A COMPREHENSIVE ANALYSIS OF JUCHEZ DISTRIBUTION: EXPLORING STRUCTURAL PROPERTIES AND APPLICATIONS

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Abstract

This article introduces an innovative extension of the Juchez distribution, referred to as the length-biased Juchez distribution. This distribution, a specific instance of the broader weighted distribution, is thoroughly explored in terms of mathematical and statistical properties. Parameter estimation is accomplished through the application of maximum likelihood estimation techniques. To highlight the practical significance of this new distribution, a comprehensive analysis is conducted using two real-life lifetime datasets. The findings underscore the relevance and applicability of the proposed distribution in modeling and analyzing diverse datasets.

Keywords: Length biased distribution, Juchez distribution, Order statistics, Survival analysis, Maximum likelihood estimation.

1. Introduction

In statistics, the theory of weighted probability distributions has retained a reputed and prominent place because it provides a new shape to the existing classical distribution by introducing an additional parameter to it. This additional parameter brings more superiority and flexibility to a class of distribution functions and it should be very significant from data analysis point of view. This extra parameter can be introduced through various techniques. One of such technique is of weighted technique. The idea of weighted distribution was propounded firstly by Fisher [10] to study how the method of ascertainment can influence the form of distribution of recorded observation. Later, Rao [16] developed this concept in a collective way in association with modeling statistical data when usual practice of using standard distributions was found to be unsuitable. The theory of weighted distributions plays a dominant and tremendous practical role in probability, statistics and mathematics. The concept of weighted distribution provides an integrative conceptualization for model stipulation and data representation problems. The weighted distributions also provide a collective approach for correction of biases that exists in unequally weighted sample data. The weighted distribution reduces to length biased distribution when the weight function considers only the length of units of interest. Length biased distribution have been applied in various biomedical areas such as survival analysis, family history, reliability analysis, clinical trials, intermediate events and population studies were a proper sampling frame is absent. In such situation items are sampled at a rate proportional to their lengths so that the larger value could be sampled with higher probability.

Many authors have described and developed some important length biased probability models along with their illustrations in various fields. Al-Omari and Alanzi [6] presented inverse length biased Maxwell distribution and obtain its statistical inference with an application. Andure (Yawale) and Ade [7] presented the new length biased Hamza distribution with statistical properties and applications. Saghir, Tazeem and Ahmad [24] discussed on the length biased weighted exponentiated inverted weibull distribution and introduce its necessary properties. Mustafa and Khan [12] developed the length biased power hazard rate distribution with some properties and applications. Abdullah et al. [2] presented the size biased Lomax distribution with applications. Alidamat and Al-Omari [5] described the length biased two parameter Mirra distribution with application to engineering data. Ahajeeth et al. [4] proposed the area biased Amarendra distribution with its application to model lifetime data. Sanat [25] derived the beta-length biased Pareto distribution and its properties. Ganaie and Rajagopalan [11] presented the length biased power quasi Lindley distribution with properties and applications of lifetime data. Abd-Elfattah et al. [1] studied the length biased Burr-XII distribution with properties and application. Reyad et al. [22] proposed the length biased weighted Erlang distribution. Rather and Subramanian [16] discussed on length biased Sushila distribution with properties and applications. Nanuwong and Bodhisuwan [14] executed the length biased beta-Pareto distribution with its structural properties and application. Saghir and Khadim [23] presented the mathematical properties of length biased weighted Maxwell distribution. Rather et al [21] enriched the research by offering a comprehensive overview, perspectives, and characterizations of a new size biased Ailamujia distribution with applications in engineering and medical science which shows more flexibility than classical distributions. Rather and Subramanian [19] discussed the characterization and estimation of length biased weighted generalized uniform distribution. Rather and Subramanian [18], obtained length biased sushila distribution with properties and its applications. Shenbagaraja et al. [26], discussed length biased Garima distribution. Rather and Subramanian [20], conducted a thorough examination of the length biased erlang-truncated exponential distribution with life time data. Subramanian and Rather [27], obtained a new extension of Shanker distribution with real life data. Rather and Ozel [17], explored a new length biased power lindley distribution with properties and its applications. Recently, Mustafa and Khan [13] developed the length biased powered inverse Rayleigh distribution with applications.

Juchez probability distribution is a recently developed one parametric continuous lifetime distribution studied by Echebiri and Mbegbu [9]. Some of statistical features including median, mode, mean, moments, coefficient of variation, skewness, kurtosis, mean residual life function, hazard function, bonferroni and lorenz curves, order statistics, stochastic ordering and Renyi entropy have been presented. Furthermore, its parameter has been estimated by using the maximum likelihood estimation.

2. Length Biased Juchez Distribution

The probability density function of Juchez distribution is given by

$$f(x;\theta)=\frac{\theta^4}{\theta^3+\theta^2+6}\left(1+x+x^3\right)e^{-\theta x};\ x>0,\ \theta>0 \tag{1}$$

and the cumulative distribution function of Juchez distribution is given by

$$F(x;\theta)=1-\left(1+\frac{\theta x[\theta^2+\theta^2x^2+3\theta x+6]}{\theta^3+\theta^2+6}\right)e^{-\theta x};\quad x>0,\ \theta>0 \tag{2}$$

Let X be the random variable represents non-negative condition with probability density function $f(x)$. Let its non-negative weight function be $w(x)$, then the probability density function of weighted random variable X_w is given by

$$f_w(x)=\frac{w(x)f(x)}{E(w(x))},\quad x>0.$$

When the non - negative weight function be $w(x)$ and $E(w(x))=\int w(x)f(x)dx<\infty$.

Depending upon the different weighted functions $w(x)$ obviously when $w(x) = x^c$, proposed distribution is known as weighted distribution. In this paper, we have to be considered the length biased version of Juchez distribution termed as length biased Juchez distribution. So the weight function considered at $w(x) = x$ by taking weight parameter c is 1 in weights x^c resulting distribution is known as length biased distribution and its probability density function given by

$$f_l(x)=\frac{x\,f(x)}{E(x)} \tag{3}$$

Where $E(x)=\int_0^\infty x\,f(x)dx$

$$E(x)=\frac{\left(\theta^3+2\theta^2+24\right)}{\theta(\theta^3+\theta^2+6)} \tag{4}$$

Now by using the equations (1) and (4) in equation (3), we will get required probability density function of length biased Juchez distribution as

$$f_l(x)=\frac{x\theta^5}{\left(\theta^3+2\theta^2+24\right)}\left(1+x+x^3\right)e^{-\theta x} \tag{5}$$

and the cumulative distribution function of length biased Juchez distribution can be determined as

$$F_l(x)=\int_0^x f_l(x)dx$$

$$=\int_0^x \frac{x\theta^5}{\left(\theta^3+2\theta^2+24\right)}\left(1+x+x^3\right)e^{-\theta x}dx$$

$$=\frac{1}{\left(\theta^3+2\theta^2+24\right)}\int_0^x x\theta^5\left(1+x+x^3\right)e^{-\theta x}dx$$

$$= \frac{1}{\left(\theta^3 + 2\theta^2 + 24\right)} \left(\theta^5 \int_0^x x e^{-\theta x} dx + \theta^5 \int_0^x x^2 e^{-\theta x} dx + \theta^5 \int_0^x x^4 e^{-\theta x} dx \right) \tag{6}$$

Put $\theta x = t \Rightarrow \theta dx = dt \Rightarrow dx = \frac{dt}{\theta}$, Also $x = \frac{t}{\theta}$

When $x \to x,\ t \to \theta x$ and when $x \to 0,\ t \to 0$

After the simplification of equation (6), we will determine cumulative distribution function of length biased Juchez distribution as

$$F_l(x) = \frac{1}{\left(\theta^3 + 2\theta^2 + 24\right)} \left(\theta^3 \gamma(2, \theta x) + \theta^2 \gamma(3, \theta x) + \gamma(5, \theta x) \right) \tag{7}$$

Figure 1 and figure 2, shows the graphical representation of the pdf and cdf plot and has been R-core version [15] for this.

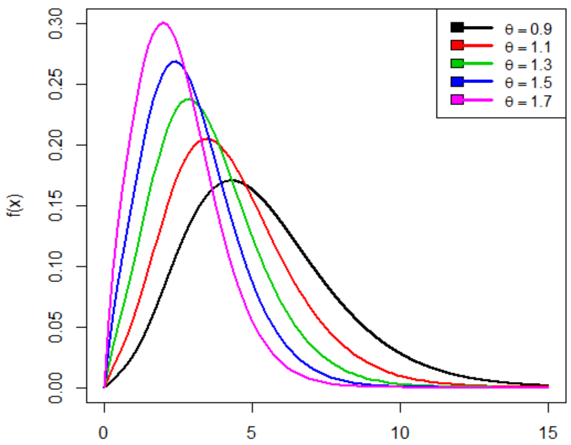

Figure 1: *Pdf plot of length biased Juchez distribution*

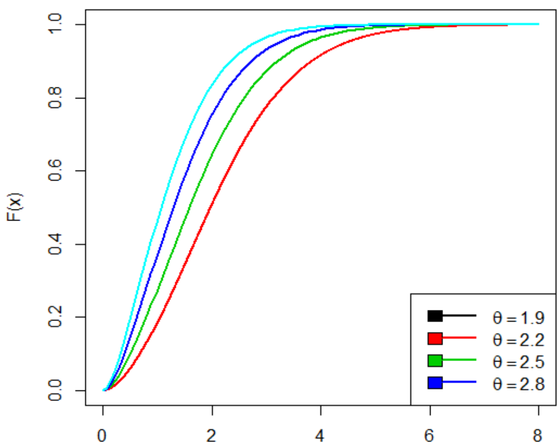

Figure 2: *Cdf plot of length biased Juchez distribution*

3. Survival Analysis

In this section, we will derive the survival function, hazard rate function, reverse hazard rate function and Mills ratio of the proposed length biased Juchez distribution. The survival or reliability function of length biased Juchez distribution can be obtained as

$$S(x) = 1 - F_l(x)$$

$$= 1 - \frac{1}{\left(\theta^3 + 2\theta^2 + 24\right)} \left(\theta^3 \gamma(2, \theta x) + \theta^2 \gamma(3, \theta x) + \gamma(5, \theta x) \right) \tag{8}$$

The hazard function is also known as hazard rate or failure rate or force of mortality and is given by

$$h(x) = \frac{f_l(x)}{1 - F_l(x)}$$

$$= \frac{x\theta^5 (1 + x + x^3)\, e^{-\theta x}}{(\theta^3 + 2\theta^2 + 24) - (\theta^3 \gamma(2, \theta x) + \theta^2 \gamma(3, \theta x) + \gamma(5, \theta x))} \tag{9}$$

The reverse hazard rate function is given by

$$h_r(x)=\frac{f_l(x)}{F_l(x)}$$

$$=\frac{x\theta^5(1+x+x^3)\,e^{-\theta x}}{(\theta^3\gamma(2,\,\theta x)+\theta^2\gamma(3,\,\theta x)+\gamma(5,\,\theta x))} \tag{10}$$

The Mills Ratio is given by

$$M.R=\frac{1}{h_r(x)}=\frac{(\theta^3\gamma(2,\,\theta x)+\theta^2\gamma(3,\,\theta x)+\gamma(5,\,\theta x))}{x\theta^5(1+x+x^3)\,e^{-\theta x}} \tag{11}$$

Figure 3 and figure 4, shows the graphical representation of the survival function and cdf plot.

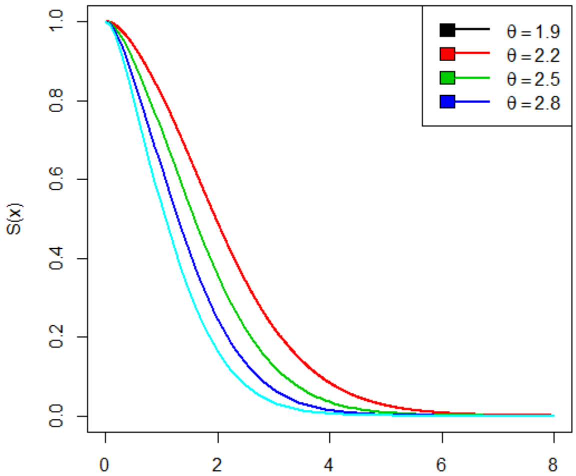

Figure 3: *Survival function of length biased Juchez distribution*

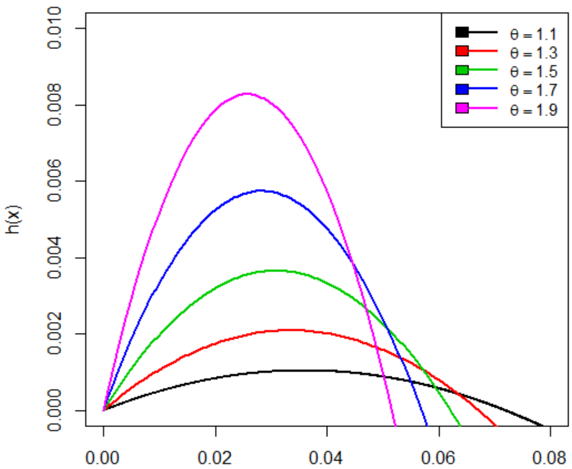

Figure 4: *Hazard function of length biased Juchez distribution*

4. Order Statistics

Order statistics is a very significant concept in statistical sciences and has wide range of applications in modeling auctions, insurance policies, car races, optimizing production processes and estimating parameters of distributions. Consider X(1), X(2),…, X(n) be the order statistics of a random sample X1, X2,…, Xn from a continuous population with probability density function fx (x) and cumulative distribution function FX(x), then the probability density function of rth order statistics X(r) is given by

$$f_{x(r)}(x)=\frac{n!}{(r-1)!(n-r)!}\,f_X(x)\left(F_X(x)\right)^{r-1}\left(1-F_X(x)\right)^{n-r} \tag{12}$$

By using the equations (5) and (7) in equation (12), we will obtain the probability density function of r^{th} order statistics $X_{(r)}$ of length biased Juchez distribution as

$$f_{x(r)}(x)=\frac{n!}{(r-1)!(n-r)!}\left(\frac{x\theta^5}{\left(\theta^3+2\theta^2+24\right)}\left(1+x+x^3\right)e^{-\theta x}\right)$$
$$\times\left(\frac{1}{\left(\theta^3+2\theta^2+24\right)}\left(\theta^3\gamma(2,\,\theta x)+\theta^2\gamma(3,\,\theta x)+\gamma(5,\,\theta x)\right)\right)^{r-1}$$
$$\times\left(1-\frac{1}{\left(\theta^3+2\theta^2+24\right)}\left(\theta^3\gamma(2,\,\theta x)+\theta^2\gamma(3,\,\theta x)+\gamma(5,\,\theta x)\right)\right)^{n-r} \tag{13}$$

Therefore, the probability density function of higher order statistic $X_{(n)}$ of length biased Juchez distribution can be obtained as

$$f_{x(n)}(x)=\frac{n\,x\,\theta^5}{\left(\theta^3+2\theta^2+24\right)}\left(1+x+x^3\right)e^{-\theta x}$$
$$\times\left(\frac{1}{\left(\theta^3+2\theta^2+24\right)}\left(\theta^3\gamma(2,\,\theta x)+\theta^2\gamma(3,\,\theta x)+\gamma(5,\,\theta x)\right)\right)^{n-1} \tag{14}$$

and the probability density function of first order statistic $X_{(1)}$ of length biased Juchez distribution can be obtained as

$$f_{x(1)}(x)=\frac{n\,x\,\theta^5}{\left(\theta^3+2\theta^2+24\right)}\left(1+x+x^3\right)e^{-\theta x}$$
$$\times\left(1-\frac{1}{\left(\theta^3+2\theta^2+24\right)}\left(\theta^3\gamma(2,\,\theta x)+\theta^2\gamma(3,\,\theta x)+\gamma(5,\,\theta x)\right)\right)^{n-1} \tag{15}$$

5. Likelihood Ratio Test

Let the random sample X1, X2,…., Xn of size n drawn from the length biased Juchez distribution. To analyze its significance, the hypothesis is to be tested

$$H_o : f(x)=f(x;\theta) \quad \text{against} \quad H_1 : f(x)=f_l(x;\theta) \tag{16}$$

In order to determine, whether the random sample of size n comes from Juchez distribution or length biased Juchez distribution, the following test statistic is employed

$$\Delta=\frac{L_1}{L_o}=\prod_{i=1}^{n}\frac{f_l(x;\theta)}{f(x;\theta)} \tag{17}$$

$$=\frac{L_1}{L_o}=\prod_{i=1}^{n}\left(\frac{x_i\,\theta\left(\theta^3+\theta^2+6\right)}{\left(\theta^3+2\theta^2+24\right)}\right) \tag{18}$$

$$= \frac{L_1}{L_o} = \left(\frac{\theta(\theta^3+\theta^2+6)}{\left(\theta^3+2\theta^2+24\right)} \right)^n \prod_{i=1}^{n} x_i \tag{19}$$

We should refuse to retain the null hypothesis, if

$$\Delta = \left(\frac{\theta(\theta^3+\theta^2+6)}{\left(\theta^3+2\theta^2+24\right)} \right)^n \prod_{i=1}^{n} x_i > k \tag{20}$$

Equivalently, we should also refuse to retain the null hypothesis, where

$$\Delta = \prod_{i=1}^{n} x_i > k \left(\frac{\left(\theta^3+2\theta^2+24\right)}{\theta(\theta^3+\theta^2+6)} \right)^n \tag{21}$$

$$\Delta = \prod_{i=1}^{n} x_i > k^*, \text{where } \; k^* = k \left(\frac{\theta^3+2\theta^2+24}{\theta(\theta^3+\theta^2+6)} \right)^n \tag{22}$$

such that $p\left(\Delta > k^*\right) = \alpha$, where α is the level of significance.

6. Structural Properties

In this section, we will derive several statistical properties of length biased Juchez distribution which include moments, harmonic mean, moment generating function and characteristic function.

6.1 Moments

Let X be the random variable following length biased Juchez distribution with parameter θ, then the r^{th} order moment $E(X^r)$ of introduced distribution can be obtained as

$$E(X^r) = \mu_r' = \int_0^\infty x^r f_l(x) dx \tag{23}$$

$$= \mu_r' = \int_0^\infty x^r \frac{x\theta^5}{\left(\theta^3+2\theta^2+24\right)} \left(1+x+x^3\right) e^{-\theta x} dx \tag{24}$$

$$= \mu_r' = \int_0^\infty \frac{x^{r+1}\theta^5}{\left(\theta^3+2\theta^2+24\right)} \left(1+x+x^3\right) e^{-\theta x} dx \tag{25}$$

$$= \mu_r' = \frac{\theta^5}{\left(\theta^3+2\theta^2+24\right)} \int_0^\infty x^{r+1} \left(1+x+x^3\right) e^{-\theta x} dx \tag{26}$$

$$= \mu_r' = \frac{\theta^5}{\left(\theta^3+2\theta^2+24\right)} \left(\int_0^\infty x^{(r+2)-1} e^{-\theta x} dx + \int_0^\infty x^{(r+3)-1} e^{-\theta x} dx + \int_0^\infty x^{(r+5)-1} e^{-\theta x} dx \right) \tag{27}$$

After the simplification of above equation, we obtain

$$E(X^r) = \mu_r' = \frac{\theta^3\Gamma(r+2)+\theta^2\Gamma(r+3)+\Gamma(r+5)}{\theta^r(\theta^3+2\theta^2+24)} \tag{28}$$

Now putting r = 1, 2, 3 and 4 in equation (28), we will obtain the first four moments of length biased Juchez distribution as

$$E(X)=\mu_1'=\frac{2\theta^3+6\theta^2+120}{\theta(\theta^3+2\theta^2+24)} \tag{29}$$

$$E(X^2)=\mu_2'=\frac{6\theta^3+24\theta^2+720}{\theta^2(\theta^3+2\theta^2+24)} \tag{30}$$

$$E(X^3)=\mu_3'=\frac{24\theta^3+120\theta^2+5040}{\theta^3(\theta^3+2\theta^2+24)} \tag{31}$$

$$E(X^4)=\mu_4'=\frac{120\theta^3+720\theta^2+40320}{\theta^4(\theta^3+2\theta^2+24)} \tag{32}$$

$$\text{Variance}=\frac{6\theta^3+24\theta^2+720}{\theta^2(\theta^3+2\theta^2+24)}-\left(\frac{2\theta^3+6\theta^2+120}{\theta(\theta^3+2\theta^2+24)}\right)^2 \tag{33}$$

$$S.D(\sigma)=\sqrt{\left(\frac{6\theta^3+24\theta^2+720}{\theta^2(\theta^3+2\theta^2+24)}-\left(\frac{2\theta^3+6\theta^2+120}{\theta(\theta^3+2\theta^2+24)}\right)^2\right)} \tag{34}$$

6.2 Harmonic mean

The harmonic mean for the executed length biased Juchez distribution can be determined as

$$H.M = E\left(\frac{1}{x}\right)=\int_0^\infty \frac{1}{x} f_l(x)dx \tag{35}$$

$$=\int_0^\infty \frac{1}{x}\frac{x\theta^5}{\left(\theta^3+2\theta^2+24\right)}\left(1+x+x^3\right)e^{-\theta x}dx \tag{36}$$

$$=\int_0^\infty \frac{\theta^5}{\left(\theta^3+2\theta^2+24\right)}\left(1+x+x^3\right)e^{-\theta x}dx \tag{37}$$

$$=\frac{\theta^5}{\left(\theta^3+2\theta^2+24\right)}\int_0^\infty \left(1+x+x^3\right)e^{-\theta x}dx \tag{38}$$

$$=\frac{\theta^5}{\left(\theta^3+2\theta^2+24\right)}\left(\int_0^\infty x^{(2)-2}e^{-\theta x}dx + \int_0^\infty x^{(2)-1}e^{-\theta x}dx + \int_0^\infty x^{(4)-1}e^{-\theta x}dx\right) \tag{39}$$

After the simplification of above equation, we obtain

$$H.M=\frac{\theta(\theta^2+\theta^2+6)}{\left(\theta^3+2\theta^2+24\right)} \tag{40}$$

6.3 Moment generating function and characteristic function

Let X be the random variable following length biased Juchez distribution with parameter θ, then the moment generating function of proposed distribution can be obtained as

$$M_X(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx} f_l(x)dx \tag{41}$$

Using Taylor's series, we obtain

$$M_X(t) = \int_0^\infty \left(1 + tx + \frac{(tx)^2}{2!} + \ldots\right) f_l(x)dx \tag{42}$$

$$= \int_0^\infty \sum_{j=0}^\infty \frac{t^j}{j!} x^j f_l(x)dx \tag{43}$$

$$= \sum_{j=0}^\infty \frac{t^j}{j!} \mu_j'$$

$$= \sum_{j=0}^\infty \frac{t^j}{j!} \left(\frac{\theta^3\Gamma(j+2) + \theta^2\Gamma(j+3) + \Gamma(j+5)}{\theta^j(\theta^3 + 2\theta^2 + 24)}\right) \tag{44}$$

$$= \frac{1}{(\theta^3 + 2\theta^2 + 24)} \sum_{j=0}^\infty \frac{t^j}{j!\theta^j} \left(\theta^3\Gamma(j+2) + \theta^2\Gamma(j+3) + \Gamma(j+5)\right) \tag{45}$$

Similarly, the characteristic function of length biased Juchez distribution can be obtained as

$$\varphi_x(t) = M_X(it)$$

$$M_X(it) = \frac{1}{(\theta^3 + 2\theta^2 + 24)} \sum_{j=0}^\infty \frac{it^j}{j!\theta^j} \left(\theta^3\Gamma(j+2) + \theta^2\Gamma(j+3) + \Gamma(j+5)\right) \tag{46}$$

7. Bonferroni and Lorenz Curves

The bonferroni and Lorenz curves also termed as income distribution curves or classical curves are frequently being applied to measure the distribution of inequality in income or poverty. The bonferroni and Lorenz curves can be executed as

$$B(p) = \frac{1}{p\mu_1'} \int_0^q x\, f_l(x)dx \tag{47}$$

$$L(p) = pB(p) = \frac{1}{\mu_1'} \int_0^q x\, f_l(x)dx \quad \text{and} \quad q = F^{-1}(p)$$

$$\text{where} \quad \mu_1' = \frac{\left(2\theta^3 + 6\theta^2 + 120\right)}{\theta(\theta^3 + 2\theta^2 + 24)}$$

$$B(p) = \frac{\theta(\theta^3 + 2\theta^2 + 24)}{p(2\theta^3 + 6\theta^2 + 120)} \int_0^q \frac{x^2\,\theta^5}{\left(\theta^3 + 2\theta^2 + 24\right)} \left(1 + x + x^3\right) e^{-\theta x} dx \tag{48}$$

$$= \frac{\theta^6}{p(2\theta^3 + 6\theta^2 + 120)} \int_0^q x^2 \left(1 + x + x^3\right) e^{-\theta x} dx \tag{49}$$

$$= \frac{\theta^6}{p(2\theta^3 + 6\theta^2 + 120)} \left(\int_0^q x^{(3)-1} e^{-\theta x} dx + \int_0^q x^{(4)-1} e^{-\theta x} dx + \int_0^q x^{(6)-1} e^{-\theta x} dx \right) \quad (50)$$

After simplification, we obtain

$$B(p) = \frac{\theta^6}{p(2\theta^3 + 6\theta^2 + 120)} \left(\gamma(3, \theta q) + \gamma(4, \theta q) + \gamma(6, \theta q) \right) \quad (51)$$

$$L(p) = \frac{\theta^6}{(2\theta^3 + 6\theta^2 + 120)} \left(\gamma(3, \theta q) + \gamma(4, \theta q) + \gamma(6, \theta q) \right) \quad (52)$$

8. Maximum Likelihood Estimation and Fisher's Information Matrix

In this section, we will discuss the technique of maximum likelihood estimation to estimate the parameters of length biased Juchez distribution. Consider $X_1, X_2, \ldots, X_n$ be a random sample of size n from length biased Juchez distribution, then the likelihood function can be defined as

$$L(x) = \prod_{i=1}^{n} f_l(x)$$

$$L(x) = \frac{\theta^{5n}}{(\theta^3 + 2\theta^2 + 24)^n} \prod_{i=1}^{n} \left(x_i \left(1 + x_i + x_i^3 \right) e^{-\theta x_i} \right) \quad (53)$$

The log likelihood function is given by

$$\log L = 5n \log \theta - n \log(\theta^3 + 2\theta^2 + 24) + \sum_{i=1}^{n} \log x_i + \sum_{i=1}^{n} \log\left(1 + x_i + x_i^3 \right) - \theta \sum_{i=1}^{n} x_i \quad (54)$$

Now differentiating log likelihood equation (54) with respect to parameter θ, we establish following normal equation

$$\frac{\partial \log L}{\partial \theta} = \frac{5n}{\theta} - n \left(\frac{\left(3\theta^2 + 4\theta\right)}{\left(\theta^3 + 2\theta^2 + 24\right)} \right) - \sum_{i=1}^{n} x_i = 0 \quad (55)$$

The above likelihood equation is too complicated to solve it algebraically. Therefore, we use numerical technique like Newton Raphson method for estimating the required parameter of proposed distribution.

In order to use the asymptotic normality results for determining the confidence interval. We have that if $(\hat{\beta} = \hat{\theta})$ denotes the MLE *of* $(\beta = \theta)$. We can execute the results as

$$\sqrt{n}(\hat{\beta} - \beta) \to N(0, I^{-1}(\beta))$$

where $I^{-1}(\beta)$ is Fisher's information matrix.i.e.,

$$I(\beta) = -\frac{1}{n} \left(E\left(\frac{\partial^2 \log L}{\partial \theta^2} \right) \right)$$

Here, we see that

$$E\left(\frac{\partial^2 \log L}{\partial \theta^2} \right) = -\frac{5n}{\theta^2} - n \left(\frac{(\theta^3 + 2\theta^2 + 24)\,(6\theta + 4) - (3\theta^2 + 4\theta)^2}{(\theta^3 + 2\theta^2 + 24)^2} \right)$$

Since β being unknown, we estimate $I^{-1}(\beta)$ by $(I^{-1}(\hat{\beta})$ and this can be used to obtain asymptotic confidence interval for θ.

9. Applications

In this section, we have fitted two real lifetime data sets in length biased Juchez distribution to determine its goodness of fit and then comparison has been developed in order to reveal that the length biased Juchez distribution provides a better result over Juchez, exponential and Lindley distributions. The two real lifetime data sets are given below as.

The following first real data set reported by Bader and priest [8] represents the strength measured in GPA for single carbon fibres and impregnated 1000-carbon fibre tows. Single fibres were tested under tension at guage length of 10mm with sample size (n = 63) and the data set is given below in table 1

Table 1: *Data regarding the strength of carbon fibres measured in GPA reported by Bader & Priest (1982)*

1.901	2.132	2.203	2.228	2.257	2.350	2.361	2.396	2.397	2.445	2.454	2.474	2.518
2.522	2.525	2.532	2.575	2.614	2.616	2.618	2.624	2.659	2.675	2.738	2.740	2.856
2.917	2.928	2.937	2.937	2.977	2.996	3.030	3.125	3.139	3.145	3.220	3.223	3.235
3.243	3.264	3.272	3.294	3.332	3.346	3.377	3.408	3.435	3.493	3.501	3.537	3.554
3.562	3.628	3.852	3.871	3.886	3.971	4.024	4.027	4.225	4.395	5.020		

The second real lifetime data set represents the waiting time (in minutes) of 65 dental patients, waiting before OPD (out Patient Diagnosis) at Halibet hospital, Asmara, from 25th to 29th December, 2017 available in the master thesis of Abebe [3] and the data set is given below in table 2.

Table 2: *Data regarding waiting time (in minutes) of 65 dental patients waiting before OPD (out Patient Diagnosis)*

2(5)	6	7	8(3)	9	10	11	12(2)	13	14(4)
15	16	17(2)	18(2)	19	20(3)	22	23(2)	26	27
28	29(2)	30(3)	31	32	33	35	36(2)	37(2)	40(2)
41(2)	42	43	44	46	47	49	52	53	55
56	58	90							

To determine the model comparison criterions along with the estimation of unknown parameters, the technique of R software is used. In order to compare the performance of length biased Juchez distribution over Juchez, exponential and Lindley distributions, we consider the criterions like *AIC* (Akaike Information Criterion), *BIC* (Bayesian Information Criterion), *AICC* (Akaike Information Criterion Corrected), *CAIC* (Consistent Akaike Information Criterion), Shannon's entropy *H(X)* and -*2logL*. The better distribution is which corresponds to lesser values of *AIC*, *BIC*, *AICC*, *CAIC*, *H(X)* and -2logL. For determining the criterions AIC, BIC, AICC, CAIC, H(X) and -2logL given below following formulas are used.

$AIC = 2k - 2\log L,$ $\quad BIC = k\log n - 2\log L,$ $\quad AICC = AIC + \dfrac{2k(k+1)}{n-k-1}$

$CAIC = -2\log L + \dfrac{2kn}{n-k-1}$ and $H(X) = -\dfrac{2\log L}{n}$

Where n is the sample size, k is number of parameters in statistical model and $-2\log L$ is the maximized value of log-likelihood function under the considered model. Table 3 shows the parameter and standard error values and table 4 shows the comparison of distributions.

Table 3: *Shows MLE and S.E Estimates for the data set 1 and data set 2*

Data sets	Distributions	MLE	S.E
1	Length Biased Juchez	$\hat{\theta} = 1.45059991$	$\hat{\theta} = 0.07765314$
	Juchez	$\hat{\theta} = 1.07320917$	$\hat{\theta} = 0.06273572$
	Exponential	$\hat{\theta} = 0.32687301$	$\hat{\theta} = 0.04118174$
	Lindley	$\hat{\theta} = 0.53923226$	$\hat{\theta} = 0.04958387$
2	Length Biased Juchez	$\hat{\theta} = 2.4289412$	$\hat{\theta} = 0.1590837$
	Juchez	$\hat{\theta} = 1.6783037$	$\hat{\theta} = 0.1224368$
	Exponential	$\hat{\theta} = 0.6615390$	$\hat{\theta} = 0.1008835$
	Lindley	$\hat{\theta} = 0.9934028$	$\hat{\theta} = 0.1144634$

Table 4: *Shows Comparison and Performance of fitted distributions*

Data sets	Distributions	-2logL	AIC	BIC	AICC	CAIC	H(X)
1	Length Biased Juchez	186.03	188.03	190.1731	188.0955	188.0955	2.9528
	Juchez	211.9434	213.9434	216.0865	214.0089	214.0089	3.3641
	Exponential	266.8915	268.8915	271.0347	268.9570	268.9570	4.2363
	Lindley	242.7153	244.7153	246.8584	244.7808	244.7808	3.8526
2	Length Biased Juchez	100.0493	102.0493	103.8105	102.1127	102.1127	1.5392
	Juchez	118.4688	120.4688	122.23	120.5322	120.5322	1.8225
	Exponential	121.5341	123.5341	125.2953	123.5975	123.5975	1.8697
	Lindley	114.5096	116.5096	118.2708	116.5730	116.5730	1.7616

From results given above in table 4, it has been clearly realized and observed that the length biased Juchez distribution has lesser *AIC*, *BIC*, *AICC*, *CAIC*, *H(X)* and *-2logL* values as compared to the

Juchez, exponential and Lindley distributions. Hence, it can be concluded that the length biased Juchez distribution provides a better fit over Juchez, exponential and Lindley distributions.

10. Conclusion

In the present study, we have developed a new class of Juchez distribution termed as length biased Juchez distribution. The proposed new distribution is executed by using the length biased technique to its classical distribution. Its various structural properties those include moments, shape of the pdf and cdf, harmonic mean, order statistics, survival function, hazard rate function, reverse hazard function, moment generating function, bonferroni and Lorenz curves have been derived. The parameter of proposed new distribution is estimated by using the maximum likelihood estimation. Finally, a new distribution has been examined and analyzed with two real data sets to demonstrate its superiority and flexibility. Hence, it is revealed from the results that the proposed length biased Juchez distribution leads to a better fit over Juchez, exponential and Lindley distributions.

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