WEIGHTED TRANSMUTED MUKHERJEE-ISLAM DISTRIBUTION WITH STATISTICAL PROPERTIES

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Abstract

In this study, we employ a weighted transformation approach to introduce a novel model that generalises the Transmuted Mukherjee-Islam distribution. The resulting generalized distribution is referred to as the Weighted Transmuted Mukherjee-Islam (WTMI) distribution The paper thoroughly explores the probability density function (PDF) and the corresponding cumulative distribution function (CDF) associated with the WTMI distribution. A thorough investigation of the distinctive structural properties of the proposed model is conducted, including survival function, conditional survival function, hazard function, cumulative hazard function, mean residual life, moments, moment generating function (MGF), characteristics function (CF), cumulant generating function (CGF), likelihood ratio test, ordered statistics, entropy measures, and Bonferroni and Lorenz curves. The maximum likelihood estimation method is employed for the precise estimation of model parameters.

Key words: Transmuted Mukherjee-Islam distribution, Reliability analysis, Maximum likelihood estimator, Ordered statistics

1. Introduction

In numerous applied sciences, including, engineering, agricultural science, biological science, biomedicine, ecology and various social science fields such as economics, finance, and population science, the modelling and analysis of lifetime data holds paramount importance. Multiple lifetime distributions have been employed to characterize such data and the effectiveness of statistical analysis procedures relies significantly on the chosen probability model or distribution. Consequently, substantial efforts have been taken for creating extensive classes of standard probability distributions along with corresponding statistical methodologies. Despite these advancements, numerous significant challenges persist, as real-world data often deviates from classical or standard probability models. Thus, the development of new forms of probability distributions remains a common objective of statistical theory. To extend the applicability of probability distributions, the literature proposes several methods that introduce additional parameter(s) to established baseline probability models. This enhances the flexibility of the models to capture the complexity of the data, leading to several generalized classes such as the Pearson Family, Burr Family, Exponentiated Family, Marshall-Olkin Family, $T-X$ Family, Transmuted Family, Weighted Family, and more.

In this article, we employ a weighted transformation approach to introduce a novel model that generalizes the Transmuted Mukherjee-Islam distribution. The new generalized distribution will be termed as WTMI distribution. The weighted family of distributions emerged from the pioneering work of Fisher in 1934 [10], subsequently refined and formalized by Rao in 1965 [16]. This concept serves as an essential tool in statistical theory, particularly in scenarios where observations are derived from non-experimental, non-replicated, and non-random conditions. In practice, Observing and recording all events is not always possible due to various factors. Some events may not be observable by the method used, may only be observable with a certain probability, or may change randomly during observation. Additionally, events produced under different mechanisms with unspecified relative frequencies may be mixed up and added to the same record. Therefore, the original event specification may not be appropriate for the recorded data unless modified. The weighted transformation approach enhances the flexibility of standard probability distributions in such scenarios. Researchers and scholars have extensively explored weighted probability models, along with the application of these models across various domains. Notably, Ghitany et al. [14] conducted a comprehensive study on the two-parameter weighted Lindley distribution, focusing on its relevance in analyzing survival data. Jain et al. [11] introduced the weighted gamma distribution, while Dey et al. [8] contributed significantly by introducing the weighted exponential distribution and its estimation techniques. In 2016 Das and Kundu [7], have obtained weighted and length biased version of exponential distribution. Kilany [13] have obtained the weighted version of lomax distribution. Subramanian and Rather [23] studied the weighted version of the exponentiated Mukherjee-Islam distribution, derived its statistical properties. Further, in 2018 Rather et al [20] explored the size-biased Ailamujia distribution with applications in engineering and medical science. Para and Jan [15] introduced the Weighted Pareto type-II distribution as a new model for handling medical science data and unveiling its statistical properties and potential applications across various fields. Rather and Subramanian [17] extensively studied the weighted Sushila distribution with properties and applications which shows more flexibility then its baseline distribution. Rather and Subramanian [19] further explored weighted distributions by offering a comprehensive overview, perspectives, and characterizations of the weighted version of Akshaya distribution with applications in engineering science. In a recent contribution, Rather and Ozel [18] discussed the weighted power Lindley distribution, demonstrating its effectiveness in analyzing lifetime data.

2. Probability density function (PDF) and cumulative distribution function (CDF)

The Transmuted Mukherjee-Islam distribution had explored by Rather and Subramanian {21] using the quadratic rank transmutation map studied first by Shaw and Buckley in 2007 [22]. Loai M. A. Al-Zou'bi [4] also obtained various properties of Transmuted Mukherjee-Islam distribution and its applications. The probability density function of a random variable say Z following Transmuted Mukherjee- Islam distribution with parameters say $(\varepsilon, \nu, \omega)$ is given by

$$f(z;\varepsilon,\nu,\omega)=\frac{\varepsilon}{\nu^{\varepsilon}}z^{\varepsilon-1}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)\ ;0<z<\nu,\varepsilon>0,\nu>0,-1\leq\omega\leq1 \tag{1}$$

And the corresponding cumulative distribution function is

$$F_Z(z)=\left(\frac{z}{\nu}\right)^{\varepsilon}\left(1+\omega-\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right) \tag{2}$$

Various researchers have attracted to the quadratic transmutation map and they have introduced the new members of this family for various choices of baseline distributions. Transmuted Weibull distribution by Aryal and Tsokos [5], transmuted inverse Rayleigh distribution by Ahmad et al. [3], transmuted Marshall-Olkin Frechet distribution by Afify et al. [2], transmuted generalized Lindley distribution by Elgarhy et al. [9], transmuted modified Weibull distribution by Cordeiro et al. [6], transmuted exponential Lomax distribution by Abdullahi and Ieren [1], transmuted Burr Type X distribution by Khan et al. [12].

Using the weighted transformation approach, the PDF $y(z)$ of a non-negative random variable Z is given by

$$y_w(z) = \frac{w(z)\,y(z)}{E(w(z))}; \qquad z > 0$$

Where $w(z)$ be a non-negative weight function and $E(w(z)) = \int_{-\infty}^{\infty} w(z)\,y(z)\,dz < \infty$.

In this paper, we will consider the weight function as $w(z) = z^s$ and PDF of the random variable Z to be Transmuted Mukherjee- Islam distribution to derive the PDF of Weighted Transmuted Mukherjee- Islam distribution. The PDF of Weighted Transmuted Mukherjee- Islam distribution distribution is given by

$$g(z;\varepsilon,\nu,\omega,s) = \frac{z^s f(z;\varepsilon,\nu,\omega)}{E(z^s)} \tag{3}$$

Now

$$E(z^s) = \int_0^{\nu} z^s \frac{\varepsilon}{\nu^{\varepsilon}} z^{\varepsilon-1}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)dz \tag{4}$$

$$E(z^s) = \frac{\varepsilon}{\nu^{\varepsilon}}\left((1+\omega)\int_0^{\nu} z^{s+\varepsilon-1}dz - \frac{2\omega}{(\nu)^{\varepsilon}}\int_0^{\nu} z^{s+2\varepsilon-1}dz\right) \tag{5}$$

After simplification we get

$$E(z^s) = \frac{\varepsilon\nu^s\left(s(1-\omega)+2\varepsilon\right)}{(s+\varepsilon)(s+2\varepsilon)} \tag{6}$$

Using (1) and (4) in (3) we get

$$g(z;\varepsilon,\nu,\omega,s) = \frac{z^s \frac{\varepsilon}{\nu^{\varepsilon}} z^{\varepsilon-1}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\frac{\varepsilon\nu^s\left(s(1-\omega)+2\varepsilon\right)}{(s+\varepsilon)(s+2\varepsilon)}} \tag{7}$$

$$g(z;\varepsilon,\nu,\omega,s) = \frac{(s+\varepsilon)(s+2\varepsilon)z^{s+\varepsilon-1}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{s+\varepsilon}\left(s(1-\omega)+2\varepsilon\right)} \tag{8}$$

The corresponding CDF of WTMI distribution is given by

$$G_Z(z) = \int_0^{z}\left(\frac{(s+\varepsilon)(s+2\varepsilon)z^{s+\varepsilon-1}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{s+\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}\right)dz \tag{9}$$

$$G_Z(z) = \frac{(s+\varepsilon)(s+2\varepsilon)}{\nu^{s+\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}\left((1+\omega)\int_0^{z} z^{s+\varepsilon-1}dz - \frac{2\omega}{(\nu)^{\varepsilon}}\int_0^{z} z^{s+2\varepsilon-1}\right) \tag{10}$$

After simplification we get

$$G_Z(z) = \frac{(s+2\varepsilon)(1+\omega)(\nu)^{\varepsilon} z^{s+\varepsilon} - 2\omega(s+\varepsilon) z^{s+2\varepsilon}}{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right)} \tag{11}$$

3. Reliability Analysis

3.1 Survival function

The survival function of WTMI distribution is given by

$$R_T(t) = P_r(T > t)$$

$$R_T(t) = 1 - P_r(T \le t)$$

$$R_T(t) = 1 - \frac{(s+2\varepsilon)(1+\omega)(\nu)^{\varepsilon} t^{s+\varepsilon} - 2\omega(s+\varepsilon) t^{s+2\varepsilon}}{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right)} \tag{12}$$

$$R_T(t) = \frac{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right) - (s+2\varepsilon)(1+\omega)(\nu)^{\varepsilon} t^{s+\varepsilon} + 2\omega(s+\varepsilon) t^{s+2\varepsilon}}{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right)} \tag{13}$$

After simplification we get

$$R_T(t) = \frac{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right) - (s+2\varepsilon)(1+\omega)(\nu)^{\varepsilon} t^{s+\varepsilon} + 2\omega(s+\varepsilon) t^{s+2\varepsilon}}{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right)} \tag{14}$$

After simplification we get

$$R_T(t) = \frac{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right) - t^{s+\varepsilon}\left((s+2\varepsilon)(1+\omega)(\nu)^{\varepsilon} - 2\omega(s+\varepsilon) t\right)}{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right)} \tag{15}$$

3.2 Conditional survival function

For an item survived for at least t_0 time (years, $t_0>0$), the probability that the item will survive additional t years is known as conditional survival function. In case of WTMI distribution the conditional survival function is given by

$$R_T(t \mid t_0) = P_r(T > t_0 + t \mid T > t_0)$$

$$R_T(t \mid t_0) = \frac{P_r(T > t_0 + t)}{P_r(T > t_0)}$$

$$R_T(t \mid t_0) = \frac{R_T(t_0 + t)}{R_T(t_0)}$$

$$R_T(t \mid t_0) = \frac{\dfrac{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right) - (t_0+t)^{s+\varepsilon}\left((s+2\varepsilon)(1+\omega)(\nu)^{\varepsilon} - 2\omega(s+\varepsilon)(t_0+t)\right)}{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}}{\dfrac{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right) - t^{s+\varepsilon}\left((s+2\varepsilon)(1+\omega)(\nu)^{\varepsilon} - 2\omega(s+\varepsilon) t\right)}{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}}$$

$$R_T(t \mid t_0) = \frac{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right) - (t_0+t)^{s+\varepsilon}\left((s+2\varepsilon)(1+\omega)(\nu)^{\varepsilon} - 2\omega(s+\varepsilon)(t_0+t)\right)}{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right) - t^{s+\varepsilon}\left((s+2\varepsilon)(1+\omega)(\nu)^{\varepsilon} - 2\omega(s+\varepsilon) t\right)} \tag{16}$$

3.3 Hazard function

The hazard function of WTMI distribution is given by

$$H_T(t) = \frac{g(t;\varepsilon,\nu,\omega,s)}{1-G_T(t)}$$

$$H_T(t) = \frac{\dfrac{(s+\varepsilon)(s+2\varepsilon)t^{s+\varepsilon-1}\left(1+\omega-2\omega\left(\dfrac{t}{\nu}\right)^{\varepsilon}\right)}{\nu^{s+\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}}{1-\dfrac{(s+2\varepsilon)(1+\omega)(\nu)^{\varepsilon}t^{s+\varepsilon}-2\omega(s+\varepsilon)t^{s+2\varepsilon}}{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}}$$

After simplification we have

$$H_T(t) = \frac{(s+\varepsilon)(s+2\varepsilon)(\nu)^{\varepsilon}\,t^{s+\varepsilon-1}\left(1+\omega-2\omega\left(\dfrac{t}{\nu}\right)^{\varepsilon}\right)}{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right)-t^{s+\varepsilon}\left((s+2\varepsilon)(1+\omega)(\nu)^{\varepsilon}-2\omega(s+\varepsilon)t^{\varepsilon}\right)} \tag{17}$$

3.4 Cumulative hazard function

The cumulative hazard function of WTMI distribution is given by

$${}_{C}H_T(t) = -\ln\left(R_T(t)\right)$$

$${}_{C}H_T(t) = -\ln\left(\frac{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right)-t^{s+\varepsilon}\left((s+2\varepsilon)(1+\omega)(\nu)^{\varepsilon}-2\omega(s+\varepsilon)t\right)}{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}\right) \tag{18}$$

Similarly, the Conditional Cumulative hazard function of WTMI distribution is given by

$${}_{C}H_T(t\mid t_0) = -\ln\left(R_T(t\mid t_0)\right)$$

$${}_{C}H_T(t\mid t_0) = -\ln\left(\frac{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right)-(t_0+t)^{s+\varepsilon}\left((s+2\varepsilon)(1+\omega)(\nu)^{\varepsilon}-2\omega(s+\varepsilon)(t_0+t)\right)}{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right)-t^{s+\varepsilon}\left((s+2\varepsilon)(1+\omega)(\nu)^{\varepsilon}-2\omega(s+\varepsilon)t\right)}\right)$$

3.5 Reverse Hazard function

The reverse hazard function of WTMI distribution is given by

$$H_r(t) = \frac{g(t;\varepsilon,\nu,\omega,s)}{G_T(t)}$$

$$H_r(t) = \frac{\dfrac{(s+\varepsilon)(s+2\varepsilon)t^{s+\varepsilon-1}\left(1+\omega-2\omega\left(\dfrac{t}{\nu}\right)^{\varepsilon}\right)}{\nu^{s+\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}}{\dfrac{(s+2\varepsilon)(1+\omega)(\nu)^{\varepsilon}t^{s+\varepsilon}-2\omega(s+\varepsilon)t^{s+2\varepsilon}}{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}}$$

After simplification we get

$$H_r(t) = \frac{(s+\varepsilon)(s+2\varepsilon)(\nu)^{\varepsilon}\, t^{s+\varepsilon-1}\left(1+\omega-2\omega\left(\frac{t}{\nu}\right)^{\varepsilon}\right)}{(s+2\varepsilon)(1+\omega)(\nu)^{\varepsilon} t^{s+\varepsilon} - 2\omega(s+\varepsilon)t^{s+2\varepsilon}} \tag{19}$$

3.6 Mills Ratio

The Mills ratio of WTMI distribution is given by

$$Mills\ ratio = \frac{1}{H_r(t)}$$

$$Mills\ ratio = \frac{(s+2\varepsilon)(1+\omega)(\nu)^{\varepsilon} t^{s+\varepsilon} - 2\omega(s+\varepsilon)t^{s+2\varepsilon}}{(s+\varepsilon)(s+2\varepsilon)(\nu)^{\varepsilon}\, t^{s+\varepsilon-1}\left(1+\omega-2\omega\left(\frac{t}{\nu}\right)^{\varepsilon}\right)} \tag{20}$$

3.7 Mean residual life

The mean residual life (MRL) in case of WTMI distribution is given by

$$MRL = \frac{1}{1-G_Z(z)}\int_z^{\nu} t\, g(t;\varepsilon,\nu,\omega,s)\,dt - z$$

$$MRL = \frac{1}{1-G_Z(z)}\int_z^{\nu} t\,\frac{(s+\varepsilon)(s+2\varepsilon)t^{s+\varepsilon-1}\left(1+\omega-2\omega\left(\frac{t}{\nu}\right)^{\varepsilon}\right)}{\nu^{s+\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}dt - z \tag{21}$$

$$MRL = \frac{1}{1-G_Z(z)}\int_z^{\nu} \frac{(s+\varepsilon)(s+2\varepsilon)t^{s+\varepsilon}\left(1+\omega-2\omega\left(\frac{t}{\nu}\right)^{\varepsilon}\right)}{\nu^{s+\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}dt - z \tag{22}$$

$$MRL = \frac{(s+\varepsilon)(s+2\varepsilon)}{\left(1-G_Z(z)\right)\nu^{s+\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}\left((1+\omega)\int_z^{\nu} t^{s+\varepsilon}dt - \frac{2\omega}{(\nu)^{\varepsilon}}\int_z^{\nu} t^{s+2\varepsilon}dt\right) - z \tag{23}$$

After simplification we get

$$MRL = \frac{(s+\varepsilon)(s+2\varepsilon)}{\left(1-G_Z(z)\right)\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right)(s+\varepsilon+1)(s+2\varepsilon+1)}\left\{\begin{array}{l}(\nu)^{s+2\varepsilon+1}\left((1-\omega)(s+\varepsilon+1)+(1+\omega)\varepsilon\right)+ \\ 2\omega(s+\varepsilon+1)z^{s+2\varepsilon+1} - (\nu)^{\varepsilon}(1+\omega)(s+2\varepsilon+1)z^{s+\varepsilon+1}\end{array}\right\} - z$$

4. Moments

The rth raw moment about origin of WTMI distributionis defined as

$$\mu_r' = \int_0^{\nu} z^r\ g(z;\varepsilon,\nu,\omega,s)\,dz$$

$$\mu_r' = \int_0^{\nu} z^r \frac{(s+\varepsilon)(s+2\varepsilon)z^{s+\varepsilon-1}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{s+\varepsilon}\left(s(1-\omega)+2\varepsilon\right)} dz$$

$$\mu_r' = \frac{(s+\varepsilon)(s+2\varepsilon)}{\nu^{s+\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}\left((1+\omega)\int_0^{\nu} z^{s+\varepsilon+r-1}dz - \frac{2\omega}{(\nu)^{\varepsilon}}\int_0^{\nu} z^{s+2\varepsilon+r-1}dz\right) \tag{24}$$

$$\mu_r' = \frac{(s+\varepsilon)(s+2\varepsilon)}{\nu^{s+\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}\left((1+\omega)\frac{(\nu)^{s+\varepsilon+r}}{s+\varepsilon+r} - \frac{2\omega(\nu)^{s+2\varepsilon+r}}{(\nu)^{\varepsilon}(s+2\varepsilon+r)}\right) \tag{25}$$

After simplification we get

$$\mu_r' = \frac{(s+\varepsilon)(s+2\varepsilon)(\nu)^r\left((1-\omega)(s+\varepsilon+r)+(1+\omega)\varepsilon\right)}{\left(s(1-\omega)+2\varepsilon\right)(s+\varepsilon+r)(s+2\varepsilon+r)} \tag{26}$$

Putting $r = 1, 2, 3, 4$ in (6) we get

$$\mu_1' = \frac{(s+\varepsilon)(s+2\varepsilon)(\nu)\left((1-\omega)(s+\varepsilon+1)+(1+\omega)\varepsilon\right)}{\left(s(1-\omega)+2\varepsilon\right)(s+\varepsilon+1)(s+2\varepsilon+1)} \tag{27}$$

$$\mu_2' = \frac{(s+\varepsilon)(s+2\varepsilon)(\nu)^2\left((1-\omega)(s+\varepsilon+2)+(1+\omega)\varepsilon\right)}{\left(s(1-\omega)+2\varepsilon\right)(s+\varepsilon+2)(s+2\varepsilon+2)} \tag{28}$$

$$\mu_3' = \frac{(s+\varepsilon)(s+2\varepsilon)(\nu)^3\left((1-\omega)(s+\varepsilon+3)+(1+\omega)\varepsilon\right)}{\left(s(1-\omega)+2\varepsilon\right)(s+\varepsilon+3)(s+2\varepsilon+3)} \tag{29}$$

$$\mu_4' = \frac{(s+\varepsilon)(s+2\varepsilon)(\nu)^4\left((1-\omega)(s+\varepsilon+4)+(1+\omega)\varepsilon\right)}{\left(s(1-\omega)+2\varepsilon\right)(s+\varepsilon+4)(s+2\varepsilon+4)} \tag{30}$$

The variance and coefficient of variance $(C.V)$ respectively are given by

$$\sigma^2 = \mu_2' - (\mu_1')^2$$

and

$$C.V = \frac{\sigma}{\mu_1'}; \qquad \text{where, } \sigma = \sqrt{\mu_2' - (\mu_1')^2}$$

5. Harmonic mean

The harmonic mean of WTMI distribution can be obtained as

$$Harmonic\,mean = E\left(\frac{1}{Z}\right)$$

$$Harmonic\,mean = \int_0^{\nu} \frac{1}{z} \frac{(s+\varepsilon)(s+2\varepsilon)z^{s+\varepsilon-1}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{s+\varepsilon}\left(s(1-\omega)+2\varepsilon\right)} dz$$

$$Harmonic\,mean = \int_0^{\nu} \frac{(s+\varepsilon)(s+2\varepsilon)z^{s+\varepsilon-2}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{s+\varepsilon}\left(s(1-\omega)+2\varepsilon\right)} dz \tag{31}$$

$$Harmonic\,mean = \frac{(s+\varepsilon)(s+2\varepsilon)}{\nu^{s+\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}\left((1+\omega)\int_0^{\nu} z^{s+\varepsilon-2}dz - \frac{2\omega}{(\nu)^{\varepsilon}}\int_0^{\nu} z^{s+2\varepsilon-2}dz\right) \tag{32}$$

After simplification we get

$$Harmonic\,mean = \frac{(s+\varepsilon)(s+2\varepsilon)\left((1-\omega)(s+\varepsilon-1)+(1+\omega)\right)}{\nu\left(s(1-\omega)+2\varepsilon\right)(s+\varepsilon-1)(s+2\varepsilon-1)} \tag{33}$$

6. MGF, CF and CGF

The MGF of WTMI distributionis equal to

$$M_Z(t) = E(e^{tz})$$

$$M_Z(t) = \int_0^{\nu} e^{tz}\, \frac{(s+\varepsilon)(s+2\varepsilon)z^{s+\varepsilon-1}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{s+\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}dz$$

$$M_Z(t) = \int_0^{\nu}\sum_{k=0}^{\infty}\frac{(tz)^k}{k!}\, \frac{(s+\varepsilon)(s+2\varepsilon)z^{s+\varepsilon-1}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{s+\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}dz \tag{34}$$

$$M_Z(t) = \sum_{k=0}^{\infty}\frac{(t)^k}{k!}\int_0^{\infty} z^k g\left(z;\varepsilon,\nu,\omega,s\right)dz$$

$$M_Z(t) = \sum_{k=0}^{\infty}\frac{(t)^k}{k!}\mu_k'$$

$$M_Z(t) = \sum_{k=0}^{\infty}\frac{(t)^k}{k!}\,\frac{(s+\varepsilon)(s+2\varepsilon)(\nu)^k\left((1-\omega)(s+\varepsilon+k)+(1+\omega)\,\varepsilon\right)}{\left(s(1-\omega)+2\varepsilon\right)(s+\varepsilon+k)(s+2\varepsilon+k)} \tag{35}$$

The CF of WTMI distribution can be obtained as

$$\phi_Z(t) = E(e^{itz})$$

$$\phi_Z(t) = \int_0^{\nu} e^{\imath tz}\, \frac{(s+\varepsilon)(s+2\varepsilon)z^{s+\varepsilon-1}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{s+\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}dz$$

$$\phi_Z(t) = \int_0^{\nu}\sum_{k=0}^{\infty}\frac{(\imath tz)^k}{k!}\, \frac{(s+\varepsilon)(s+2\varepsilon)z^{s+\varepsilon-1}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{s+\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}dz \tag{36}$$

$$\phi_Z(t) = \sum_{k=0}^{\infty}\frac{(\imath)^k(t)^k}{k!}\int_0^{\infty} z^k g\left(z;\varepsilon,\nu,\omega,s\right)dz \tag{37}$$

$$\phi_Z(t) = \sum_{k=0}^{\infty}\frac{(\imath t)^k}{k!}\mu_k'$$

$$\phi_Z(t)=\sum_{k=0}^{\infty}\frac{(\iota t)^k}{k!}\frac{(s+\varepsilon)(s+2\varepsilon)(\nu)^k\left((1-\omega)(s+\varepsilon+k)+(1+\omega)\,\varepsilon\right)}{\left(s(1-\omega)+2\varepsilon\right)(s+\varepsilon+k)(s+2\varepsilon+k)} \tag{38}$$

The CGF of WTMI distribution is given by

$$\kappa_Z(t)=\log\left(M_Z(t)\right)$$

$$\kappa_Z(t)=\log\left(\sum_{k=0}^{\infty}\frac{(t)^k}{k!}\frac{(s+\varepsilon)(s+2\varepsilon)(\nu)^k\left((1-\omega)(s+\varepsilon+k)+(1+\omega)\,\varepsilon\right)}{\left(s(1-\omega)+2\varepsilon\right)(s+\varepsilon+k)(s+2\varepsilon+k)}\right) \tag{39}$$

7. Estimation of Parameters

Let $z_1, z_2, z_3, \dots, z_n$ be a random sample of size n from WTMI distribution. Then The likelihood function is defined as the joint density of the random sample, which is given as

$$L(\varepsilon,\nu,\omega,s)=\prod_{l=1}^{n}g\left(z_l;\varepsilon,\nu,\omega,s\right)=\frac{(s+\varepsilon)(s+2\varepsilon)z^{s+\varepsilon-1}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{s+\varepsilon}\left(s(1-\omega)+2\varepsilon\right)} \tag{40}$$

$$L(\varepsilon,\nu,\omega,s)=\prod_{l=1}^{n}\frac{(s+\varepsilon)(s+2\varepsilon)z_l^{\,s+\varepsilon-1}\left(1+\omega-2\omega\left(\frac{z_l}{\nu}\right)^{\varepsilon}\right)}{\nu^{s+\varepsilon}\left(s(1-\omega)+2\varepsilon\right)} \tag{41}$$

$$L(\varepsilon,\nu,\omega,s)=\frac{(s+\varepsilon)^n(s+2\varepsilon)^n}{\nu^{n(s+\varepsilon)}\left(s(1-\omega)+2\varepsilon\right)^n}\left(\prod_{l=1}^{n}z_l^{\,s+\varepsilon-1}\right)\left(\prod_{l=1}^{n}\left(1+\omega-2\omega\left(\frac{z_l}{\nu}\right)^{\varepsilon}\right)\right) \tag{42}$$

Taking logarithm on both sides we get

$$\log L(\varepsilon,\nu,\omega,s)=n\log(s+\varepsilon)+n\log(s+2\varepsilon)-n(s+\varepsilon)\log(\nu)-n\log\left(s(1-\omega)+2\varepsilon\right)$$
$$+(s+\varepsilon-1)\sum_{l=1}^{n}\log z_l+\sum_{l=1}^{n}\log\left(1+\omega-\frac{2\omega}{(\nu)^{\varepsilon}}(z_l)^{\varepsilon}\right) \tag{43}$$

Differentiating equation (43) partially with respect to ε and equating to zero we get

$$\frac{n}{s+\varepsilon}+\frac{2n}{s+2\varepsilon}-n\log(\nu)-\frac{2n}{s(1-\omega)+2\varepsilon}+\sum_{l=1}^{n}\log z_l-\sum_{l=1}^{n}\frac{2\,\omega}{\left(1+\omega-\frac{2\omega}{(\nu)^{\varepsilon}}(z_l)^{\varepsilon}\right)}\left(\frac{z}{\nu}\right)^{\varepsilon}\log\left(\frac{z}{\nu}\right)=0 \tag{44}$$

Differentiating equation (43) partially with respect to ν and equating to zero we get

$$\sum_{l=1}^{n}\frac{2\,\omega\,(z_l)\,\varepsilon}{\left(1+\omega-\frac{2\omega}{(\nu)^{\varepsilon}}(z_l)^{\varepsilon}\right)(\nu)^{\varepsilon+1}}-\frac{n(s+\varepsilon)}{\nu}=0 \tag{45}$$

Differentiating equation (43) partially with respect to ω and equating to zero we get

$$\frac{n\,s}{s(1-\omega)+2\varepsilon}+\sum_{l=1}^{n}\frac{1-2\left(\frac{z_l}{\nu}\right)^{\varepsilon}}{\left(1+\omega-\frac{2\omega}{(\nu)^{\varepsilon}}(z_l)^{\varepsilon}\right)}=0 \tag{46}$$

Differentiating equation (43) partially with respect to s and equating to zero we get

$$\frac{n}{s+\varepsilon}+\frac{n}{s+2\varepsilon}-n\log(\nu)-\frac{n(1-\omega)}{s(1-\omega)+2\varepsilon}+\sum_{l=1}^{n}\log z_l=0 \tag{47}$$

On solving equation (44), (45), (46), and (47) simultaneously, we obtain the maximum likelihood estimators of parameters involved in the given distribution. However, the above system of non-linear equations cannot be evaluated directly. So, to get the maximum likelihood estimates for the distribution parameters, we have to solve these system of equations using Newton-Raphson method, Mathematica, or Secant method.

8. Distribution of ordered statistics

Let $z_1, z_2, z_3, \ldots, z_n$ be a random sample of size n from WTMI distribution. Then $Z_{(1)}, Z_{(2)}, Z_{(3)}, \ldots, Z_{(n)}$ be the ordered statistics associated with the given sample such that $Z_{(1)} \le Z_{(2)} \le Z_{(3)} \le \ldots \le Z_{(n)}$, Where

$Z_{(1)} = \min(z_1, z_2, z_3, \ldots, z_n)$ and $Z_{(n)} = \max(z_1, z_2, z_3, \ldots, z_n)$

The probability density function of k^{th} ordered statistics from the given distribution is given by

$$g_{Z_{(k)}}(z)=\frac{n!}{(k-1)!(n-k)!}g(z;\varepsilon,\nu,\omega,s)\left(G_Z(z)\right)^{k-1}\left(1-G_Z(z)\right)^{n-k}$$

$$g_{Z_{(k)}}(z)=\frac{n!}{(k-1)!(n-k)!}\frac{(s+\varepsilon)(s+2\varepsilon)z^{s+\varepsilon-1}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{s+\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}\left(\frac{(s+2\varepsilon)(1+\omega)(\nu)^{\varepsilon}z^{s+\varepsilon}-2\omega(s+\varepsilon)z^{s+2\varepsilon}}{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}\right)^{k-1}$$
$$\times\left(1-\frac{(s+2\varepsilon)(1+\omega)(\nu)^{\varepsilon}z^{s+\varepsilon}-2\omega(s+\varepsilon)z^{s+2\varepsilon}}{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}\right)^{n-k} \tag{48}$$

$$g_{Z_{(k)}}(z)=\frac{n!}{(k-1)!(n-k)!}\frac{(s+\varepsilon)(s+2\varepsilon)z^{s+\varepsilon-1}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{s+\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}\left(\frac{(s+2\varepsilon)(1+\omega)(\nu)^{\varepsilon}z^{s+\varepsilon}-2\omega(s+\varepsilon)z^{s+2\varepsilon}}{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}\right)^{k-1}$$
$$\times\left(\frac{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right)-z^{s+\varepsilon}\left((s+2\varepsilon)(1+\omega)(\nu)^{\varepsilon}-2\omega(s+\varepsilon)z^{\varepsilon}\right)}{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}\right)^{n-k} \tag{49}$$

And the corresponding cumulative distribution function of k^{th} ordered statistics is

$$G_{Z_{(k)}}(z)=\sum_{j=k}^{n}\left({}^{n}C_j\right)\left(G_Z(z)\right)^{j}\left(1-G_Z(z)\right)^{n-j}$$

$$G_{Z_{(k)}}(z)=\sum_{j=k}^{n}\left(\begin{array}{l}\left({}^{n}C_j\right)\left(\frac{(s+2\varepsilon)(1+\omega)(\nu)^{\varepsilon}z^{s+\varepsilon}-2\omega(s+\varepsilon)z^{s+2\varepsilon}}{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}\right)^{j}\\ \times\left(\frac{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right)-z^{s+\varepsilon}\left((s+2\varepsilon)(1+\omega)(\nu)^{\varepsilon}-2\omega(s+\varepsilon)z^{\varepsilon}\right)}{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}\right)^{n-j}\end{array}\right) \tag{50}$$

On substituting $k=1,n$ in equation (49) we get the probability density functions of smallest and highest ordered statistics respectively and are given as

$$g_{Z_{(1)}}(z)=n\frac{(s+\varepsilon)(s+2\varepsilon)z^{s+\varepsilon-1}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{s+\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}\times\left(\frac{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right)-z^{s+\varepsilon}\left((s+2\varepsilon)(1+\omega)(\nu)^{\varepsilon}-2\omega(s+\varepsilon)z^{\varepsilon}\right)}{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}\right)^{n-1} \tag{51}$$

and

$$g_{Z_{(n)}}(z)=\frac{(s+\varepsilon)(s+2\varepsilon)z^{s+\varepsilon-1}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{s+\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}\left(\frac{(s+2\varepsilon)(1+\omega)(\nu)^{\varepsilon}z^{s+\varepsilon}-2\omega(s+\varepsilon)z^{s+2\varepsilon}}{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}\right)^{n-1} \tag{52}$$

Their corresponding cumulative density functions are obtained on substituting $k=1,n$ in equation (50) and are given by

$$G_{Z_{(1)}}(z)=1-\left(1-\left(\frac{(s+2\varepsilon)(1+\omega)(\nu)^{\varepsilon}z^{s+\varepsilon}-2\omega(s+\varepsilon)z^{s+2\varepsilon}}{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}\right)\right)^{n} \tag{53}$$

And

$$G_{Z_{(n)}}(z)=\left(\frac{(s+2\varepsilon)(1+\omega)(\nu)^{\varepsilon}z^{s+\varepsilon}-2\omega(s+\varepsilon)z^{s+2\varepsilon}}{\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}\right)^{n} \tag{54}$$

9. Likelihood ratio test

In the context of probability distributions, the likelihood ratio test is often employed to compare whether two distributions adequately describe the observed data or not. Suppose $z_1, z_2, z_3, ..., z_n$ be a random sample of size n from WTMI distribution.. To test the hypothesis

$H_0 : g(z)=g(z;\varepsilon,\nu,\omega,)$ against $H_1 : g(z)=g(z;\varepsilon,\nu,\omega,s)$

The likelihood ratio test is defined as

$$\ell=\prod_{k=1}^{n}\frac{g(z_k;\varepsilon,\nu,\omega,s)}{f(z_k;\varepsilon,\nu,\omega)}$$

$$\ell=\prod_{k=1}^{n}\frac{\dfrac{(s+\varepsilon)(s+2\varepsilon)z_k^{\,s+\varepsilon-1}\left(1+\omega-2\omega\left(\frac{z_k}{\nu}\right)^{\varepsilon}\right)}{\nu^{s+\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}}{\dfrac{\varepsilon}{\nu^{\varepsilon}}z_k^{\,\varepsilon-1}\left(1+\omega-2\omega\left(\frac{z_k}{\nu}\right)^{\varepsilon}\right)} \tag{55}$$

$$\ell=\left(\frac{(s+\varepsilon)(s+2\varepsilon)}{\varepsilon\nu^{s}\left(s(1-\omega)+2\varepsilon\right)}\right)^{n}\prod_{k=1}^{n}z_k^{\,s} \tag{56}$$

So, we reject null hypothesis at α level of significance if $\ell>\mathrm{K}^*$ such that $P(\ell>\mathrm{K}^*)=\alpha$, where K^* is the critical value at α level of significance of the given test statistics. That is,

$$\left(\frac{(s+\varepsilon)(s+2\varepsilon)}{\varepsilon\nu^{s}\left(s(1-\omega)+2\varepsilon\right)}\right)^{n}\prod_{k=1}^{n}z_k^{\,s}>\mathrm{K}^* \tag{57}$$

$$\prod_{k=1}^{n}z_k^{\,s}>\mathrm{K}^*\left(\frac{\varepsilon\nu^{s}\left(s(1-\omega)+2\varepsilon\right)}{(s+\varepsilon)(s+2\varepsilon)}\right)^{n} \tag{58}$$

For large sample size n, $-2\log(\ell)$ is distributed as Chi-square distribution with one degree of freedom. Also p-value is calculated from the chi-square distribution. On the basis of p-value, we reject the null hypothesis when the p-value is less than level of significance.

10. Entropy measures

10.1 Renyi entropy and Tsallis entropy

By definition, the Renyi entropy is given by

$$R(\tau)=\frac{1}{1-\tau}\log\left(\int_0^{\nu}\left(g\left(z_k;\varepsilon,\nu,\omega,s\right)\right)^{\tau}dz\right) \tag{59}$$

$$R(\tau)=\frac{1}{1-\tau}\log\left(\int_0^{\nu}\left(\frac{(s+\varepsilon)(s+2\varepsilon)z^{s+\varepsilon-1}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{s+\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}\right)^{\tau}dz\right) \tag{60}$$

$$R(\tau)=\frac{1}{1-\tau}\log\left(\left(\frac{(s+\varepsilon)(s+2\varepsilon)}{\nu^{s+\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}\right)^{\tau}\int_0^{\nu}\left(z^{\tau(s+\varepsilon-1)}\sum_{k=0}^{\tau}({}^{\tau}C_k)(-1)^k(1+\omega)^{\tau-k}\left(2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)^k\right)dz\right)$$

$$R(\tau)=\frac{1}{1-\tau}\log\left(\left(\frac{(s+\varepsilon)(s+2\varepsilon)}{\nu^{s+\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}\right)^{\tau}\sum_{k=0}^{\tau}({}^{\tau}C_k)(-1)^k(1+\omega)^{\tau-k}\left(\frac{2\omega}{(\nu)^{\varepsilon}}\right)^k\int_0^{\nu}z^{\varepsilon k+\tau(s+\varepsilon-1)}dz\right) \tag{61}$$

After simplification we get

$$R(\tau)=\frac{1}{1-\tau}\log\left(\frac{\left((s+\varepsilon)(s+2\varepsilon)\right)^{\tau}(\nu)^{1-\tau}}{\left(s(1-\omega)+2\varepsilon\right)^{\tau}}\sum_{k=0}^{\tau}\frac{({}^{\tau}C_k)(-1)^k(1+\omega)^{\tau-k}(2\omega)^k}{\varepsilon k+\tau(s+\varepsilon-1)+1}\right) \tag{62}$$

Similarly, the Tsallis entropy associated with the given distribution is given by

$$T_s(\xi)=\frac{1}{\xi-1}\left(1-\int_0^{\nu}\left(g\left(z_k;\varepsilon,\nu,\omega,s\right)\right)^{\xi}dz\right)$$

$$T_s(\xi)=\frac{1}{\xi-1}\left(1-\frac{\left((s+\varepsilon)(s+2\varepsilon)\right)^{\xi}(\nu)^{1-\xi}}{\left(s(1-\omega)+2\varepsilon\right)^{\xi}}\sum_{k=0}^{\xi}\frac{({}^{\xi}C_k)(-1)^k(1+\omega)^{\xi-k}(2\omega)^k}{\varepsilon k+\xi(s+\varepsilon-1)+1}\right) \tag{63}$$

11. Bonferroni and Lorenz curves

The Bonferroni curve of the given distribution is given by

$$\Psi(\zeta)=\frac{1}{\zeta\,\mu_1'}\int_0^{\varphi}z\,g\left(z;\varepsilon,\nu,\omega,s\right)dz$$

Where $\mu_1'=\dfrac{(s+\varepsilon)(s+2\varepsilon)(\nu)\left((1-\omega)(s+\varepsilon+1)+(1+\omega)\,\varepsilon\right)}{\left(s(1-\omega)+2\varepsilon\right)(s+\varepsilon+1)(s+2\varepsilon+1)}$ and $\varphi=F^{-1}(\zeta)$

$$\Psi(\zeta)=\frac{1}{\zeta\,\mu_1'}\int_0^{\varphi}z\,\frac{(s+\varepsilon)(s+2\varepsilon)z^{s+\varepsilon-1}\left(1+\omega-2\omega\left(\frac{z}{\nu}\right)^{\varepsilon}\right)}{\nu^{s+\varepsilon}\left(s(1-\omega)+2\varepsilon\right)} \tag{64}$$

$$\Psi(\zeta)=\frac{1}{\zeta\ \mu_1'}\frac{(s+\varepsilon)(s+2\varepsilon)}{\nu^{s+\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}\left((1+\omega)\int_0^{\varphi} z^{s+\varepsilon}dz-\frac{2\omega}{(\nu)^{\varepsilon}}\int_0^{\varphi} z^{s+2\varepsilon}dz\right) \tag{65}$$

$$\Psi(\zeta)=\frac{1}{\zeta\ \mu_1'}\frac{(s+\varepsilon)(s+2\varepsilon)}{\nu^{s+\varepsilon}\left(s(1-\omega)+2\varepsilon\right)}\left((1+\omega)\left(\frac{(\varphi)^{s+\varepsilon+1}}{s+\varepsilon+1}\right)-\frac{2\omega}{(\nu)^{\varepsilon}}\left(\frac{(\varphi)^{s+2\varepsilon+1}}{s+2\varepsilon+1}\right)\right) \tag{66}$$

After simplification we get

$$\Psi(\zeta)=\frac{(s+\varepsilon)(s+2\varepsilon)(\varphi)^{s+\varepsilon+1}\left((1+\omega)(s+2\varepsilon+1)(\nu)^{\varepsilon}-2\omega(s+\varepsilon+1)(\varphi)^{\varepsilon}\right)}{\zeta\ \mu_1'\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right)(s+\varepsilon+1)(s+2\varepsilon+1)} \tag{67}$$

Also, the Lorenz curve of the given distribution is given by

$$\Phi(\zeta)=\zeta\ \Psi(\zeta)$$

$$\Phi(\zeta)=\zeta\left(\frac{(s+\varepsilon)(s+2\varepsilon)(\varphi)^{s+\varepsilon+1}\left((1+\omega)(s+2\varepsilon+1)(\nu)^{\varepsilon}-2\omega(s+\varepsilon+1)(\varphi)^{\varepsilon}\right)}{\zeta\ \mu_1'\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right)(s+\varepsilon+1)(s+2\varepsilon+1)}\right) \tag{68}$$

$$\Phi(\zeta)=\frac{(s+\varepsilon)(s+2\varepsilon)(\varphi)^{s+\varepsilon+1}\left((1+\omega)(s+2\varepsilon+1)(\nu)^{\varepsilon}-2\omega(s+\varepsilon+1)(\varphi)^{\varepsilon}\right)}{\mu_1'\nu^{s+2\varepsilon}\left(s(1-\omega)+2\varepsilon\right)(s+\varepsilon+1)(s+2\varepsilon+1)} \tag{69}$$

12. Conclusion

In this research paper, we have explored an innovative extension of the Transmuted Mukherjee-Islam distribution, known as the weighted Transmuted Mukherjee-Islam distribution. This distribution is formulated by incorporating a weighted model, utilizing the three-parameter Transmuted Mukherjee-Islam distribution as the base distribution. We thoroughly examine and discuss the newly introduced weighted Transmuted Mukherjee-Islam distribution, exploring its mathematical and statistical properties. The parameters of this novel distribution are determined through the application of maximum likelihood estimation techniques.

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