BAYESIAN ESTIMATION OF $P(X \leq Y)$ FOR POWER-SERIES MODEL

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Abstract

When the probability distributions for the stress (X) and strength (Y) are different members of the power series family, the expressions of the stress-strength reliability function, $R = P(X \leq Y)$, are derived. Apart from stress-strength reliability, it has applications in statistical tolerancing, measurement of demand-supply system performance, genetic trait hereditary measure, bio-equivalence study, etc. The Bayes' estimates of R under squared error and Precautionary losses are derived for various combinations of distributions of X and Y like binomial, Poisson, negative binomial, and geometric. As in practice, the availability of prior parameters is difficult; the empirical Bayes estimation procedure has been adopted to get their estimates from observed data. Simulation results have been reported, and estimates of posterior risks are compared. In the context of real Soccer games, the Bayes estimates are enumerated and compared with their classical counterparts.

Keywords: Empirical Bayes' estimate, Estimated Posterior risk, Precautionary loss, Squared error loss, Stress-strength reliability.

1. Introduction

It is a satisfactory fact that the strength of a manufactured product is a variable quantity. When ascertaining the reliability of equipment or the viability of a material, it is also necessary to consider the stress conditions of the operating environment. The uncertainty of the stressful environment leads us to take stress as a random component. In the stress-strength model, X is the stress applied on the unit by the operating system, and Y is the unit's strength, which is the in-built capacity of the unit to withstand the applied stress. No doubt, any unit can perform its actual function if its strength is greater than the stress given to it. In this context, we consider the reliability (R) as

$$R = P(X \leq Y),$$

which is the probability that the unit performs its task satisfactorily. Also, we can say this is the probability of the unit overcoming the stress. In this paper, the estimation of R when X and Y are independently distributed but not necessarily identical follows the power-series distribution. The quantity R has many applications, for example, statistical tolerance, measurement of demand-supply system performance, stress-strength reliability, genetic trait hereditary measure, a study of bio-equivalence, etc.

Several authors explore different statistical properties concerning R for several members of the power series distributions. A study on the various estimation methods of R when X and Y are independently distributed geometric random variables was made by [11]. [2] worked the Bayes estimation of it. A study on different estimations of the negative binomial distribution was considered by [8] and [15] in the context of a system reliability estimation. [5] and [4] used Poisson distribution to explore different estimates of R. [13] studied the application of log series distribution.

In most of the papers, the Bayes estimate for the stress-strength problems was found for the continuous distributions, like [16] found the estimate of R for the Gompertz case. [14] found for Weibull distribution, [1] used type-II censoring for Rayleigh distribution for finding Bayes estimate. [12] used exponential-Poisson distribution. [9] used generalized exponential distribution, and [7] used inverted gamma distribution to estimate it using the Bayes method.

In the current work, we consider the Bayesian estimate of R for the stress and strength distributions as general members of the power series family of distributions. Section 2 starts with the exact expression of R for the generalized form of power series for both X and Y. The Bayesian estimates of R for power series distributions under squared error loss and precautionary loss, as well as some prerequisites, are discussed in section 3. The Bayesian estimates and their estimated posterior risks for particular choices of distributions like binomial, Poisson, negative binomial, and geometric of X and Y are discussed in section 4. Section 5 is devoted to searching estimates of the hyper-parameters of the prior distributions and hence is engaged in finding the empirical Bayes estimates of R. Simulation study results have been reported in section 6. An application of R in real soccer games is discussed in section 7, where we find the R estimate with the estimated posterior risk. Section 8 draws concluding remarks.

2. The initial set-up

Let X and Y be two random variables belonging to the power-series family of distributions, which are defined as follows.

$$P(X=x)=\frac{a(x)\alpha^x}{f(\alpha)}, x=0,1,... \quad \alpha>0 \text{ and } f(\alpha)=\sum_{x=0}^{\infty}a(x)\alpha^x. \tag{1}$$

$$P(Y=y)=\frac{b(y)\beta^y}{g(\beta)}, y=0,1,... \quad \beta>0 \text{ and } g(\beta)=\sum_{x=0}^{\infty}b(y)\beta^y. \tag{2}$$

The stress-strength reliability R is given by

$$R=P(X\leq Y)=\frac{1}{f(\alpha)g(\beta)}\sum_{y=0}^{\infty}\Big\{\sum_{x=0}^{y}a(x)\alpha^x\Big\}b(y)\beta^y. \tag{3}$$

Let $x_1, x_2, \ldots, x_{n_1}$ be a sample of size n_1 from the distribution of X and $y_1, y_2, \ldots, y_{n_2}$ be a sample of size n_2 from that of Y. Then, the likelihood function is

$$\begin{aligned} L(\alpha,\beta|\underline{x},\underline{y}) &= \prod_{i=1}^{n_1}P(X=x_i)\prod_{j=1}^{n_2}P(Y=y_j) \\ &= \frac{\alpha^{\sum_{i=1}^{n_1}x_i}}{f^{n_1}(\alpha)}\prod_{i=1}^{n_1}a(x_i)\frac{\beta^{\sum_{j=1}^{n_2}y_j}}{g^{n_2}(\beta)}\prod_{j=1}^{n_2}b(y_j) \\ &= \alpha^{t_x}\{f(\alpha)\}^{-n_1}\prod_{i=1}^{n_1}a(x_i)\beta^{t_y}\{g(\beta)\}^{-n_2}\prod_{j=1}^{n_2}b(y_j) \\ &= L(\alpha|\underline{x})L(\beta|\underline{y}), \end{aligned}$$

where $t_x = \sum_{i=1}^{n_1} x_i$, $t_y = \sum_{j=1}^{n_2} y_j$.
The joint posterior density function of α, β corresponding to prior distributions h(α), k(β) is

$$\prod(\alpha, \beta | t_x, t_y) = \frac{L(\alpha, \beta | t_x, t_y) h(\alpha) k(\beta)}{\int_0^\infty \int_0^\infty L(\alpha, \beta | t_x, t_y) h(\alpha) k(\beta) d\alpha d\beta} = \frac{L(\alpha | t_x) h(\alpha)}{\int_0^\infty L(\alpha | t_x) h(\alpha) d\alpha} \cdot \frac{L(\beta | t_y) k(\beta)}{\int_0^\infty L(\beta | t_y) k(\beta)}.$$

3. Bayes estimation of R for power-series distributions

The main objective of this section is to find Bayes' estimates of R, the stress-strength reliability. We have considered two different loss functions, and the Bayes estimates corresponding to each loss function are given for the general cases. The joint distributions, the prior distributions of the parameters, and some preliminaries, which are required to find the Bayes estimates of the different combinations of the power-series model, are discussed.

3.1. The Bayes Estimates under different loss functions

The squared error loss (SEL) and Pre-cautionary loss (PL) functions have been considered. If d is the estimate of the parameter θ, the SEL is given by

$$L(d, \theta) = (d - \theta)^2. \tag{4}$$

Under the SEL, the posterior mean is the Bayes estimate of the parameter θ and is denoted by

$$d = \hat{\theta} = E(\theta).$$

The Posterior risk, in this case, is

$$PR_{SEL} = E(\theta^2) - E^2(\theta).$$

The PL is defined as

$$L(\theta, d) = \frac{(d - \theta)^2}{d}.$$

The Bayes estimate under the PL is defined as

$$d = \hat{\theta} = \sqrt{E(\theta^2)}.$$

The Posterior risk, in this case, is

$$PR_{PL} = 2[\sqrt{E(\theta^2)} - E(\theta)].$$

So, the Bayes estimate of R under the SEL is given by

$$\begin{aligned} \hat{R}_{SEL} &= E(R | t_x, t_y) \\ &= \int_0^\infty \int_0^\infty \sum_{y=0}^{\infty} \sum_{x=0}^{y} \frac{a(x) b(y) \alpha^x \beta^y}{f(\alpha) g(\beta)} \prod(\alpha, \beta | t_x, t_y) d\alpha d\beta \\ &= \sum_{y=0}^{\infty} \sum_{x=0}^{y} \frac{\int_0^\infty \frac{a(x)\alpha^x}{f(\alpha)} L(\alpha) h(\alpha) d\alpha}{\int_0^\infty L(\alpha) h(\alpha) d\alpha} \frac{\int_0^\infty \frac{b(y)\beta^y}{g(\beta)} L(\beta) k(\beta) d\beta}{\int_0^\infty L(\beta) k(\beta) d\beta}, \end{aligned} \tag{5}$$

and the same under the PL is $\hat{R}_{PL} = \sqrt{E(R^2|t_x, t_y)}$, where

$$\begin{aligned} E(R^2|t_x,t_y) &= \int_0^\infty \int_0^\infty [\sum_{y=0}^{\infty}\sum_{x=0}^{y} \frac{a(x)b(y)\alpha^x\beta^y}{f(\alpha)g(\beta)}]^2 \prod(\alpha,\beta|t_x,t_y)d\alpha d\beta \\ &= \int_0^\infty \int_0^\infty \sum_{y=0}^{\infty}\sum_{x=0}^{y} \frac{a^2(x)\alpha^{2x}}{f^2(\alpha)}\frac{b^2(y)\beta^{2y}}{g^2(\beta)} \prod(\alpha,\beta|t_x,t_y)d\alpha d\beta + \\ & \int_0^\infty \int_0^\infty \sum_{y_j=0}^{\infty}\sum_{x_i=0}^{y_j}\sum_{y_l=0}^{\infty}\sum_{x_k=0}^{y_l} \frac{a(x_i)a(x_k)\alpha^{x_i+x_k}}{f^2(\alpha)}\frac{b(y_j)b(y_l)\beta^{y_j+y_l}}{g^2(\beta)} \prod(\alpha,\beta|t_x,t_y)d\alpha d\beta. \end{aligned} \tag{6}$$

The estimated posterior risks of Bayes estimate of R under SEL and PL are $\hat{PR}_{SEL} = \hat{E}(R^2|t_x,t_y) - \hat{E}^2(R|t_x,t_y)$ and $\hat{PR}_{PL} = 2[\sqrt{\hat{E}(R^2|t_x,t_y)} - \hat{E}(R|t_x,t_y)]$, respectively.

3.2. Some particular distributions and joint distributions

Four different distributions have been considered; all belong to the power-series family. These are binomial (m,p), Poisson (λ), negative binomial (r,η), and geometric (μ) distributions. Let z_1,z_2,...,z_n be a sample from the distribution of Z. Then, $t_z=\sum_{i=0}^{n} z_i$ be the complete sufficient statistic for estimating the distribution parameter. The probability mass function (pmf) of different distributions and their joint distributions for a sample of size n are shown in Table 1.

Table 1: *The pmfs of the different distributions and their joint distributions*

Distribution (Z)	pmf, P(Z=z)	Joint distribution
Bin(m,p)	P(Z=z)=$\binom{m}{p}p^z(1-p)^{m-z}$	$p^{t_z}(1-p)^{mn-t_z}$
Poisson(λ)	P(Z=z)=$\frac{e^{-\lambda}\lambda^z}{z!}$	$\frac{e^{-n\lambda}\lambda^{t_z}}{\prod_{i=1}^{n} z_i!}$
Negative binomial (r,η)	P(Z=z)=$\binom{r+z-1}{z}\eta^z(1-\eta)^r$	$\eta^{t_z}(1-\eta)^{nr}$
Geometric (μ)	P(Z=z)=$\mu^z(1-\mu)$	$\mu^{t_z}(1-\mu)^n$

3.3. Prior distributions and their probability distribution functions

Each distribution characteristic depends on the value of the single parameter it evolves. The parameter itself follows a distribution. The probability distribution function (pdf) of the prior parameters and some forms related to the posterior distribution of the parameters have been found, and presented in Table 2.

Table 2: *The pdfs of the prior parameters and some forms related to the posterior distribution*

Distribution	Prior distribution	pdf of prior dist	Terms required for posterior dist
Bin(m,p)	Beta(α,β)	$\frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha,\beta)}$	$I_{bin}(m,p)$= $\frac{p^{t_z+\alpha-1}(1-p)^{mn-t_z+\beta-1}}{B(t_z+\alpha,mn-t_z+\beta)}$
Poisson(λ)	Gamma(γ,δ)	$\frac{\delta^{\gamma}e^{-\gamma\lambda}\lambda^{\delta-1}}{\Gamma\delta}$	$I_{pois}(\lambda)$= $\frac{(n+\gamma)^{(t_z+\delta)}e^{-(n+\gamma)\lambda}\lambda^{t_z+\delta-1}}{\Gamma(t_z+\delta)}$
Neg bin (r,η)	Beta(c,d)	$\frac{\eta^{c-1}(1-\eta)^{d-1}}{B(c,d)}$	$I_{neg}(\eta)$= $\frac{\eta^{t_z+c-1}(1-\eta)^{nr+d-1}}{B(t_z+c,nr+d)}$
Geometric (μ)	Beta(a,b)	$\frac{\mu^{a-1}(1-\mu)^{b-1}}{B(a,b)}$	$I_{geo}(\mu)$= $\frac{\mu^{t_z+a-1}(1-\mu)^{n+b-1}}{B(t_z+a,n+b)}$

3.4. Some preliminaries

The expression of R is given in 3. The expression for Bayes estimates under SEL and PL are shown in 5 and 6, respectively. To find the explicit forms, we require some terms defined in this section and presented in Tables 3 and 4. These are to be used for finding E(R) and E(R^2).

Table 3: *Prerequisite for finding E(R) and E(R^2)*

Distribution	Terms related to E(R) (R_{dist})	Terms for R^2 ($z_i = z_k$) (Rsq_{dist})
Bin(m,p)	$\binom{m}{z}\frac{B(t_z+z+\alpha,m(n+1)-t_z-z+\beta)}{B(t_z+\alpha,mn-t_z+\beta)}$	$\binom{m}{z}^2\frac{B(2z+t_z+\alpha,2m-2z+n_1m-t_z+\beta)}{B(t_z+\alpha,mn_1-t_z+\beta)}$
Poisson(λ)	$\frac{(n+\gamma)^{(t_z+\delta)}}{z!\Gamma(t_z+\delta)}\frac{\Gamma(t_z+z+\delta)}{(n+\gamma+1)^{(t_z+z+\delta)}}$	$\frac{(n_2+\gamma)^{(t_z+\delta)}\Gamma 2z+t_z+\delta}{(2+n_2+\delta)^{(2z+t_z+\delta)}\Gamma t_z+\delta}$
Negative binomial (r,η)	$\binom{r+z-1}{z}\frac{B(t_z+z+c,nr+r+d)}{B(t_z+c,nr+d)}$	$\binom{r+z-1}{y}^2\frac{B(2z+t_z+c,2r+n_2r+d)}{B(t_z+c,n_2r+d)}$
Geometric (μ)	$\frac{B(t_z+z+a,n+1+b)}{B(t_z+a,n+b)}$	$\frac{B(t_z+2z+a,n_2+b+2)}{B(t_z+a,n_2+b)}$

Table 4: *Prerequisite for finding E(R) and E(R^2)*

Distribution	Terms for R^2 ($z_j \neq z_k$) ($Rcov_{dist}$)
Bin(m,p)	$\binom{m}{z_i}\binom{m}{z_j}\frac{B(z_i+z_j+t_z+\alpha,2n-z_i-z_j+n_1m-t_z+\beta)}{B(t_z+\alpha,mn_1-t_z+\beta)}$
Poisson(λ)	$\frac{(n_2+\gamma)^{(t_z+\delta)}\Gamma(z_i+z_j+t_z+\delta)}{(2+n_2+\delta)^{(z_i+z_j+t_z+\delta)}\Gamma t_z+\delta}$
Negative binomial (r,η)	$\binom{r+z_i-1}{z_i}\binom{r+z_j-1}{z_j}\frac{B(z_i+z_j+t_z+c,2r+n_2r+d)}{B(t_z+c,n_2r+d)}$
Geometric(μ)	$\frac{B(t_z+z_i+z_j+a,n_2+b+2)}{B(t_z+a,n_2+b)}$

4. Theoretical expression of R for different stress-strength models

The theoretical expressions of R for general power series stress-strength models have been derived earlier. Such expressions are of theoretical interest because specific members of the power series family are used for modeling stress and/or strength. Being restricted to the family of power series distributions, we provide simplified expressions of such quantities for several stress and strength distribution choices. This section found the posterior distributions for different combinations of the parameters attached to the different distributions. In each subsection, we have mentioned the pmfs of the random variables used for stress and strength, the prior distributions of the parameters involved in the pmfs, the joint posterior distribution of the parameters, the $E(R|t_x, t_y)$ and $E(R^2|t_x, t_y)$ that are required for the Bayes estimates $\hat{R}$.

4.1. X and Y both follow binomial distributions

Let X$\sim$binomial(m_1, p_1) and Y$\sim$binomial(m_2, p_2), where $p_1 \sim$beta(α_1, β_1) and $p_2 \sim$beta(α_2, β_2). The joint prior distribution of p_1 and p_2 is

$$\pi(p_1, p_2) = g(p_1).h(p_2).$$

The posterior distribution of p_1 and p_2 is given by

$$\prod(p_1, p_2|t_x, t_y) = I_{bin}(m_1, p_1) I_{bin}(m_2, p_2). \tag{7}$$

The Bayes estimate under SEL is given by

$$\begin{aligned}
\hat{R}_{SEL} &= E(R_{BB}|t_x, t_y) \\
&= \int_0^1 \int_0^1 \sum_{y=0}^{m_2} \sum_{x=0}^{min(y,m_1)} \binom{m_1}{x} p_1^x (1-p_1)^{m_1-x} \binom{m_2}{y} p_2^y (1-p_2)^{m_2-y} \prod(p_1, p_2|t_x, t_y) dp_1 dp_2 \\
&= \sum_{y=0}^{m_2} \sum_{x=0}^{min(y,m_1)} R_{bin}(m_1, p_1) R_{bin}(m_2, p_2). \qquad (8)
\end{aligned}$$

The Bayes estimate under PL is $\hat{R}_{PL} = \sqrt{E(R^2)}$, where $E(R^2)$ is given by

$$\begin{aligned} E(R^2_{BB}|t_x,t_y) &= \int_0^1\int_0^1 \left[\sum_{y=0}^{m_2}\sum_{x=0}^{min(y,m_1)} \binom{m_1}{x} p_1^x(1-p_1)^{m_1-x}\binom{m_2}{y} p_2^y(1-p_2)^{m_2-y}\right]^2 \\ &\quad \times \prod(p_1,p_2|t_x,t_y)dp_1dp_2 \\ &= \sum_{y=0}^{m_2}\sum_{x=0}^{min(y,m_1)} Rsq_{bin}(m_1,p_1)Rsq_{bin}(m_2,p_2) \\ &\quad + \sum_{y_j=0}^{m_2}\sum_{x_i=0}^{min(y_j,m_1)}\sum_{y_l=0}^{m_2}\sum_{x_k=0}^{min(y_l,m_1)} Rcov_{bin}(m_1,p_1)Rcov_{bin}(m_2,p_2). \end{aligned}$$

The estimated posterior risks are as follows.

$$\hat{PR}_{SEL} = \hat{E}(R^2_{BB}) - \hat{E}^2(R_{BB}).$$
$$\hat{PR}_{PL} = 2\left(\sqrt{\hat{E}(R^2_{BB})} - \hat{E}(R_{BB})\right).$$

4.2. X and Y both follow Poisson distributions

Let X∼Poisson(λ_1) and Y∼Poisson(λ_2), where λ_1 ∼Gamma(γ_1, δ_1) and λ_2 ∼Gamma(γ_2, δ_2).
The joint prior distribution of λ_1 and λ_2 is

$$\pi(\lambda_1,\lambda_2) = g(\lambda_1).h(\lambda_2).$$

The posterior distribution of λ_1 and λ_2 is given by

$$\prod(\lambda_1,\lambda_2|t_x,t_y) = I_{pois}(\lambda_1)I_{pois}(\lambda_2).$$

Hence, under the SEL

$$\begin{aligned} E(R_{PP}|t_x,t_y) &= \int_0^1\int_0^1 \sum_{y=0}^{m_2}\sum_{x=0}^{y} \frac{e^{-\lambda_1}\lambda_1^x}{x!}\frac{e^{-\lambda_2}\lambda_2^y}{y!}\prod(\lambda_1,\lambda_2|t_x,t_y)d\lambda_1 d\lambda_2 \\ &= \sum_{y=0}^{m_2}\sum_{x=0}^{y} R_{pois}(\lambda_1)R_{pois}(\lambda_2), \end{aligned}$$

and under the PL

$$\begin{aligned} E(R^2_{PP}|t_x,t_y) &= \int_0^1\int_0^1 \left[\sum_{y=0}^{m_2}\sum_{x=0}^{y} \frac{e^{-\lambda_1}\lambda_1^x}{x!}\frac{e^{-\lambda_2}\lambda_2^y}{y!}\right]^2\prod(\lambda_1,\lambda_2|t_x,t_y)d\lambda_1 d\lambda_2 \\ &= \sum_{y=0}^{m_2}\sum_{x=0}^{y} Rsq_{pois}(\lambda_1)Rsq_{pois}(\lambda_2) + \sum_{y_j=0}^{m_2}\sum_{x_i=0}^{y_j}\sum_{y_l=0}^{m_2}\sum_{x_k=0}^{y_l} Rcov_{pois}(\lambda_1)Rcov_{pois}(\lambda_2). \end{aligned}$$

4.3. X and Y both follow negative binomial distributions

Let X∼negative binomial(r_1,η_1) and Y∼negative binomial(r_2,η_2), where η_1 ∼beta(c_1, d_1) and η_2 ∼beta(c_2, d_2).
The joint prior distribution of η_1 and η_2 is

$$\pi(\eta_1,\eta_2) = g(\eta_1).h(\eta_2).$$

The posterior distribution of η_1 and η_2 is given by

$$\prod(\eta_1,\eta_2|t_x,t_y) = I_{neg}(\eta_1)I_{neg}(\eta_2).$$

Therefore, under SEL

$$\begin{aligned} E(R_{NN}|t_x,t_y) &= \int_0^1\int_0^1 \sum_{y=0}^{\infty}\sum_{x=0}^{y} \binom{r_1+x-1}{x}\eta_1^x(1-\eta_1)_1^r\binom{r_2+y-1}{y}\eta_2^y(1-\eta_2)_2^r \\ &\quad \times \prod(\eta_1,\eta_2|t_x,t_y)d\eta_1 d\eta_2 \\ &= \sum_{y=0}^{\infty}\sum_{x=0}^{y} R_{neg}(\eta_1)R_{neg}(\eta_2), \end{aligned}$$

and under PL

$$\begin{aligned} E(R_{NN}^2|t_x,t_y) &= \int_0^1\int_0^1 \left[\sum_{y=0}^{\infty}\sum_{x=0}^{y} \binom{r_1+x-1}{x}\eta_1^x(1-\eta_1)_1^r\binom{r_2+y-1}{y}\eta_2^y(1-\eta_2)_2^r\right]^2 \\ &\quad \times \prod(\eta_1,\eta_2|t_x,t_y)d\eta_1 d\eta_2 \\ &= \sum_{y=0}^{\infty}\sum_{x=0}^{y} Rsq_{neg}(\eta_1)Rsq_{neg}(\eta_2) + \sum_{y_j=0}^{\infty}\sum_{x_i=0}^{y_j}\sum_{y_l=0}^{\infty}\sum_{x_k=0}^{y_l} Rcov_{neg}(\eta_1)Rcov_{neg}(\eta_2). \end{aligned}$$

4.4. X and Y both follow geometric distributions

Let X$\sim$geometric(μ_1) and Y$\sim$geometric(μ_2), where $\mu_1 \sim$beta(a_1, b_1) and $\mu_2 \sim$beta(c_2, d_2). The joint prior distribution of μ_1 and μ_2 is

$$\pi(\mu_1,\mu_2) = g(\mu_1).h(\mu_2).$$

The posterior distribution of μ_1 and μ_2 is given by,

$$\prod(\mu_1,\mu_2|t_x,t_y) = I_{geo}(\mu_1)I_{geo}(\mu_2).$$

Therefore, under SEL

$$\begin{aligned} E(R_{GG}|t_x,t_y) &= \int_0^1\int_0^1 \sum_{y=0}^{\infty}\sum_{x=0}^{y} \mu_1^x(1-\mu_1)\mu_2^y(1-\mu_2)\prod(\mu_1,\mu_2|t_x,t_y)d\mu_1 d\mu_2 \\ &= \sum_{y=0}^{\infty}\sum_{x=0}^{y} R_{geo}(\mu_1)R_{geo}(\mu_2), \end{aligned}$$

and under PL

$$\begin{aligned} E(R_{GG}^2|t_x,t_y) &= \int_0^1\int_0^1 \left[\sum_{y=0}^{\infty}\sum_{x=0}^{y} \mu_1^x(1-\mu_1)\mu_2^y(1-\mu_2)\right]^2\prod(\mu_1,\mu_2|t_x,t_y)d\mu_1 d\mu_2 \\ &= \sum_{y=0}^{\infty}\sum_{x=0}^{y} Rsq_{geo}(\mu_1)Rsq_{geo}(\mu_2) + \sum_{y_j=0}^{\infty}\sum_{x_i=0}^{y_j}\sum_{y_l=0}^{\infty}\sum_{x_k=0}^{y_l} Rcov_{geo}(\mu_1)Rcov_{geo}(\mu_2). \end{aligned}$$

4.5. X follows binomial distribution and Y follows Poisson distribution

Let X$\sim$binomial(m,p) and Y$\sim$Poisson(λ), where p$\sim$beta(α, β) and $\lambda \sim$gamma(γ, δ). The joint prior distribution of p and λ is

$$\pi(p,\lambda) = g(p).h(\lambda).$$

The posterior distribution of p and λ is given by

$$\prod(p,\lambda|t_x,t_y) = I_{bin}(p)I_{pois}(\lambda).$$

Under SEL

$$
\begin{aligned}
E(R_{BP}|t_x, t_y) &= \int_0^1 \int_0^1 \sum_{y=0}^{\infty} \sum_{x=0}^{y} \binom{m}{x} p^x (1-p)^{m-x} \frac{e^{-\lambda}\lambda^y}{y!} \prod(p, \lambda|t_x, t_y) dp d\lambda \\
&= \sum_{y=0}^{\infty} \sum_{x=0}^{y} R_{bin}(p) R_{pois}(\lambda),
\end{aligned}
$$

and under PL

$$
\begin{aligned}
E(R^2_{BP}|t_x, t_y) &= \int_0^1 \int_0^1 \left[\sum_{y=0}^{\infty} \sum_{x=0}^{y} \binom{m}{x} p^x (1-p)^{m-x} \frac{e^{-\lambda}\lambda^y}{y!} \right]^2 \prod(p, \lambda|t_x, t_y) dp d\lambda \\
&= \sum_{y=0}^{\infty} \sum_{x=0}^{y} Rsq_{bin}(p) Rsq_{pois}(\lambda) + \sum_{y_j=0}^{\infty} \sum_{x_i=0}^{y_j} \sum_{y_l=0}^{\infty} \sum_{x_k=0}^{y_l} Rcov_{bin}(p) Rcov_{pois}(\lambda).
\end{aligned}
$$

4.6. X follows binomial distribution and Y follows negative binomial distribution

Let X$\sim$binomial(m,p) and Y$\sim$negative binomial(r,η), where p$\sim$beta(α, β) and η $\sim$beta(c,d). The joint prior distribution of p and η is

$$\pi(p, \eta) = g(p).h(\eta).$$

The posterior distribution of p and η is

$$\prod(p, \eta|t_x, t_y) = I_{bin}(p) I_{neg}(\eta).$$

So, under SEL

$$
\begin{aligned}
E(R_{BN}|t_x, t_y) &= \int_0^1 \int_0^1 \sum_{y=0}^{\infty} \sum_{x=0}^{y} \binom{m}{x} p^x (1-p)^{m-x} \binom{r+y-1}{y} \eta^y (1-\eta)^r \prod(p, \eta|t_x, t_y) dp d\eta \\
&= \sum_{y=0}^{\infty} \sum_{x=0}^{y} R_{bin}(p) R_{neg}(\eta),
\end{aligned}
$$

and under PL

$$
\begin{aligned}
E(R^2_{BN}|t_x, t_y) &= \int_0^1 \int_0^1 \left[\sum_{y=0}^{\infty} \sum_{x=0}^{y} \binom{m}{x} p^x (1-p)^{m-x} \binom{r+y-1}{y} \eta^y (1-\eta)^r \right]^2 \prod(p, \eta|t_x, t_y) dp d\eta \\
&= \sum_{y=0}^{\infty} \sum_{x=0}^{y} Rsq_{bin}(p) Rsq_{neg}(\eta) + \sum_{y_j=0}^{\infty} \sum_{x_i=0}^{y_j} \sum_{y_l=0}^{\infty} \sum_{x_k=0}^{y_l} Rcov_{bin}(p) Rcov_{neg}(\eta).
\end{aligned}
$$

4.7. X follows binomial distribution and Y follows geometric distribution

Let X$\sim$binomial(m,p) and Y$\sim$geometric (μ), where p$\sim$beta(α, β) and μ $\sim$beta(a,b).
The joint prior distribution of p and μ is

$$\pi(p, \mu) = g(p).h(\mu).$$

The posterior distribution of p and μ is given by

$$\prod(p, \mu|t_x, t_y) = I_{bin}(p) I_{geo}(\mu).$$

Under SEL,

$$\begin{aligned} E(R_{BG}|t_x, t_y) &= \int_0^1 \int_0^1 \sum_{y=0}^{\infty} \sum_{x=0}^{y} \binom{m}{x} p^x (1-p)^{m-x} \mu^y (1-\mu) \prod(p, \mu|t_x, t_y) dp d\mu \\ &= \sum_{y=0}^{\infty} \sum_{x=0}^{y} R_{bin}(p) R_{geo}(\mu), \end{aligned}$$

and under PL,

$$\begin{aligned} E(R_{BG}^2|t_x, t_y) &= \int_0^1 \int_0^1 \left[\sum_{y=0}^{\infty} \sum_{x=0}^{y} \binom{m}{x} p^x (1-p)^{m-x} \mu^y (1-\mu) \right]^2 \prod(p, \mu|t_x, t_y) dp d\mu \\ &= \sum_{y=0}^{\infty} \sum_{x=0}^{y} Rsq_{bin}(p) Rsq_{geo}(\mu) + \sum_{y_j=0}^{\infty} \sum_{x_i=0}^{y_j} \sum_{y_l=0}^{\infty} \sum_{x_k=0}^{y_l} Rcov_{bin}(p) Rcov_{geo}(\mu). \end{aligned}$$

4.8. X follows Poisson distribution and Y follows binomial distribution

Let X∼Poisson(λ) and Y∼binomial (n,p), where λ ∼gamma(δ, γ) and p ∼beta(α, β). The joint prior distribution of λ and p is

$$\pi(\lambda, p) = g(\lambda).h(p).$$

The posterior distribution of λ and p is given by

$$\prod(\lambda, p|t_x, t_y) = I_{pois}(\lambda) I_{bin}(p).$$

Therefore, under SEL

$$\begin{aligned} E(R_{PB}|t_x, t_y) &= \int_0^1 \int_0^1 \sum_{y=0}^{m} \sum_{x=0}^{y} \frac{e^{-\lambda} \lambda^x}{x!} \binom{m}{y} p^y (1-p)^{m-y} \prod(\lambda, p|t_x, t_y) d\lambda dp \\ &= \sum_{y=0}^{\infty} \sum_{x=0}^{y} R_{pois}(\lambda) R_{bin}(p), \end{aligned}$$

and under PL

$$\begin{aligned} E(R_{PB}^2|t_x, t_y) &= \int_0^1 \int_0^1 \left[\sum_{y=0}^{m} \sum_{x=0}^{y} \frac{e^{-\lambda} \lambda^x}{x!} \binom{m}{y} p^y (1-p)^{m-y} \right]^2 \prod(\lambda, p|t_x, t_y) d\lambda dp \\ &= \sum_{y=0}^{\infty} \sum_{x=0}^{y} Rsq_{pois}(\lambda) Rsq_{bin}(p) + \sum_{y_j=0}^{\infty} \sum_{x_i=0}^{y_j} \sum_{y_l=0}^{\infty} \sum_{x_k=0}^{y_l} Rcov_{pois}(\lambda) Rcov_{bin}(p). \end{aligned}$$

4.9. X follows Poisson distribution and Y follows negative binomial distribution

Let X∼Poisson(λ) and Y∼negative binomial (r,η), where λ ∼gamma(δ, γ) and η ∼beta(c,d). The joint prior distribution of λ and η is

$$\pi(\lambda, \eta) = g(\lambda).h(\eta).$$

The posterior distribution of λ and η is

$$\prod(\lambda, \eta|t_x, t_y) = I_{pois}(\lambda) I_{neg}(\eta).$$

So, under SEL

$$\begin{aligned} E(R_{PN}|t_x, t_y) &= \int_0^1 \int_0^1 \sum_{y=0}^{\infty} \sum_{x=0}^{y} \frac{e^{-\lambda} \lambda^x}{x!} \binom{r+y-1}{y} \eta^y (1-\eta)^r \prod(\lambda, \eta|t_x, t_y) d\lambda d\eta \\ &= \sum_{y=0}^{\infty} \sum_{x=0}^{y} R_{pois}(\lambda) R_{neg}(\eta), \end{aligned}$$

and under PL

$$\begin{aligned} E(R_{PN}^2|t_x,t_y) &= \int_0^1\int_0^1 \left[\sum_{y=0}^{\infty}\sum_{x=0}^{y}\frac{e^{-\lambda}\lambda^x}{x!}\binom{r+y-1}{y}\eta^y(1-\eta)^r\right]^2 \prod(\lambda,\eta|t_x,t_y)d\lambda d\eta \\ &= \sum_{y=0}^{\infty}\sum_{x=0}^{y} Rsq_{pois}(\lambda)Rsq_{neg}(\eta) + \sum_{y_j=0}^{\infty}\sum_{x_i=0}^{y_j}\sum_{y_l=0}^{\infty}\sum_{x_k=0}^{y_l} Rcov_{pois}(\lambda)Rcov_{neg}(\eta). \end{aligned}$$

4.10. X follows Poisson distribution and Y follows geometric distribution

Let X$\sim$Poisson(λ) and Y$\sim$geometric (μ), where λ $\sim$gamma(δ,γ) and μ $\sim$beta(a,b).
The joint prior distribution of λ and μ is

$$\pi(\lambda,\mu) = g(\lambda).h(\mu).$$

The posterior distribution of λ and μ is

$$\prod(\lambda,\mu|t_x,t_y) = I_{pois}(\lambda)I_{geo}(\mu).$$

Under SEL

$$\begin{aligned} E(R_{PG}|t_x,t_y) &= \int_0^1\int_0^1 \sum_{y=0}^{\infty}\sum_{x=0}^{y}\frac{e^{-\lambda}\lambda^x}{x!}\mu^y(1-\mu)\prod(\lambda,\mu|t_x,t_y)d\lambda d\mu \\ &= \sum_{y=0}^{\infty}\sum_{x=0}^{y} R_{pois}(\lambda)R_{geo}(\mu), \end{aligned}$$

and under PL

$$\begin{aligned} E(R_{PG}^2|t_x,t_y) &= \int_0^1\int_0^1 \left[\sum_{y=0}^{\infty}\sum_{x=0}^{y}\frac{e^{-\lambda}\lambda^x}{x!}\mu^y(1-\mu)\right]^2 \prod(\lambda,\mu|t_x,t_y)d\lambda d\mu \\ &= \sum_{y=0}^{\infty}\sum_{x=0}^{y} Rsq_{pois}(\lambda)Rsq_{geo}(\mu) + \sum_{y_j=0}^{\infty}\sum_{x_i=0}^{y_j}\sum_{y_l=0}^{\infty}\sum_{x_k=0}^{y_l} Rcov_{pois}(\lambda)Rcov_{geo}(\mu). \end{aligned}$$

4.11. X follows negative binomial distribution and Y follows binomial distribution

Let X$\sim$negative binomial(r,η) and Y$\sim$binomial (n,p), where η $\sim$beta(c,d) and p $\sim$beta(α,β).
The joint prior distribution of η and p is

$$\pi(\eta,p) = g(\eta).h(p).$$

The posterior distribution of η and p is given by

$$\prod(\eta,p|t_x,t_y) = I_{neg}(\eta)I_{bin}(p).$$

So, under SEL

$$\begin{aligned} E(R_{NB}|t_x,t_y) &= \int_0^1\int_0^1 \sum_{y=0}^{m}\sum_{x=0}^{y}\binom{r+x-1}{x}\eta^x(1-\eta)^r\binom{m}{y}p^y(1-p)^{m-y}\prod(\eta,p|t_x,t_y)d\eta dp \\ &= \sum_{y=0}^{\infty}\sum_{x=0}^{y} R_{neg}(\eta)R_{bin}(p), \end{aligned}$$

and under PL

$$\begin{aligned} E(R^2_{NB}|t_x,t_y) &= \int_0^1\int_0^1 \left[\sum_{y=0}^{m}\sum_{x=0}^{y}\binom{r+x-1}{x}\eta^x(1-\eta)^r\binom{m}{y}p^y(1-p)^{m-y}\right]^2 \\ &\quad\times\prod(\eta,p|t_x,t_y)d\eta dp \\ &= \sum_{y=0}^{\infty}\sum_{x=0}^{y} Rsq_{neg}(\eta)Rsq_{bin}(p) + \sum_{y_j=0}^{\infty}\sum_{x_i=0}^{y_j}\sum_{y_l=0}^{\infty}\sum_{x_k=0}^{y_l} Rcov_{neg}(\eta)Rcov_{bin}(p). \end{aligned}$$

4.12. X follows negative binomial distribution and Y follows Poisson distribution

Let X∼negative binomial(r,η) and Y∼Poisson (λ), where η ∼beta(c,d) and λ ∼gamma(δ,γ). The joint prior distribution of η and λ is

$$\pi(\eta,\lambda) = g(\eta).h(\lambda).$$

The posterior distribution of η and λ is given by

$$\prod(\eta,\lambda|t_x,t_y) = I_{neg}(\eta)I_{pois}(\lambda).$$

Under SEL

$$\begin{aligned} E(R|t_x,t_y) &= \int_0^1\int_0^1 \sum_{y=0}^{\infty}\sum_{x=0}^{y}\binom{r+x-1}{x}\eta^x(1-\eta)^r\frac{e^{-\lambda}\lambda^y}{y!}\prod(\eta,\lambda|t_x,t_y)d\eta d\lambda \\ &= \sum_{y=0}^{\infty}\sum_{x=0}^{y} R_{neg}(\eta)R_{pois}(\lambda), \end{aligned}$$

and under PL

$$\begin{aligned} E(R^2_{NP}|t_x,t_y) &= \int_0^1\int_0^1 \left[\sum_{y=0}^{\infty}\sum_{x=0}^{y}\binom{r+x-1}{x}\eta^x(1-\eta)^r\frac{e^{-\lambda}\lambda^y}{y!}\right]^2\prod(\eta,\lambda|t_x,t_y)d\eta d\lambda \\ &= \sum_{y=0}^{\infty}\sum_{x=0}^{y} Rsq_{neg}(\eta)Rsq_{pois}(\lambda) + \sum_{y_j=0}^{\infty}\sum_{x_i=0}^{y_j}\sum_{y_l=0}^{\infty}\sum_{x_k=0}^{y_l} Rcov_{neg}(\eta)Rcov_{pois}(\lambda). \end{aligned}$$

4.13. X follows negative binomial distribution and Y follows geometric distribution

Let X∼negative binomial(r,η) and Y∼geometric (μ), where η ∼beta(c,d) and μ ∼beta(a,b). The joint prior distribution of η and μ is

$$\pi(\eta,\mu) = g(\eta).h(\mu).$$

The posterior distribution of η and μ is given by

$$\prod(\eta,\mu|t_x,t_y) = I_{neg}(\eta)I_{geo}(\mu).$$

Under SEL

$$\begin{aligned} E(R_{NG}|t_x,t_y) &= \int_0^1\int_0^1 \sum_{y=0}^{\infty}\sum_{x=0}^{y}\binom{r+x-1}{x}\eta^x(1-\eta)^r\mu^y(1-\mu)\prod(\eta,\mu|t_x,t_y)d\eta d\mu \\ &= \sum_{y=0}^{\infty}\sum_{x=0}^{y} R_{neg}(\eta)R_{geo}(\mu), \end{aligned}$$

and under PL

$$E(R_{NG}^2|t_x,t_y) = \int_0^1\int_0^1\left[\sum_{y=0}^{\infty}\sum_{x=0}^{y}\binom{r+x-1}{x}\eta^x(1-\eta)^r\mu^y(1-\mu)\right]^2\prod(\eta,\mu|t_x,t_y)d\eta d\mu$$
$$= \sum_{y=0}^{\infty}\sum_{x=0}^{y}Rsq_{neg}(\eta)Rsq_{geo}(\mu)+\sum_{y_j=0}^{\infty}\sum_{x_i=0}^{y_j}\sum_{y_l=0}^{\infty}\sum_{x_k=0}^{y_l}Rcov_{neg}(\eta)Rcov_{geo}(\mu).$$

4.14. X follows geometric distribution and Y follows binomial distribution

X$\sim$geometric(μ) and Y$\sim$binomial (n,p), where $\mu\sim$beta(a,b) and $p\sim$beta(α,β).
The joint prior distribution of μ and p is given by

$$\pi(\mu,p)=g(\mu).h(p).$$

The posterior distribution of μ and p is given by

$$\prod(\mu,p|t_x,t_y)=I_{geo}(\mu)I_{bin}(p).$$

Under SEL

$$E(R_{GB}|t_x,t_y) = \int_0^1\int_0^1\sum_{y=0}^{m}\sum_{x=0}^{y}\mu^x(1-\mu)^r\binom{m}{y}p^y(1-p)^{m-y}\prod(\mu,p|t_x,t_y)d\mu dp$$
$$= \sum_{y=0}^{\infty}\sum_{x=0}^{y}R_{geo}(\mu)R_{bin}(p),$$

and under PL

$$E(R_{GB}^2|t_x,t_y) = \int_0^1\int_0^1\left[\sum_{y=0}^{m}\sum_{x=0}^{y}\mu^x(1-\mu)^r\binom{m}{y}p^y(1-p)^{m-y}\right]^2\prod(\mu,p|t_x,t_y)d\mu dp$$
$$= \sum_{y=0}^{\infty}\sum_{x=0}^{y}Rsq_{geo}(\mu)Rsq_{bin}(p)+\sum_{y_j=0}^{\infty}\sum_{x_i=0}^{y_j}\sum_{y_l=0}^{\infty}\sum_{x_k=0}^{y_l}Rcov_{geo}(\mu)Rcov_{bin}(p).$$

4.15. X follows geometric distribution and Y follows negative binomial distribution

Let X$\sim$geometric(μ) and Y$\sim$negative binomial (r,η), where $\mu\sim$beta(a,b) and $\eta\sim$beta(c,d).
The joint prior distribution of μ and η is

$$\pi(\mu,\eta)=g(\mu).h(\eta).$$

The posterior distribution of μ and η is given by

$$\prod(\mu,\eta|t_x,t_y)=I_{geo}(\mu)I_{neg}(\eta).$$

Under SEL

$$E(R_{GN}|t_x,t_y) = \int_0^1\int_0^1\sum_{y=0}^{\infty}\sum_{x=0}^{y}\mu^x(1-\mu)^r\binom{r+y-1}{y}\eta^y(1-\eta)^r\prod(\mu,\eta|t_x,t_y)d\mu d\eta$$
$$= \sum_{y=0}^{\infty}\sum_{x=0}^{y}R_{geo}(\mu)R_{neg}(\eta),$$

and under PL

$$E(R_{GN}^2|t_x,t_y) = \int_0^1\int_0^1\left[\sum_{y=0}^{\infty}\sum_{x=0}^{y}\mu^x(1-\mu)^r\binom{r+y-1}{y}\eta^y(1-\eta)^r\right]^2\prod(\mu,\eta|t_x,t_y)d\mu d\eta$$
$$= \sum_{y=0}^{\infty}\sum_{x=0}^{y}Rsq_{geo}(\mu)Rsq_{neg}(\eta)+\sum_{y_j=0}^{\infty}\sum_{x_i=0}^{y_j}\sum_{y_l=0}^{\infty}\sum_{x_k=0}^{y_l}Rcov_{geo}(\mu)Rcov_{neg}(\eta).$$

4.16. X follows geometric distribution and Y follows Poisson distribution

X∼geometric (μ) and Y∼Poisson (λ), where μ ∼beta(a,b) and λ ∼gamma(δ, γ).
The joint prior distribution of μ and λ is

$$\pi(\mu, \lambda) = g(\mu).h(\lambda).$$

The posterior distribution of μ and λ is given by

$$\prod(\mu, \lambda | t_x, t_y) = I_{geo}(\mu) I_{pois}(\lambda).$$

Under SEL

$$\begin{aligned} E(R|t_x, t_y) &= \int_0^1 \int_0^1 \sum_{y=0}^{\infty} \sum_{x=0}^{y} \mu^x (1-\mu) \frac{e^{-\lambda} \lambda^y}{y!} \prod(\mu, \lambda | t_x, t_y) d\mu d\lambda \\ &= \sum_{y=0}^{\infty} \sum_{x=0}^{y} R_{geo}(\mu) R_{pois}(\lambda), \end{aligned}$$

and under PL

$$\begin{aligned} E(R_{GP}^2 | t_x, t_y) &= \int_0^1 \int_0^1 \left[\sum_{y=0}^{\infty} \sum_{x=0}^{y} \mu^x (1-\mu) \frac{e^{-\lambda} \lambda^y}{y!} \right]^2 \prod(\mu, \lambda | t_x, t_y) d\mu d\lambda \\ &= \sum_{y=0}^{\infty} \sum_{x=0}^{y} Rsq_{geo}(\mu) Rsq_{pois}(\lambda) + \sum_{y_j=0}^{\infty} \sum_{x_i=0}^{y_j} \sum_{y_l=0}^{\infty} \sum_{x_k=0}^{y_l} Rcov_{geo}(\mu) Rcov_{pois}(\lambda). \end{aligned}$$

5. Estimation of the hyper-parameters

Though the prior distributions are assumed to be known, most of the time, in practice, those need to be estimated based on observed data. In this section, the estimates of the hyper-parameters of the prior distributions have been found. These parameters could be estimated using the empirical Bayes procedure [see [10] and [3]]. Given the observations, the joint likelihood distributions have been compared with the joint prior distributions. The joint likelihood distributions are just the multiplication of the likelihood distribution of X and Y, and joint prior distributions are the multiplication of the prior distributions of the parameters. We can estimate the prior parameters by comparing them individually with their corresponding likelihood functions. The estimates of the hyper-parameters are shown in Table 5.

Table 5: *Estimate of the hyper-parameters*

Distribution	Joint distribution	Prior distribution	Estimate of hyper-parameters
Binomial(m,p)	$p^{t_z}(1-p)^{mn-t_z}$	$\frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha,\beta)}$	$\hat{\alpha} = t_x + 1,\ \hat{\beta} = mn - t_x + 1$
Poisson(λ)	$\frac{e^{-n\lambda}\lambda^{t_z}}{\prod_{i=1}^{n} z_i!}$	$\frac{\delta^{\gamma} e^{-\gamma\lambda} \lambda^{\delta-1}}{\Gamma\delta}$	$\hat{\delta} = t_x + 1,\ \hat{\gamma} = n$
Neg. binomial (r, η)	$\eta^{t_z}(1-\eta)^{nr}$	$\frac{\eta^{c-1}(1-\eta)^{d-1}}{B(c,d)}$	$\hat{c} = t_x + 1,\ \hat{d} = nr + 1$
Geometric (μ)	$\mu^{t_z}(1-\mu)^{n}$	$\frac{\mu^{a-1}(1-\mu)^{b-1}}{B(a,b)}$	$\hat{a} = t_x + 1,\ \hat{b} = n + 1$

6. Simulation study

The properties of the Bayesian estimations of R for the different combinations of stress-strength models within the power series family have been explored empirically. The estimated posterior risks for those different combinations have been found.

Table 6: *Geometric-Geometric*

Sample size		Distribution of μ_1			Distribution of μ_2			Actual R	E(R)	$E(R^2)$	Bayes estimate		Posterior risk	
n_1	n_2	c	d	$E(\mu_1)$	a	b	$E(\mu_2)$				SEL	PL	SEL	PL
15	15	3	7	0.3	2	8	0.2	0.745	0.727	0.532	0.727	0.729	0.00411	0.00564
15	15	3	7	0.3	5	5	0.5	0.824	0.827	0.686	0.827	0.828	0.00217	0.00263
15	15	3	7	0.3	8	2	0.8	0.921	0.934	0.873	0.934	0.935	0.00046	0.00049
15	15	5	5	0.5	2	8	0.2	0.556	0.561	0.319	0.561	0.565	0.00418	0.00742
15	15	5	5	0.5	5	5	0.5	0.667	0.672	0.455	0.672	0.675	0.00383	0.00569
15	15	5	5	0.5	8	2	0.8	0.833	0.830	0.691	0.830	0.831	0.00192	0.00232
15	15	8	2	0.8	2	8	0.2	0.238	0.233	0.056	0.233	0.236	0.00153	0.00652
15	15	8	2	0.8	5	5	0.5	0.333	0.341	0.119	0.341	0.345	0.00277	0.00807
15	15	8	2	0.8	8	2	0.8	0.556	0.554	0.311	0.554	0.558	0.00384	0.00691
15	30	3	7	0.3	2	8	0.2	0.745	0.747	0.563	0.747	0.750	0.00393	0.00525
15	30	3	7	0.3	5	5	0.5	0.824	0.829	0.689	0.829	0.830	0.00173	0.00208
15	30	3	7	0.3	8	2	0.8	0.921	0.922	0.851	0.922	0.922	0.00043	0.00047
15	30	5	5	0.5	2	8	0.2	0.556	0.583	0.344	0.583	0.586	0.00410	0.00702
15	30	5	5	0.5	5	5	0.5	0.667	0.671	0.454	0.671	0.674	0.00338	0.00502
15	30	5	5	0.5	8	2	0.8	0.833	0.849	0.723	0.849	0.850	0.00126	0.00149
15	30	8	2	0.8	2	8	0.2	0.238	0.238	0.058	0.238	0.241	0.00142	0.00591
15	30	8	2	0.8	5	5	0.5	0.333	0.335	0.115	0.335	0.339	0.00239	0.00708
15	30	8	2	0.8	8	2	0.8	0.556	0.585	0.346	0.585	0.588	0.00315	0.00537
30	30	3	7	0.3	2	8	0.2	0.745	0.735	0.542	0.735	0.736	0.00201	0.00274
30	30	3	7	0.3	5	5	0.5	0.823	0.823	0.678	0.823	0.824	0.00131	0.00159
30	30	3	7	0.3	8	2	0.8	0.921	0.928	0.861	0.928	0.928	0.00029	0.00032
30	30	5	5	0.5	2	8	0.2	0.556	0.550	0.305	0.550	0.552	0.00216	0.00391
30	30	5	5	0.5	5	5	0.5	0.667	0.659	0.436	0.659	0.661	0.00201	0.00305
30	30	5	5	0.5	8	2	0.8	0.833	0.847	0.718	0.847	0.847	0.00076	0.00090
30	30	8	2	0.8	2	8	0.2	0.238	0.229	0.053	0.229	0.231	0.00074	0.00322
30	30	8	2	0.8	5	5	0.5	0.333	0.322	0.105	0.322	0.324	0.00139	0.00431
30	30	8	2	0.8	8	2	0.8	0.556	0.534	0.287	0.534	0.536	0.00203	0.00380

Table 7: *Negative binomial-Poisson*

Sample size		Distribution of η			Distribution of λ			Actual R	E(R)	$E(R^2)$	Bayes estimate		Posterior risk	
n_1	n_2	c	d	$E(\eta)$	δ	γ	$E(\lambda)$				SEL	PL	SEL	PL
15	15	3	7	0.3	2	10	0.2	0.039	0.032	0.001	0.032	0.034	$7.61 * 10^{-5}$	$2.30 * 10^{-3}$
15	15	3	7	0.3	5	10	0.5	0.058	0.054	0.003	0.054	0.056	$2.14 * 10^{-4}$	$3.90 * 10^{-3}$
15	15	3	7	0.3	8	10	0.8	0.079	0.080	0.007	0.080	0.083	$4.36 * 10^{-4}$	$5.36 * 10^{-3}$
15	15	5	5	0.5	2	10	0.2	0.162	0.156	0.025	0.156	0.160	$1.06 * 10^{-3}$	$6.70 * 10^{-3}$
15	15	5	5	0.5	5	10	0.5	0.218	0.214	0.048	0.214	0.218	$1.89 * 10^{-3}$	$8.75 * 10^{-3}$
15	15	5	5	0.5	8	10	0.8	0.273	0.283	0.083	0.283	0.289	$2.94 * 10^{-3}$	$1.03 * 10^{-2}$
15	15	8	2	0.8	2	10	0.2	0.570	0.592	0.355	0.592	0.596	$4.00 * 10^{-3}$	$6.74 * 10^{-3}$
15	15	8	2	0.8	5	10	0.5	0.645	0.639	0.413	0.645	0.642	$4.39 * 10^{-3}$	$6.85 * 10^{-3}$
15	15	8	2	0.8	8	10	0.8	0.707	0.693	0.484	0.693	0.696	$4.17 * 10^{-3}$	$6.00 * 10^{-3}$
15	30	3	7	0.3	2	10	0.2	0.039	0.043	0.002	0.043	0.044	$7.27 * 10^{-5}$	$1.64 * 10^{-3}$
15	30	3	7	0.3	5	10	0.5	0.058	0.054	0.003	0.054	0.056	$1.93 * 10^{-4}$	$3.48 * 10^{-3}$
15	30	3	7	0.3	8	10	0.8	0.079	0.075	0.006	0.075	0.077	$3.42 * 10^{-4}$	$4.51 * 10^{-3}$
15	30	5	5	0.5	2	10	0.2	0.162	0.163	0.028	0.163	0.167	$1.35 * 10^{-3}$	$8.19 * 10^{-3}$
15	30	5	5	0.5	5	10	0.5	0.218	0.245	0.062	0.245	0.249	$2.01 * 10^{-3}$	$8.15 * 10^{-3}$
15	30	5	5	0.5	8	10	0.8	0.273	0.273	0.077	0.273	0.277	$2.37 * 10^{-3}$	$8.60 * 10^{-3}$
15	30	8	2	0.8	2	10	0.2	0.570	0.578	0.338	0.578	0.581	$3.77 * 10^{-3}$	$6.51 * 10^{-3}$
15	30	8	2	0.8	5	10	0.5	0.645	0.642	0.416	0.642	0.645	$3.81 * 10^{-3}$	$5.91 * 10^{-3}$
15	30	8	2	0.8	8	10	0.8	0.707	0.716	0.516	0.716	0.718	$3.43 * 10^{-3}$	$4.78 * 10^{-3}$
30	30	3	7	0.3	2	10	0.2	0.039	0.040	0.0017	0.040	0.041	$5.61 * 10^{-5}$	$9.27 * 10^{-4}$
30	30	3	7	0.3	5	10	0.5	0.058	0.063	0.004	0.063	0.064	$1.43 * 10^{-4}$	$2.24 * 10^{-3}$
30	30	3	7	0.3	8	10	0.8	0.079	0.078	0.006	0.078	0.079	$2.11 * 10^{-4}$	$2.69 * 10^{-3}$
30	30	5	5	0.5	2	10	0.2	0.162	0.169	0.029	0.169	0.171	$6.16 * 10^{-4}$	$3.61 * 10^{-3}$
30	30	5	5	0.5	5	10	0.5	0.218	0.240	0.059	0.240	0.243	$1.27 * 10^{-3}$	$5.27 * 10^{-3}$
30	30	5	5	0.5	8	10	0.8	0.273	0.282	0.081	0.282	0.284	$1.41 * 10^{-3}$	$4.97 * 10^{-3}$
30	30	8	2	0.8	2	10	0.2	0.570	0.565	0.321	0.565	0.567	$2.02 * 10^{-3}$	$3.57 * 10^{-3}$
30	30	8	2	0.8	5	10	0.5	0.645	0.637	0.407	0.637	0.638	$2.19 * 10^{-3}$	$3.44 * 10^{-3}$
30	30	8	2	0.8	8	10	0.8	0.707	0.700	0.491	0.700	0.701	$2.17 * 10^{-3}$	$3.10 * 10^{-3}$

For computation, we draw $n_1(n_2)$ independent observations from the prior distribution(s) of the parameters of the stress (strength) distribution(s). For different combinations of those hyper-parameters of the stress (strength) distributions considered, the hyper-parameters are so chosen that the expected value of the parameters is equal to the values of the parameters selected combinations in [6]. We draw a random sample from the distribution(s) of the stress (strength) distribution(s). Then we compute $\hat{R_{SEL}}$, $\hat{R_{PL}}$ and their estimated posterior risk, where $\hat{R_{SEL}}$ and $\hat{R_{PL}}$ are the Bayesian estimates under squared error and precautionary loss functions. All these calculations are done using R-Software, and two representative tables are reported in Tables $6-7$ for space limitation. Some more tables are prepared and may be available from the corresponding author on request. The figures of the tables compared to [6]. We can say that the estimates under the Bayesian method are closer than those under MLE and UMVUE. Also, the estimated posterior risks under the empirical Bayesian method are smaller than the variance of UMVUE and the MSE of the MLE. So, we can infer that the Bayesian estimation gives better estimates than the MLE and UMVUE.

7. Real Life Data Analysis

We have provided a real dataset as an application of the stress-strength reliability model, which is related to soccer matches where the defenders and the goalkeeper are responsible for not allowing the opposition to score goals. They protect the team from continuous attacks from the opponent team. The number of goals conceded by the team can be treated as the stress put on the system's defence, whereas the number of goals saved by the defence acts as the strength of the team's defence. We want to estimate the reliability of the team's defence, i.e., $R = P(X \leq Y)$, where X is the stress on the system, and Y is the system's strength. We have considered the same dataset used in [6], where Manchester United played 38 matches against various teams of the EPL in the season 2017-2018. The data is presented in the Table 8.

Among those 38 matches, in 33 matches, they have saved more goals than they conceded. So,

Table 8: *EPL data for Manchester United during 2017-2018*

Match	Goals conceded	Goals saved	Match	Goals conceded	Goals saved
1	0	1	20	2	0
2	0	0	21	0	3
3	0	4	22	0	0
4	2	3	23	0	5
5	0	3	24	0	3
6	0	4	25	2	4
7	0	1	26	0	0
8	0	5	27	1	2
9	2	1	28	1	6
10	0	4	29	2	6
11	1	7	30	1	1
12	1	4	31	0	2
13	0	2	32	2	4
14	2	1	33	1	3
15	1	0	34	0	2
16	2	5	35	1	2
17	0	7	36	1	3
18	1	4	37	0	2
19	2	1	38	0	3

in about 86% cases, the defence system has worked above the stress given by the opposition. The reliability of the system is estimated using the theory defined above. The data are discrete.

The authors have shown that the "number of goals conceded" follows a geometric distribution, whereas the "number of goals saved" follows the Poisson distribution. We have found the estimates of R under SEL and PL as $\hat{R}_{SEL}$= 0.84683 and $\hat{R}_{PL}$= 0.84737 with estimated posterior risks $5.91 * 10^{-4}$ and $6.98 * 10^{-4}$, respectively.

8. Concluding remarks

In this article, the Bayesian estimation of $R = P(X \leq Y)$ has been considered when X and Y belong to a power-series family of distributions under two loss functions: (i) squared error loss, a symmetric and (ii) precautionary loss, an asymmetric one. The conjugate prior distribution(s) of the parameter(s) are chosen for deriving the Bayes estimates of R. Though the prior distributions are assumed to be known, the practice is not such. Generally, the prior distribution(s) parameters are estimated based on observations. The empirical Bayes method has been used to estimate the prior parameters and hence get the estimate of R. Simulation study results slightly favour the Bayes estimate of R under SEL than that under PL concerning estimated posterior risk sense. Data analysis result also affirms this. Scientists and practitioners are recommended to use the proposed Bayes estimate of R.

Declarations

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