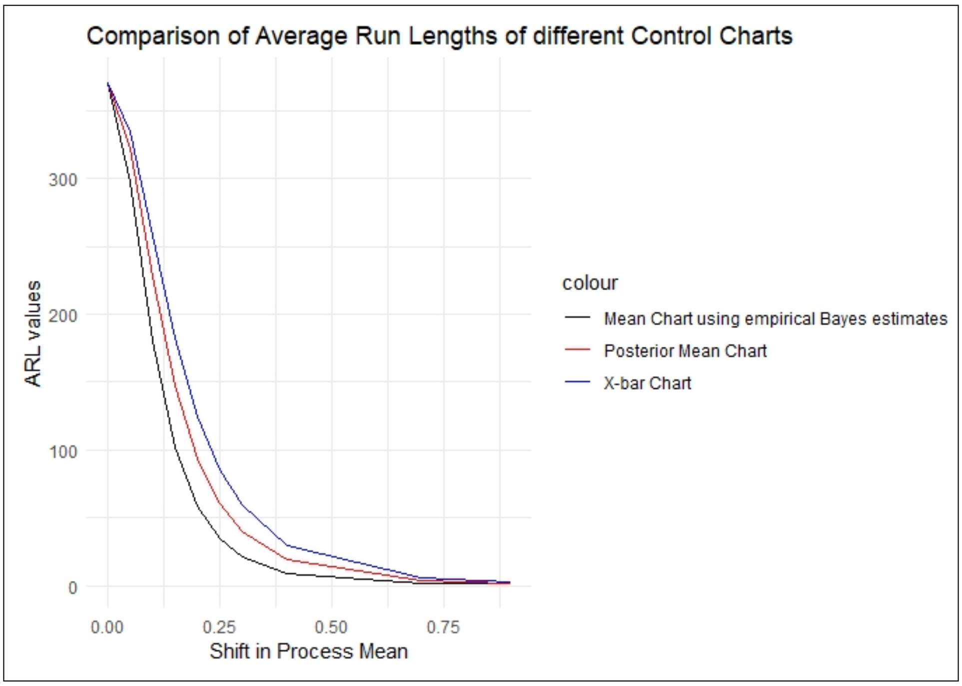

Figure 4: *Comparison of ARLs of Empirical Bayes Mean Control Chart with other Control Charts for the example dataset*

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