TRANSIENT AND METAHEURISTIC COST SCRUTINY OF $M^X/G(A,B)/1$ RETRIAL QUEUE WITH RANDOM FAILURE UNDER EXTENDED BERNOULLI VACATION WITH IMPATIENT CUSTOMERS

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Abstract

The transient and metaheuristic cost analysis of a $M^X/G(a,b)/1$ retrial queue with random failure during an extended Bernoulli vacation with impatient clients is covered in this study. Any batch that arrives and discovers the server is busy, down, or on vacation joins an orbit. In the alternative, only one new customer from the group joins the service right away, while the others join the orbit. After providing each service, the server either waits to serve the following customer with probability $(1-\theta)$ or goes on vacation with probability θ. It has been found that these systems express steady-state solutions and are dependent on time probability generating functions in consideration of their Laplace transforms. We also discuss a few exceptional and particular instances. After that, the impact of different parameters on the system's effectiveness is evaluated. We are also talking about ANFIS. Additional approaches employed in this study to swiftly determine the system's optimum cost include genetic algorithms (GA), artificial bee colonies (ABC), and particle swarm optimization (PSO). We also examined the graph-based convergence of several optimization algorithms.

Keywords: Batch arrival, Retrial queues, Feedback, Extended Bernoulli Vacation, ANFIS, Cost Optimization.

1. Introduction

For the development, capacity planning, performance assessment, and optimization of numerous real-world systems, queueing theory offers a potent tool. Chaudhry and Templeton[1] provided a comprehensive analysis of bulk queuing. Bulk arrival analysis, a condensed form of customer examination, is a great place to start with customised models. Bulk service queuing models were created by Bailey [2]. He invented the process known as "fixed-batch service". The server continuously offers a specific batch of services to each set of users in fixed-batch service queueing systems (QS).

The "retrial queueing"system, which is used when a customer enters and the server is occupied, requires the customer to leave the appropriate area and repeat his request after a certain period of time. This property is essential for network technologies, cognitive networks, online computing systems, manufacturing systems, and other systems.

Sumitha and Udaya Chandrika [3] investigated a retrial queuing system with starting failure, single vacation, and orbital search. In batch arrival retrial queues, Radha et al. [4] studied some system performance measures are evaluated using the supplementary variable technique (SVT) and the steady-state (SS) probability generating function (PGF) for system size.

Gomez-Corral has talked a lot about a retrial QS with FCFS discipline and typical retrial periods. The M/G/1 retrial queue with feedback and starting failures was described by Krishna Kumar et al. [5]. Yang, Tao, and Hui Li[6] investigated an M/G/1 retrial queue with a starting failure-prone server. An analysis of a feedback retrial queuing system with starting failures and a single vacation was studied by Mokaddis et al. [7].

In a Vacation, queueing system the server could be temporarily unavailable for a number of reasons, including maintenance monitoring, tending to other queues, or simply taking a break. When the server is unavailable to users, that time period is referred to as a "vacation ". A single server batch arrival Bernoulli feedback QS with a waiting server, K-variant vacations, and anxious clients was examined by Bouchentouf et al. [8]. The transient behaviour of a batch arrival feedback retrial queue with starting failure and Bernoulli vacation (BV) was investigated by Ayyappan and Sathiya [9]. Assuming that repair, service, and vacation times are randomly distributed, the time-dependent PGF are also computed in relation to their Laplace transforms(LT).

Numerous academics who have studied queueing techniques with interruptions have as their primary tenet that, in the event of a failure, the service channel will be promptly repaired. A transient analysis of the $M^{[X_1]}, M^{[X_2]}/G1, G2/1$ retrial QS's with priority services, working breakdown, start up/close down time, BV, reneging, and balking was studied by Ayyappan et al. [10]. Kulkarni et al. [11] established a retrial queue with a server prone to failures and maintenance. Ayyappan and Shyamala [12] created an $M^{[X]}/G/1$ with Bernoulli schedule, server vacation, random break down and second optional repair. And also calculate the typical length of the line and the typical wait period in closed form. When the repair is finished, a number of consumers who had previously used the services wait for the remainder to be provided. Jau-Chuan Ke et al. [13] demonstrated a waiting line with customers complaining and providing feedback the servers malfunctioned. Furthermore, if all servers are already in use when a customer arrives, he will either join a retrial orbit or decline. When a service is finished, the client can exit the system or rejoin the retrial group to receive more services. They can also design a cost function to determine the system's ideal parameter settings under the stability condition. Computer telecommunication systems is a example of application for these types.

A consumer may try again until they are happy if they are not satisfied with the service they received. Takacs [14] investigates this at first, allowing the consumer who has finished the service to provide feedback to the rear of the line. An M/(G1, G2)/1 feedback retrial queue with two phase service, variant vacation policy under delaying repair for impatient Customers was analysed by Rajadurai et al. [15].

Many real-world systems have impatient customers as a built-in feature, particularly when the customer is a human, a perishable product, or some moving object that can depart the service area and their waiting period in the queue reaches certain pre-defined threshold values. This clearly explains why queueing literature frequently discusses the impatience phenomenon. Accounting impatience is crucial in the setting of lines for group service because a client could spend a large amount of time in the system while waiting for the accumulation of a sufficient number of customers.

More focus has been placed on the numerous retrial lineups with non-persistent (impatient) consumers. A discussion about the study of a retrial queue with group service of impatient clients involved D'rienzo et al. [16]. A batch arrival retrial queuing model with starting failures and customer impatience was addressed by Nila and Sumitha [17]. Customers arrive in batches in line with the Poisson process. In certain situations, the clients refuse and break their promises. The analysis of a retrial QS with priority services, working breakdown, BV, admission control, and balking was explained by Ayyappan et al. [18]. Ayyappan and Nirmala [19] explored an analysis of customer's impatience on bulk service QS's with an unreliable server, setup time and two types of multiple vacations. Sethi.R et al. [20] investigated the cost optimization and ANFIS computing of an unreliable M/M/1 queueing system with customers' impatience under n-policy. The ideal Cost Analysis for Discrete-Time Recurrent Queue with Bernoulli Feedback and Emergency Vacation was described by M. Vaishnawi [21]. In order to calculate costs, PSO, ABC, and GA are also used. To ensure the best deal, these methods compare and contrast the outputs.

The paper's structure is as follows: Section 2 provides a detailed explanation of the mathematical model. Section 3 discusses the ideas and formulae governing our system as well as how to obtain the time-dependent solution of our model. The PGF for the queue length at each given epoch and the SS performance of the system are explicitly determined in Section 4. In Section 5, the pertinent stability condition has been uncovered. In Section 6, we precisely estimate the mean queue size, mean queue waiting time, and efficiency features for each state of the system. In Section 7, we present a practical illustration. We offer a numerical study and associated graphs in Section 8. Furthermore, an ANFIS was provided in Section 9. The Cost optimization is offered by Section 10. The conclusion is presented in Section 11.

2. Model Description and Analysis

We suppose that the underlying queueing model is as follows:
Arrival process: Customers enter a poisson stream, and bulk service is offered on an FCFS basis. Considering that a batch of "i"customers enters the system, $\Lambda > 0$ represents the average batch arrival rate, and $\Lambda c_i dt (i \geq 1)$ represents the first order probability during the short interval of time $(\varpi, \varpi + d\varpi]$. We define a batch arrival and a bulk service as having a smallest batch size of "a"and a highest batch size of "b".
Retrial process: When a customer arrives and discovers that the server is busy, unavailable, or broken, the customer has two options: (1) leave the service area with a probability of d and join a pool of blocked customers known as an orbit; or (2) balk the system with a probability of $\bar{d}$ in accordance with FCFS, which implies that only the customer at the head of the orbit queue is permitted access to the server.
When the server is idle, the customer at the head of the retrial queue engages with potential primary customers to see who can cancel their service request and, with prob., g, either move up in the retrial queue or leave the system with prob., $(1-g)$.
A general (arbitray) distribution with the distribution function $A(u)$ and the density function $a(u)$ determines the retrial interval.
Let $g(\varsigma)d\varsigma$ be the conditional prob., density of completing the retrial within the range $(\varsigma, \varsigma + d\varsigma]$, where ς is the elapsed retrial time.

$$g(\varsigma) = \frac{a(\varsigma)}{1 - A(\varsigma)}$$

and therefore,

$$a(u) = g(u)e^{-\int_0^u g(\varsigma)d\varsigma}$$

Inter-retrial times have an arbitrary dist., $A(\varsigma)$ with correponding Laplace-Stieltijes transforms (LST) $A^*(u)$.
Service process: The server enters an idle state wherever a fresh or returning user comes before quickly resuming regular operations for the newcomers. A generic (arbitrary) distance with the distance function $B(\varsigma)$ and the density function $b(\varsigma)$ follows the service time.
Given the elapsed retrial time ς, define $\phi(\varsigma)d\varsigma$ as the conditional probability of service completion within the range $(\varsigma, \varsigma + d\varsigma]$.

$$\phi(\varsigma) = \frac{b(\varsigma)}{1 - B(\varsigma)}$$

and therefore,

$$b(\varpi) = \phi(\varpi)e^{-\int_0^{\varpi} \phi(\varsigma)d\varsigma}$$

The random variable B with the dist., function $B(\varsigma)$ and LST $B^*(\varpi)$ denotes the service time.
Random failure: Failures are anticipated to occur sporadically throughout the system and ought to follow a poisson stream with an average failure rate of $\tau > 0$. The repair times follow a general dist.,, which is represented by the random variable D and the dist., function $D(\varsigma)$, with the LST $D^*(\varpi)$.
The length of repairs is determined by a general (arbitrary) dist., with a dist., function $D(\varsigma)$ and a density function $d(\varsigma)$. Given an elapsed repair time of ς, define $\alpha(\varsigma)d\varsigma$ as the conditional probability of completing repairs within the range $(\varsigma, \varsigma + d\varsigma]$.

$$\alpha(\varsigma) = \frac{d(\varsigma)}{1 - D(\varsigma)}$$

and therefore,

$$d(\varpi) = \alpha(\varpi)e^{-\int_0^{\varpi} \alpha(\varsigma)d\varsigma}$$

Extended Bernoulli vacation: If there are any unfinished parts of the service, the server has two options: either accept the BV with a probability of θ or keep serving them with a probability of $(1 - \theta)$. After the vacation is over, the server either undertakes the second type of optional extended Bernoulli vacation with a prob., of μ or continues to serve the remaining batches with a prob., of $(1 - \mu)$.
The random variable F with the distance function $F(\varsigma)$ and LST $F^*(\varpi)$ is employed to represent the server's leisure time. This arbitrary variable F follows a general distribution.
The server's vacation time follows a general(arbitrary) dist., function $F(\varpi)$ and density function $f(\varpi)$. Let $\beta(\varsigma)d\varsigma$ be the conditional prob., of a completion of a vacation during the interval $(\varsigma, \varsigma + d\varsigma]$, given that the elapsed repair time is ς, so that

$$\beta(\varsigma) = \frac{f(\varsigma)}{1 - F(\varsigma)}$$

and therefore,

$$f(\varpi) = \beta(\varpi)e^{-\int_0^{\varpi} \beta(\varsigma)d\varsigma}$$

The system's stochastic processes are all considered to be independent of one another.
Feedback Rule: Clients who are unhappy with their offerings can re-join the line once they've been completed, give feedback to receive another service with minimal difficulty, or both p $(0 \leq p \leq 1)$, otherwise the system must be terminated with complement prob. $q = (1 - p)$

3. DEFINITIONS:

We define
1. $P_n(\varsigma, \varpi)$= Prob., that the server will be idle at time ϖ with $n(n \geq 0)$ customers in the orbit and ς for the customer's elapsed retrial time.
2. $Q_n(\varsigma, \varpi)$= Prob., that the server will be busy at time ϖ with $n(n \geq 0)$ customers in the orbit and η for the customer's elapsed retrial time.
3. $R_n(\varsigma, \varpi)$= Prob., that at time ϖ, there are $n(n \geq 0)$ customers in the orbit and the server is offline due to system repair and waiting for repairs to start with elapsed repair time ς.

4. $V_n(\varsigma,\varpi)$= Prob., that there are $n(n \geq 0)$ consumers in orbit at time ϖ and the server is on vacation with elapsed vacation time ς.
5. There are no customers in the orbit at time ϖ, and the server is inactive but still available in the system, according to the probability $P_0(\varpi)$.
The following differential-difference equations regulate the model:

$$\frac{d}{d\varpi}P_0(\varpi) = -\Lambda P_0(\varpi) + (1-\theta)\bar{d}\int_0^\infty Q_0(\varsigma,\varpi)\phi(\varsigma)d\varsigma + (1-\mu)\int_0^\infty V_0(\varsigma,\varpi)\beta(\varsigma)d\varsigma \tag{1}$$

$$\frac{\partial}{\partial\varsigma}P_n(\varsigma,\varpi) + \frac{\partial}{\partial\varpi}P_n(\varsigma,\varpi) = -[\Lambda + g(\varsigma)]P_n(\varsigma,\varpi), n \geq 1 \tag{2}$$

$$\frac{\partial}{\partial\varsigma}Q_0(\varsigma,\varpi) + \frac{\partial}{\partial\varpi}Q_0(\varsigma,\varpi) = -[\Lambda + \tau + \phi(\varsigma)]Q_0(\varsigma,\varpi) \tag{3}$$

$$\frac{\partial}{\partial\varsigma}Q_n(\varsigma,\varpi) + \frac{\partial}{\partial\varpi}Q_n(\varsigma,\varpi) = -[\Lambda + \tau + \phi(\varsigma)]Q_n(\varsigma,\varpi) + \Lambda\sum_{k=1}^{n} C_k Q_{n-k}(\varsigma,\varpi), n \geq 1 \tag{4}$$

$$\frac{\partial}{\partial\varsigma}R_0(\varsigma,\iota,\varpi) + \frac{\partial}{\partial\varpi}R_0(\varsigma,\varpi) = -[\Lambda + \alpha(\varsigma)]R_0(\varsigma,\varpi), n = 0 \tag{5}$$

$$\frac{\partial}{\partial\varsigma}R_n(\varsigma,\iota,\varpi) + \frac{\partial}{\partial\varpi}R_n(\varsigma,\varpi) = -[\Lambda + \alpha(\varsigma)]R_n(\varsigma,\varpi) + \Lambda\sum_{k=1}^{n} C_k R_{n-k}(\varsigma,\varpi), n \geq 1 \tag{6}$$

$$\frac{\partial}{\partial\varsigma}V_0(\varsigma,\varpi) + \frac{\partial}{\partial\varpi}V_0(\varsigma,\varpi) = -[\Lambda + \beta(\varsigma)]V_0(\varsigma,\varpi), n = 0 \tag{7}$$

$$\frac{\partial}{\partial\varsigma}V_n(\varsigma,\varpi) + \frac{\partial}{\partial\varpi}V_n(\varsigma,\varpi) = -[\Lambda + \beta(\varsigma)]V_n(\varsigma,\varpi) + \Lambda\sum_{k=1}^{n} C_k V_{n-k}(\varsigma,\varpi), n \geq 1 \tag{8}$$

The following boundary conditions must be met in order to answer the given equation:

$$\begin{aligned} P_n(0,\varpi) =& (1-\theta)\bar{d}\int_0^\infty Q_n(\varsigma,\varpi)\phi(\varsigma)d\varsigma + (1-\theta)d\int_0^\infty Q_{n-1}(\varsigma,\varpi)\phi(\varsigma)d\varsigma \\ &+ \int_0^\infty R_n(\varsigma,\varpi)\alpha(\varsigma)d\varsigma + (1-\mu)\int_0^\infty V_n(\varsigma,\varpi)\beta(\varsigma)d\varsigma, n \geq 1 \end{aligned} \tag{9}$$

$$\begin{aligned} Q_0(0,\varpi) =& \Lambda p(1-g)\sum_{r=a}^{b}\sum_{k=0}^{a-1} C_k\int_0^\infty P_{n-k+b}(\varsigma,\varpi)d\varsigma \\ &+ (1-\theta)p\sum_{r=a}^{b}\int_0^\infty P_r(\varsigma,\varpi)g(\varsigma)d\varsigma + \sum_{r=a}^{b}\int_0^\infty V_r(\varsigma,\varpi)\beta(\varsigma)d\varsigma \end{aligned} \tag{10}$$

$$\begin{aligned} Q_n(0,\varpi) =& \Lambda p(1-g)\sum_{k=0}^{a-1} C_k\int_0^\infty P_{n-k+b}(\varsigma,\varpi)d\varsigma + p\int_0^\infty P_{n+b}(\varsigma,\varpi)g(\varsigma)d\varsigma \\ &+ \Lambda g\int_0^\infty P_{n+b}(\varsigma,\varpi)d\varsigma + \int_0^\infty V_{n+b}(\varsigma,\varpi)\beta(\varsigma)d\varsigma \end{aligned} \tag{11}$$

$$R_0(\varsigma,0,\varpi) = \tau Q_0(\varsigma,\varpi), n = 0 \tag{12}$$

$$R_n(\varsigma,0,\varpi) = \tau Q_n(\varsigma,\varpi), n \geq 1 \tag{13}$$

$$V_n(0,\varpi) = \theta\int_0^\infty Q_n(\varsigma,\varpi)\phi(\varsigma)d\varsigma, n \geq 1 \tag{14}$$

We presume that the system is initially empty of users and that the server is idle. Thus, the initial conditions are

$$\begin{aligned} V_n(0) = R_n(0) = Q_n(0) = 0, n \geq 0 \\ P_0(0) = 1, P_n^i(0) = 0, n \geq 1 \end{aligned} \tag{15}$$

Generating functions of the queue length (The time-dependent solution):

$$P(\varsigma,\Psi,\varpi) = \sum_{n=0}^{\infty} \Psi^n P_n^i(\varsigma,\varpi); P(\Psi,\varpi) = \sum_{n=0}^{\infty} \Psi^n P_n(\varpi)$$
$$Q(\varsigma,\Psi,\varpi) = \sum_{n=0}^{\infty} \Psi^n Q_n(\varsigma,\varpi);\ Q(\Psi,\varpi) = \sum_{n=0}^{\infty} \Psi^n Q_n(\varpi)$$
$$R(\varsigma,\iota,\Psi,\varpi) = \sum_{n=0}^{\infty} \Psi^n R_n(\varsigma,\iota,\varpi);\ R(\varsigma,\Psi,\varpi) = \sum_{n=0}^{\infty} \Psi^n R_n(\varsigma,\varpi)$$
$$V(\varsigma,\Psi,\varpi) = \sum_{n=0}^{\infty} \Psi^n V_n(\varsigma,\varpi);\ V(\Psi,\varpi) = \sum_{n=0}^{\infty} \Psi^n V_n(\varpi)$$
$$C(\Psi) = \sum_{n=1}^{\infty} C_n \Psi^n;\ Q(\Psi) = \sum_{r=0}^{a-1} Q_r \Psi^r \tag{16}$$

which define the LT of a function $f(\varpi)$ as it converges within the circle defined by $z \leq 1$.

$$\bar{f}(s) = \int_0^{\infty} e^{-s\varpi} f(\varpi) d\varpi, \mathcal{R}(s) \geq 0 \tag{17}$$

Using (15) and the LT from equations (1) through (14), we arrive at

$$(s+\Lambda)\bar{p}_0(s) = 1 + (1-\theta)\bar{d}\int_0^{\infty} \bar{Q}_0(\varsigma,s)\phi(\varsigma)d\varsigma + (1-\mu)\int_0^{\infty} \bar{V}_0(\varsigma,s)\beta(\varsigma)d\varsigma \tag{18}$$
$$\frac{\partial}{\partial\varsigma}\bar{P}_n(\varsigma,s) + [s+\Lambda+g(\varsigma)]\bar{P}_n(\varsigma,s) = 0, n \geq 1 \tag{19}$$
$$\frac{\partial}{\partial\varsigma}\bar{Q}_0(\varsigma,s) + [s+\Lambda+\phi(\varsigma)]\bar{Q}_0(\varsigma,s) = 0 \tag{20}$$
$$\frac{\partial}{\partial\varsigma}\bar{Q}_n(\varsigma,s) + [s+\Lambda+\phi(\varsigma)]\bar{Q}_n(\varsigma,s) = \Lambda\sum_{k=1}^{n} C_k\bar{Q}_{n-k}(\varsigma,s), n \geq 1 \tag{21}$$
$$\frac{\partial}{\partial\varsigma}\bar{R}_0(\varsigma,\iota,s) + [s+\Lambda+\alpha(\varsigma)]\bar{R}_0(\varsigma,s) = 0 \tag{22}$$
$$\frac{\partial}{\partial\varsigma}\bar{R}_n(\varsigma,\iota,s) + [s+\Lambda+\alpha(\varsigma)]\bar{R}_n(\varsigma,s) = \Lambda\sum_{k=1}^{n} C_k\bar{R}_{n-k}(\varsigma,s), n \geq 1 \tag{23}$$
$$\frac{\partial}{\partial\varsigma}\bar{V}_0(\varsigma,s) + [s+\Lambda+\beta(\varsigma)]\bar{V}_0(\varsigma,s) = 0 \tag{24}$$
$$\frac{\partial}{\partial\varsigma}\bar{V}_n(\varsigma,s) + [s+\Lambda+\beta(\varsigma)]\bar{V}_n(\varsigma,s) = \Lambda\sum_{k=1}^{n} C_k\bar{V}_{n-k}(\varsigma,s), n \geq 1 \tag{25}$$

$$\bar{P}_n(0,s) = (1-\theta)\bar{d}\int_0^\infty \bar{Q}_n(\varsigma,s)\phi(\varsigma)d\varsigma + (1-\theta)d\int_0^\infty \bar{Q}_{n-1}(\varsigma,s)\phi(\varsigma)d\varsigma + \int_0^\infty \bar{R}_n(\varsigma,s)\alpha(\varsigma)d\varsigma + (1-\mu)\int_0^\infty \bar{V}_n(\varsigma,s)\beta(\varsigma)d\varsigma, n\geq 1 \tag{26}$$

$$\bar{Q}_0(0,s) = \Lambda p(1-g)\sum_{r=a}^{b}\sum_{k=0}^{a-1} C_k\int_0^\infty \bar{P}_{n-k+b}(\varsigma,s)d\varsigma + (1-\theta)p\sum_{r=a}^{b}\int_0^\infty \bar{P}_r(\varsigma,s)g(\varsigma)d\varsigma + \sum_{r=a}^{b}\int_0^\infty \bar{V}_r(\varsigma,s)\beta(\varsigma)d\varsigma \tag{27}$$

$$\bar{Q}_n(0,s) = \Lambda p(1-g)\sum_{k=0}^{a-1} C_k\int_0^\infty \bar{P}_{n-k+b}(\varsigma,s)d\varsigma + p\int_0^\infty \bar{P}_{n+b}(\varsigma,s)g(\varsigma)d\varsigma + \Lambda g\int_0^\infty \bar{P}_{n+b}(\varsigma,s)d\varsigma + \int_0^\infty \bar{V}_{n+b}(\varsigma,s)\beta(\varsigma)d\varsigma \tag{28}$$

$$\bar{R}_0(\varsigma,0,s) = \tau\bar{Q}_0(\varsigma,s), n=0 \tag{29}$$

$$\bar{R}_n(\varsigma,0,s) = \tau\bar{Q}_n(\varsigma,s), n\geq 1 \tag{30}$$

$$\bar{V}_n(0,s) = \theta\int_0^\infty \bar{Q}_n(\varsigma,s)\phi(\varsigma)d\varsigma, n\geq 1 \tag{31}$$

By multiplying equations (19) through (31) by Ψ^n and adding the results over n, we can obtain using the generating function mentioned in equation (16).

$$\frac{\partial}{\partial\varsigma}\bar{P}(\varsigma,\Psi,s) + [s+\Lambda+g(\varsigma)]\bar{P}(\varsigma,\Psi,s) = 0 \tag{32}$$

$$\frac{\partial}{\partial\varsigma}\bar{Q}(\varsigma,\Psi,s) + [s+\Lambda(1-C(\Psi))+\phi(\varsigma)]\bar{Q}(\varsigma,\Psi,s) = 0 \tag{33}$$

$$\frac{\partial}{\partial\varsigma}\bar{R}(\varsigma,\iota,\Psi,s) + [s+\Lambda(1-C(\Psi))+\alpha(\varsigma)]\bar{R}(\varsigma,\Psi,s) = 0 \tag{34}$$

$$\frac{\partial}{\partial\varsigma}\bar{V}(\varsigma,\Psi,s) + [s+\Lambda(1-C(\Psi))+\beta(\varsigma)]\bar{V}(\varsigma,\Psi,s) = 0 \tag{35}$$

$$\bar{P}(0,\Psi,s) = (1-\theta)(\bar{d}+d\Psi)\int_0^\infty \bar{Q}(\varsigma,\Psi,s)\phi(\varsigma)d\varsigma + \int_0^\infty \bar{R}(\varsigma,\Psi,s)\alpha(\varsigma)d\varsigma + (1-\mu)\int_0^\infty \bar{V}(\varsigma,\Psi,s)\beta\varsigma d\varsigma - \bar{d}(1-\theta)\int_0^\infty \bar{Q}_0(\varsigma,s)\phi(\varsigma)d\varsigma - (1-\mu)\int_0^\infty \bar{V}_0(\varsigma,s)\beta(\varsigma)d\varsigma, n\geq 1 \tag{36}$$

$$\Psi^b\bar{Q}(0,\Psi,s) = \Lambda(1-g)pC(\Psi)\int_0^\infty \bar{P}(\varsigma,\Psi,s)d\varsigma + p\int_0^\infty \bar{P}(\varsigma,\Psi,s)g(\varsigma)d\varsigma + \Lambda g\int_0^\infty \bar{P}(\varsigma,\Psi,s)d\varsigma + \int_0^\infty \bar{V}(\varsigma,\Psi,s)\beta(\varsigma)d\varsigma \tag{37}$$

$$\bar{R}(\varsigma,0,\Psi,s) = \tau\bar{Q}(\varsigma,\Psi,s), n\geq 1 \tag{38}$$

$$\bar{V}(0,\Psi,s) = \theta\int_0^\infty \bar{Q}(\varsigma,\Psi,s)\phi(\varsigma)d\varsigma, n\geq 1 \tag{39}$$

Equation (18) in (36) gives us

$$\bar{P}(0,\Psi,s) = [1-(s+\Lambda)\bar{P}_0(s)] + (1-\theta)(\bar{d}+d\Psi)\int_0^\infty \bar{Q}(\varsigma,\Psi,s)\phi(\varsigma)d\varsigma + \int_0^\infty \bar{R}(\varsigma,\Psi,s)\alpha(\varsigma)d\varsigma + (1-\mu)\int_0^\infty \bar{V}(\varsigma,\Psi,s)\beta(\varsigma)d\varsigma \tag{40}$$

Equation (32), when integrated between 0 and ς, yields

$$\bar{P}(\varsigma,\Psi,s) = \bar{P}(0,\Psi,s)e^{-(s+\Lambda)\varsigma-\int_0^\varsigma g(\varpi)d\varpi} \tag{41}$$

Once more, integrating equation (41) by parts with respect to ς yiedls,

$$\bar{P}(\Psi,s) = \bar{P}(0,\Psi,s)\left[\frac{1-\bar{A}(s+\Lambda)}{s+\Lambda}\right] \tag{42}$$

where,

$$\bar{A}(s+\Lambda) = \int_0^\infty e^{-(s+\Lambda)\varsigma}dA(\varsigma)$$

When integrating equations (33) to (35) from 0 to ς, similar outcomes are found.

$$\bar{Q}(\varsigma,\Psi,s) = \bar{Q}(0,\Psi,s)e^{-\zeta(\Psi,s)\varsigma-\int_0^\varsigma \phi(\varpi)d\varpi} \tag{43}$$

$$\bar{R}(\varsigma,\iota,\Psi,s) = \bar{R}(\varsigma,0,\Psi,s)e^{-\zeta(\Psi,s)\varsigma-\int_0^\varsigma \alpha(\varpi)d\varpi}$$

$$\bar{R}(\varsigma,\Psi,s) = \bar{R}(\varsigma,0,\Psi,s)\left[\frac{1-\bar{D}(\zeta(\Psi,s))}{\zeta(\Psi,s)}\right] \tag{44}$$

$$\bar{V}(\varsigma,\Psi,s) = \bar{V}(0,\Psi,s)e^{-\zeta(\Psi,s)\varsigma-\int_0^\varsigma \beta(\varpi)d\varpi} \tag{45}$$

where the values of $\bar{P}(0,\Psi,s)$,$\bar{Q}(0,\Psi,s)$,$\bar{R}(0,\Psi,s)$ and $\bar{V}(0,\Psi,s)$ are given by (37) to (40).
Taking into account ς yiedls, integrate equations (43) to (45) by parts once more.

$$\bar{Q}(\Psi,s) = \bar{Q}(0,\Psi,s)\left[\frac{1-\bar{B}(\zeta(\Psi,s))}{\zeta(\Psi,s)}\right] \tag{46}$$

$$\bar{R}(\Psi,s) = \tau\bar{Q}(0,\Psi,s)\left[\frac{1-\bar{B}(\zeta(\Psi,s))}{\zeta(\Psi,s)}\right]\left[\frac{1-\bar{D}(\zeta(\Psi,s))}{\zeta(\Psi,s)}\right] \tag{47}$$

$$\bar{V}(\Psi,s) = \bar{V}(0,\Psi,s)\left[\frac{1-\bar{F}(\zeta(\Psi,s))}{\zeta(\Psi,s)}\right] \tag{48}$$

Where,

$$\bar{B}(\zeta(\Psi,s)) = \int_0^\infty e^{-\zeta(\Psi,s)\varsigma}dB(\varsigma)$$

$$\bar{D}(\zeta(\Psi,s)) = \int_0^\infty e^{-\zeta(\Psi,s)\varsigma}dD(\varsigma)$$

$$\bar{F}(\zeta(\Psi,s)) = \int_0^\infty e^{-\zeta(\Psi,s)\varsigma}dF(\varsigma)$$

are, in order, the LST of the following values: retrial time $A(\varsigma)$, service time $B(\varsigma)$, repair time $D(\varsigma)$, and vacation time $F(\varsigma)$.
Now, multiplying both side of equations (41),(43) to (45) by $g(\varsigma)$,$\phi(\varsigma)$,$\alpha(\varsigma)$ and $\beta(\varsigma)$ and integrating over ς, we obtain

$$\int_0^\infty \bar{P}(\varsigma,\Psi,s)g(\varsigma)d\varsigma = \bar{P}(0,\Psi,s)\bar{A}(s+\Lambda) \tag{49}$$

$$\int_0^\infty \bar{Q}(\varsigma,\Psi,s)\phi(\varsigma)d\varsigma = \bar{Q}(0,\Psi,s)\bar{B}(\zeta(\Psi,s)) \tag{50}$$

$$\int_0^\infty \bar{R}(\varsigma,\iota,\Psi,s)\alpha(\varsigma)d\varsigma = \bar{R}(\varsigma,0,\Psi,s)\bar{D}(\zeta(\Psi,s)) \tag{51}$$

$$\int_0^\infty \bar{V}(\varsigma,\Psi,s)\beta(\varsigma)d\varsigma = \bar{V}(0,\Psi,s)\bar{F}(\zeta(\Psi,s)) \tag{52}$$

Using equations (50) in (39)

$$\bar{V}(0,\Psi,s) = \theta\bar{Q}(0,\Psi,s)\bar{B}(\zeta(\Psi,s)) \tag{53}$$

Using equations (49) in (37) and (38), we get

$$\bar{Q}(0,\Psi,s) = \frac{\bar{P}(0,\Psi,s)}{\Psi^b - \theta\bar{F}(\zeta(\Psi,s))\bar{B}(\zeta(\Psi,s))} \left[\Lambda(1-g)pC(\Psi)\left(\frac{1-\bar{A}(s+\Lambda)}{s+\Lambda}\right) + p\bar{A}(s+\Lambda) + \Lambda g\left(\frac{1-\bar{A}(s+\Lambda)}{s+\Lambda}\right)\right] \tag{54}$$

$$\bar{R}(\varsigma,0,\Psi,s) = \tau\bar{Q}(0,\Psi,s)\left(\frac{1-\bar{B}(\zeta(\Psi,s))}{(\zeta(\Psi,s))}\right) \tag{55}$$

Using equation (50) to (52) in (40) we get

$$\bar{P}(0,\Psi,s) = \frac{Nr(\Psi)}{Dr(\Psi)} \tag{56}$$

$$\begin{aligned}
Nr(\Psi) =& [1-(s+\Lambda)\bar{P}_0(s)][\Psi^b - \theta\bar{F}(\zeta(\Psi,s))\bar{B}(\zeta(\Psi,s))] \\
Dr(\Psi) =& \Psi^b - \theta\bar{F}(\zeta(\Psi,s))\bar{B}(\zeta(\Psi,s)) \\
& - \left[\Lambda(1-g)pC(\Psi)\left(\frac{1-\bar{A}(s+\Lambda)}{s+\Lambda}\right) + p\bar{A}(s+\Lambda) + \Lambda g\left(\frac{1-\bar{A}(s+\Lambda)}{s+\Lambda}\right)\right] \\
& \left[(1-\theta)(\bar{d}+d\Psi)\bar{B}(\zeta(\Psi,s)) + \tau\bar{D}(\zeta(\Psi,s))\left(\frac{1-\bar{B}(\zeta(\Psi,s))}{(\zeta(\Psi,s))}\right)\right. \\
& \left. + \theta(1-\mu)\bar{F}(\zeta(\Psi,s))\bar{B}(\zeta(\Psi,s))\right]
\end{aligned}$$

where,

$$\zeta(\Psi,s) = s + \Lambda(1-C(\Psi))$$

Subs/- $\bar{P}(0,\Psi,s)$ from equation (56) into equation (53) to (55)

$$\bar{Q}(0,\Psi,s) = \left[\frac{\Lambda(1-g)pC(\Psi)\left(\frac{1-\bar{A}(s+\Lambda)}{s+\Lambda}\right) + p\bar{A}(s+\Lambda) + \Lambda g\left(\frac{1-\bar{A}(s+\Lambda)}{s+\Lambda}\right)}{\Psi^b - \theta\bar{F}(\zeta(\Psi,s))\bar{B}(\zeta(\Psi,s))}\right] \left[\frac{Nr(\Psi)}{Dr(\Psi)}\right] \tag{57}$$

$$\bar{R}(\varsigma,0,\Psi,s) = \tau\left(\frac{1-\bar{B}(\zeta(\Psi,s))}{(\zeta(\Psi,s))}\right)\left[\frac{Nr(\Psi)}{Dr(\Psi)}\right] \left[\frac{\Lambda(1-g)pC(\Psi)\left(\frac{1-\bar{A}(s+\Lambda)}{s+\Lambda}\right) + p\bar{A}(s+\Lambda) + \Lambda g\left(\frac{1-\bar{A}(s+\Lambda)}{s+\Lambda}\right)}{\Psi^b - \theta\bar{F}(\zeta(\Psi,s))\bar{B}(\zeta(\Psi,s))}\right] \tag{58}$$

$$\bar{V}(0,\Psi,s) = \theta\bar{B}(\zeta(\Psi,s))\left[\frac{Nr(\Psi)}{Dr(\Psi)}\right] \left[\frac{\Lambda(1-g)pC(\Psi)\left(\frac{1-\bar{A}(s+\Lambda)}{s+\Lambda}\right) + p\bar{A}(s+\Lambda) + \Lambda g\left(\frac{1-\bar{A}(s+\Lambda)}{s+\Lambda}\right)}{\Psi^b - \theta\bar{F}(\zeta(\Psi,s))\bar{B}(\zeta(\Psi,s))}\right] \tag{59}$$

Updating equations (56) to (59) in (42), (46) to (48) We determine the PGF of various conditions in the system under a transient condition.

4. The Steady state's findings:

To define the SS prob., we disregard the argument ϖ wherever it appears in the time-dependent analysis.

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{\varpi\to\infty} f(\varpi)$$

$$P(\Psi) = P(0,\Psi)\left(\frac{1-\bar{A}(\Lambda)}{\Lambda}\right) \tag{60}$$

$$Q(\Psi) = \left(\frac{1-\bar{B}(\zeta(\Psi))}{\zeta(\Psi)}\right) P(0,\Psi) \tag{61}$$

$$\left[\frac{\Lambda(1-g)pC(\Psi)\left(\frac{1-\bar{A}(\Lambda)}{\Lambda}\right) + p\bar{A}(\Lambda) + \Lambda g\left(\frac{1-\bar{A}(\Lambda)}{\Lambda}\right)}{\Psi^b - \theta\bar{F}(\zeta(\Psi))\bar{B}(\zeta(\Psi))}\right] \tag{62}$$

$$R(\Psi) = \tau\left(\frac{1-\bar{B}(\zeta(\Psi))}{\zeta(\Psi)}\right)\left(\frac{1-\bar{D}(\zeta(\Psi))}{\zeta(\Psi)}\right)$$

$$P(0,\Psi)\left[\frac{\Lambda(1-g)pC(\Psi)\left(\frac{1-\bar{A}(\Lambda)}{\Lambda}\right) + p\bar{A}(\Lambda) + \Lambda g\left(\frac{1-\bar{A}(\Lambda)}{\Lambda}\right)}{\Psi^b - \theta\bar{F}(\zeta(\Psi))\bar{B}(\zeta(\Psi))}\right] \tag{63}$$

$$V(\Psi) = \theta\bar{B}(\zeta(\Psi))\left(\frac{1-\bar{F}(\zeta(\Psi))}{\zeta(\Psi)}\right)$$

$$P(0,\Psi)\left[\frac{\Lambda(1-g)pC(\Psi)\left(\frac{1-\bar{A}(\Lambda)}{\Lambda}\right) + p\bar{A}(\Lambda) + \Lambda g\left(\frac{1-\bar{A}(\Lambda)}{\Lambda}\right)}{\Psi^b - \theta\bar{F}(\zeta(\Psi))\bar{B}(\zeta(\Psi))}\right] \tag{64}$$

where,

$$\begin{aligned}
P(0,\Psi) &= \frac{Nr(\Psi)}{Dr(\Psi)} \\
Nr(\Psi) &= [1-\Lambda\bar{P}_0][\Psi^b - \theta\bar{F}(\zeta(\Psi))\bar{B}(\zeta(\Psi))] \\
Dr(\Psi) &= \Psi^b - \theta\bar{F}(\zeta(\Psi))\bar{B}(\zeta(\Psi)) \\
&\quad - \left[\Lambda(1-g)pC(\Psi)\left(\frac{1-\bar{A}(\Lambda)}{\Lambda}\right) + p\bar{A}(\Lambda) + \Lambda g\left(\frac{1-\bar{A}(\Lambda)}{\Lambda}\right)\right] \\
&\quad \left[(1-\theta)(\bar{d}+d\Psi)\bar{B}(\zeta(\Psi)) + \tau\bar{D}(\zeta(\Psi))\left(\frac{1-\bar{B}(\zeta(\Psi))}{(\zeta(\Psi))}\right)\right. \\
&\quad \left.+\theta(1-\mu)\bar{F}(\zeta(\Psi))\bar{B}(\zeta(\Psi))\right]
\end{aligned} \tag{65}$$

4.1. Queue sizes distribution at a certain epoch:

The PGF is a of the queue size dist., at a random interval, is obtained by adding (60) to (63) with the idle term.

$$K(\Psi) = \frac{Nr(\Psi)}{Dr(\Psi)} \tag{66}$$

$$\begin{aligned}
Nr(\Psi) &= \Lambda P_0\zeta(\Psi)\left(\Psi^b - \theta\bar{F}(\zeta(\Psi))\bar{B}(\zeta(\Psi)) - [(1-g)pC(\Psi)(1-\bar{A}(\Lambda)) + p\bar{A}(\Lambda)\right.\\
&+g(1-\bar{A}(\Lambda))]\left[(1-\theta)(\bar{d}+d\Psi)\bar{B}(\zeta(\Psi)) + \tau\bar{D}(\zeta(\Psi))\left(\frac{1-\bar{B}(\zeta(\Psi))}{(\zeta(\Psi))}\right)\right.\\
&+\theta(1-\mu)\bar{F}(\zeta(\Psi))\bar{B}(\zeta(\Psi))]]) - (1-\bar{A}(\Lambda))\zeta(\Psi)[\Psi^b - \theta\bar{F}(\zeta(\Psi))\bar{B}(\zeta(\Psi))\\
&+\Lambda[(1-g)pC(\Psi)(1-\bar{A}(\Lambda)) + p\bar{A}(\Lambda) + g(1-\bar{A}(\Lambda))]\\
&[(1-\bar{B}\zeta(\Psi)) + \tau(1-\bar{B}\zeta(\Psi))(1-\bar{D}\zeta(\Psi)) + \theta\bar{B}\zeta(\Psi)(1-\bar{F}\zeta(\Psi))]]\\
&+(1-\bar{A}(\Lambda))\zeta(\Psi)[\Psi^b - \theta\bar{F}(\zeta(\Psi))\bar{B}(\zeta(\Psi))\\
&+\Lambda[(1-g)pC(\Psi)(1-\bar{A}(\Lambda)) + p\bar{A}(\Lambda) + g(1-\bar{A}(\Lambda))]]\\
&[(1-\bar{B}\zeta(\Psi)) + \tau(1-\bar{B}\zeta(\Psi))(1-\bar{D}\zeta(\Psi)) + \theta\bar{B}\zeta(\Psi)(1-\bar{F}\zeta(\Psi))]]
\end{aligned}$$

$$\begin{aligned}
Dr(\Psi) &= \zeta(\Psi)\Lambda\left\{\Psi^b - \theta\bar{F}(\zeta(\Psi))\bar{B}(\zeta(\Psi)) - [(1-g)pC(\Psi)(1-\bar{A}(\Lambda)) + p\bar{A}(\Lambda)\right.\\
&+g(1-\bar{A}(\Lambda))]\left((1-\theta)(\bar{d}+d\Psi)\bar{B}(\zeta(\Psi)) + \tau\bar{D}(\zeta(\Psi))\left(\frac{1-\bar{B}(\zeta(\Psi))}{(\zeta(\Psi))}\right)\right.\\
&\left.\left.+\theta(1-\mu)\bar{F}(\zeta(\Psi))\bar{B}(\zeta(\Psi))\right)\right\}
\end{aligned}$$

5. Stability Condition

The PGF needs to meet P(1)=1. Applying the L'Hopital rules and equating the expression to 1 results in the result that satisfies the requirement.

$$\begin{aligned}
&b - [(1-g)pE(I)(1-\bar{A}(\Lambda))][(1-\theta)(d+\bar{d}) + \theta(1-\mu)] + p(1-g)(1-\bar{A}(\Lambda))\\
&+p\bar{A}(\Lambda) + g(1-\bar{A}(\Lambda))[(1-\theta)(d+\bar{d})(1-\Lambda E(I)E(B)) + \tau E(B)\\
&-\theta\Lambda(1-\mu)E(I)A_1] + \Lambda\theta E(I)A_1
\end{aligned}$$

Now we can determine the prob., that are unknown. P(1)=1 is therefore fulfilled if

$$\begin{aligned}
&\Psi^b - \theta\bar{F}(\zeta(\Psi))\bar{B}(\zeta(\Psi)) - [(1-g)pC(\Psi)(1-\bar{A}(\Lambda)) + p\bar{A}(\Lambda) + g(1-\bar{A}(\Lambda))]\\
&\left[(1-\theta)(\bar{d}+d\Psi)\bar{B}(\zeta(\Psi)) + \tau\bar{D}(\zeta(\Psi))\left(\frac{1-\bar{B}(\zeta(\Psi))}{(\zeta(\Psi))}\right) + \theta(1-\mu)\bar{F}(\zeta(\Psi))\bar{B}(\zeta(\Psi))\right] > 0
\end{aligned}$$

$$\rho = \frac{\begin{aligned}&[(1-g)pE(I)(1-\bar{A}(\Lambda))][(1-\theta)(d+\bar{d}) + \theta(1-\mu)] + [p(1-g)(1-\bar{A}(\Lambda))\\ &+p\bar{A}(\Lambda) + g((1-\bar{A}(\Lambda))][(1-\theta)(d+\bar{d})(1-\Lambda E(I)E(B))\\ &+\tau E(B) - \theta\Lambda(1-\mu)E(I)A_1] + \Lambda\theta E(I)A_1\end{aligned}}{b} \tag{67}$$

then $\rho < 1$ is the condition to be satisfied for the existence of the SS for the model under consideration.

6. Performance Evaluation:

This section includes system performance metrics, a model stability study, and some unique system prob., while the system is in various states.
We obtain the following prob., if the system fulfills the stability requirement $\rho < 1$.

- Let P be the SS Prob., that the server is idle during the retrial time.

$$P = \lim_{\Psi \to 1} P(\Psi) = P(1) = \frac{(1-\theta)(1-\Lambda p_0)(1-\bar{A}(\Lambda))}{\Lambda(1-\theta) - [p(1-g)(1-\bar{A}(\Lambda)) + p\bar{A}(\Lambda) + g(1-\bar{A}(\Lambda))] \\ [(1-\theta)(d+\bar{d}) + \theta(1-\mu)]}$$

- If the server is busy, let Q be the SS Prob.,

$$Q = \lim_{\Psi \to 1} Q(\Psi)$$
$$Q(1) = E(B) \times \left\{ \frac{(1-\Lambda p_0)[p(1-g)(1-\bar{A}(\Lambda))E(I)]}{b + \Lambda\theta E(I)A_1 - [(1-g)pE(I)(1-\bar{A}(\Lambda))][(1-\theta)(d+\bar{d}) + \theta(1-\mu)] \\ +[p(1-g)(1-\bar{A}(\Lambda)) + p\bar{A}(\Lambda) + g(1-\bar{A}(\Lambda))] \\ (1-\theta)(d+\bar{d})(1-\Lambda E(I)E(B)) + \tau E(B) - \Lambda\theta(1-\mu)E(I)A_1} \right\}$$

- R ought to indicate the SS Prob., that the server is being repaired.

$$R = \lim_{\Psi \to 1} R(\Psi)$$
$$R(1) = \tau E(B)E(D) \times \left\{ \frac{(1-\Lambda p_0)[p(1-g)(1-\bar{A}(\Lambda))E(I)]}{b + \Lambda\theta E(I)A_1 - ([(1-g)pE(I)(1-\bar{A}(\Lambda))][(1-\theta)(d+\bar{d}) + \theta(1-\mu)] \\ +[p(1-g)(1-\bar{A}(\Lambda)) + p\bar{A}(\Lambda) + g(1-\bar{A}(\Lambda))] \\ [(1-\theta)(d+\bar{d})(1-\Lambda E(I)E(B)) + \tau E(B) - \Lambda\theta(1-\mu)E(I)A_1])} \right\}$$

- Using V as the SS Prob., we may assume that the server is on vacation.

$$V = \lim_{\Psi \to 1} V(\Psi)$$
$$V(1) = \theta E(F)E(I) \times \left\{ \frac{(1-\Lambda p_0)(-\Lambda E(B)p(1-g)(1-\bar{A}(\Lambda)) + p\bar{A}(\Lambda) + g(1-\bar{A}(\Lambda)) \\ + p(1-g)((1-\bar{A}(\Lambda))))}{b + \Lambda\theta E(I)A_1 - ([(1-g)pE(I)(1-\bar{A}(\Lambda))][(1-\theta)(d+\bar{d}) + \theta(1-\mu)] \\ +[p(1-g)(1-\bar{A}(\Lambda)) + p\bar{A}(\Lambda) + g(1-\bar{A}(\Lambda))][(1-\theta)(d+\bar{d}) \\ (1-\Lambda E(I)E(B)) + \tau E(B) - \Lambda\theta(1-\mu)E(I)A_1])} \right\}$$

6.1. **Average queue length:**

Computing at $\Psi = 1$ and differentiating (65) with regard to Ψ yields the mean number of users in the queue (L_q) under SS conditions.

$$L_q = \lim_{\Psi \to 1} \frac{d}{d\Psi} P(\Psi)$$
$$P'(1) = \frac{Nr''(1)Dr'(1) - Dr''(1)Nr'(1)}{2(Dr'(1))^2}$$

$$
\begin{aligned}
D^{'}(1) = & -\Lambda^2 E(I)\left\{\Psi^b - \theta\bar{F}(\zeta(\Psi))\bar{B}(\zeta(\Psi)) - [(1-g)pC(\Psi)\left(1-\bar{A}(\Lambda)\right) + p\bar{A}(\Lambda)\right. \\
& +g\left(1-\bar{A}(\Lambda)\right)]\left[(1-\theta)(\bar{d}+d\Psi)\bar{B}(\zeta(\Psi)) + \tau\bar{D}(\zeta(\Psi))\left(\frac{1-\bar{B}(\zeta(\Psi))}{(\zeta(\Psi))}\right)\right. \\
& +\theta(1-\mu)\bar{F}(\zeta(\Psi))\bar{B}(\zeta(\Psi))]\} \\
D^{''}(1) = & -\Lambda^2\left\{E(I(I-1))[1-\theta-[p-g(p-1)(1-\bar{A}(\Lambda))][1-\theta\mu]]\right. \\
& +2E(I)[b+\Lambda\theta E(I)A_1 - \left([(1-g)pE(I)(1-\bar{A}(\Lambda))][(1-\theta)(d+\bar{d})+\theta(1-\mu)]\right. \\
& +[p(1-g)(1-\bar{A}(\Lambda)) + p\bar{A}(\Lambda) + g(1-\bar{A}(\Lambda))][(1-\theta)(d+\bar{d})(1-\Lambda E(I)E(B)) \\
& +\tau E(B) - \Lambda\theta(1-\mu)E(I)A_1])]\}
\end{aligned}
$$

$$
\begin{aligned}
N'(1) = & -\Lambda^2 E(I)\left\{1-\theta-[p(1-g)(1-\bar{A}(\Lambda)) + p\bar{A}(\Lambda) + g(1-\bar{A}(\Lambda))][(1-\theta)(d+\bar{d})\right. \\
& +\theta(1-\mu)]\} + (1-\Lambda)\left\{-\Lambda E(I)(1-\theta)(1-\bar{A}(\Lambda)) + \Lambda^2 E(I)E(B)\right. \\
& (p(1-g)(1-\bar{A}(\Lambda)) + p\bar{A}(\Lambda) + g(1-\bar{A}(\Lambda))) + \theta\Lambda E(I)E(F)\} \\
N''(1) = & -\Lambda^2\left\{E(I(I-1))\left(1-\theta-A_4(1-\theta\mu)\right) + 2E(I)\left(b+\theta\Lambda E(I)A_1 - A_2(1-\theta\mu)\right.\right. \\
& +A_4[(1-\theta)\left(1-\Lambda E(I)E(B)\right)] + \tau E(B) - \theta\Lambda(1-\mu)E(I)A_1)\} \\
& +(1-\Lambda)\left\{(1-\bar{A}(\Lambda))[-(1-\theta)\Lambda E(I(I-1)) - \Lambda E(I)(b+\theta\Lambda E(I)A_1)]\right. \\
& +\Lambda^2 E(I)E(B)A_2 + \Lambda^2[E(I(I-1))E(B) + E(I)E^2(B)][p(1-g)((1-\bar{A}(\Lambda))) \\
& +p\bar{A}(\Lambda) + g((1-\bar{A}(\Lambda)))] - \theta\Lambda^2 E(B)E(F)E(I)^2 \\
& \left.+\Lambda\theta[E(I(I-1))E(F) + E(I)E^2(F)]\right\}
\end{aligned}
$$

where,

$$
\begin{aligned}
A_1 = & E(B) + E(F) \\
A_2 = & p(1-g)E(I)((1-\bar{A}(\Lambda))) \\
A_4 = & p - g(p-1)(1-\bar{A}(\Lambda))
\end{aligned}
$$

- The Little's formula (W_q) is used to determine how long an average customer waits in queue.

$$
W_q = \frac{L_q}{\Lambda E(I)}
$$

7. Practical application of the model:

The field of telecommunications networks may be able to use the suggested model. This system manages a lot of consumer telephone communications. Call takers are referred to as servers and callers as customers in this context. A consumer may elect to exit the system if he calls and discovers that all the servers are occupied (impatience). Customers wait in orbit while the server is overloaded, out of commission, or undergoing maintenance. If a server has any questions or concerns that fall outside of their area of expertise, they may need to refer them to other servers who are available or speak with a senior in order to acquire the answers. A service failure can be used to represent this circumstance. The speed at which the agent receives responses from the expert in this case is known as the repair rate. Additionally, the server may do various maintenance procedures known as "vacations." Additionally, after each customer's service is finished, dissatisfied customers may re-join the line and be classified as feedback consumers.

8. Numerical Results

In this section, we'll use MATLAB to demonstrate how different parameters affect observations of system behavior. The batch size distance of the arrivals in this section is geometry; with a mean of 2. Here, the exponential distance is followed by the service, vacation, and repair stages. By creating erroneous assumptions about the parameters, we make sure that the stability criterion is satisfied. Tables 1 to 3 present estimated values for our queueing system's utilization factor (ρ), average queue length (L_q), and average waiting time (W_q).

Table 1: *The effects of arrival rate* (Λ) *on* ρ, L_q, *and* W_q

$$g = 0.5,\ p = 1.5,\ E = 0.6,\ G = 2.2,\ \theta = 3,\ d = 3,$$
$$e = 0.6,\ \mu = 0.9,\ B = 1.5, D = 1,\ F = 0.7,\ z = 1, b = 2,\ \tau = 1.8$$

Arrival rate (Λ)	ρ	L_q	W_q
0.30	0.022680	3.505127	5.841879
0.31	0.096336	4.616984	7.446748
0.32	0.169992	6.010404	9.391256
0.33	0.243648	7.742148	11.730527
0.34	0.317304	9.878047	14.526540
0.35	0.390960	12.494169	17.848813
0.36	0.464616	15.678118	21.775164

Table 2: *The effects of the service rate* $\phi(\varsigma)$ *on* ρ, L_q, W_q

$$g = 7.8,\ p = 0.7,\ E = 0.8,\ G = 6,\ \theta = 1,\ d = 3,\ e = 4.6,$$
$$\mu = 0.7,\ D = 1,\ F = 0.7,\ \Lambda = 0.3,\ z = 1,\ b = 2,\ \tau = 1$$

service rate (B)	ρ	L_q	W_q
0.50	0.737200	0.223375	0.372292
0.51	0.687360	0.189247	0.315412
0.52	0.637520	0.160488	0.267480
0.53	0.587680	0.135989	0.226649
0.54	0.537840	0.114943	0.191572
0.55	0.488000	0.096748	0.161247
0.56	0.438160	0.080947	0.134912

Table 3: *The effects of the Breakdown rate* (τ) *on* ρ, L_q, W_q

$$g = 0.2,\ p = 0.7,\ E = 2.9,\ G = 9,\ \theta = 1,$$
$$d = 7,\ e = 8.6,\ \mu = 0.2,\ B = 7, D = 2,\ F = 0.7,\ \Lambda = 0.4, z = 2,\ b = 4$$

breakdown rate (τ)	ρ	L_q	W_q
1.0	0.264592	5.619523	7.024404
1.1	0.303092	6.095933	7.619916
1.2	0.341592	6.575234	8.219042
1.3	0.380092	7.057428	8.821784
1.4	0.418592	7.542516	9.428145
1.5	0.457092	8.030500	10.038125
1.6	0.495592	8.521381	10.651727

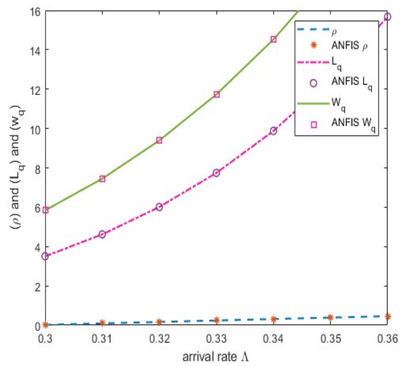
The two-dimensional graph that represents the system measurement of performance is shown in Figure 1 $(a-c)$.

- The figure 1 (a) demonstrates how the utilization factor (ρ), estimated queue length (L_q), and expected waiting time (W_q) all increase as the arrival rate (Λ) does.
- The figure 1 (b) shows that while the utilization factor (ρ) decreases, the service rate $\phi(\varsigma)$ rises. Expected waiting time (W_q) and queue length (L_q) decrease.
- The breakdown rate (τ), utilization factor (ρ), expected queue size (L_q), and expected waiting time (W_q) all show increasing trends in the figure 1 (c).

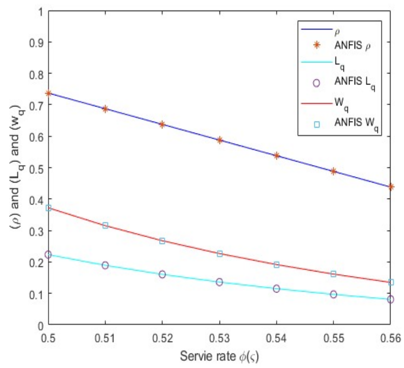
The three-dimensional graph of the system indicators of performance is shown in Figure 2 $(a-c)$.

- The surface in figure 2 (a) shows the growth of the arrival rate (Λ), estimated length of the line (L_q), and estimated wait time (W_q).
- Figure 2 (b) shows that as the service rate $\phi(\varsigma)$ rises, the estimated queue size (L_q) and waiting time (W_q) both decrease.
- Figure 2 (c) shows that as the breakdown rate τ rises, expected queue lengths (L_q) and waiting times (W_q) also rise.

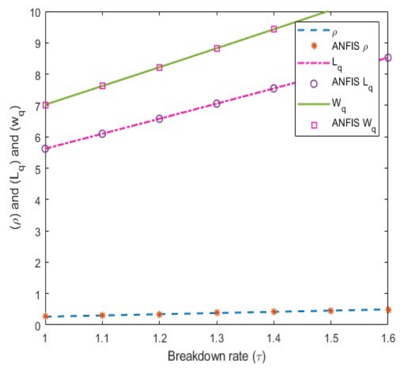
The numerical results above allow us to determine the influence of attributes on the system's evaluation criteria, and we can be assured that they are representative of realistic conditions.



(a) ρ, L_q, W_q verses arrival rate Λ

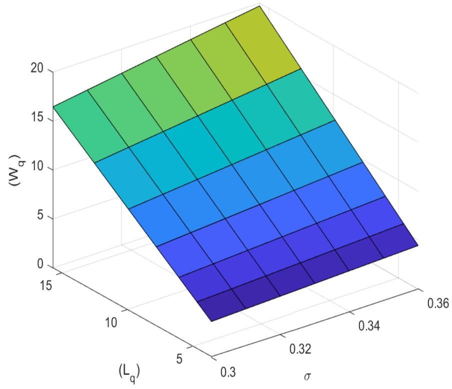


(b) ρ, L_q, W_q verses Service rate $\phi(\varsigma)$

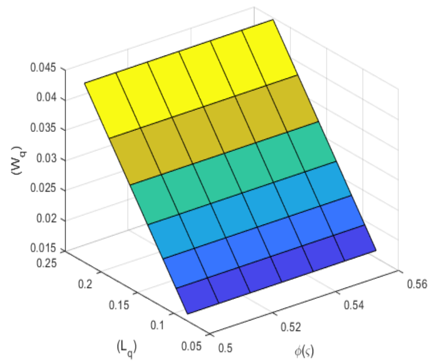


(c) ρ, L_q, W_q verses Breakdown rate τ

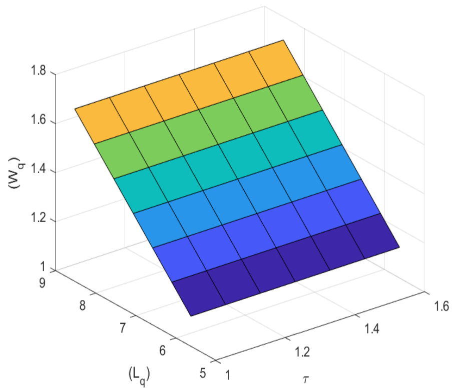
Figure 1: *2D representation effects*



(a) ρ,L_q, W_q verses arrival rate Λ



(b) ρ,L_q, W_q verses Service rate $\phi(\varsigma)$



(c) ρ,L_q, W_q verses Breakdown rate τ

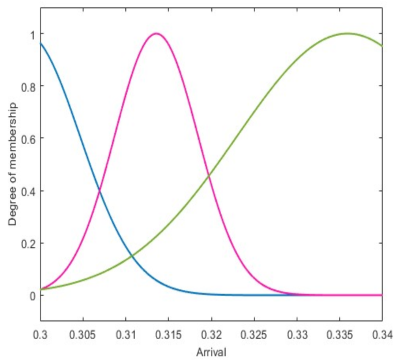
Figure 2: *3D representation effects*

9. Adaptive Neuro-Fuzzy Inference System (ANFIS)

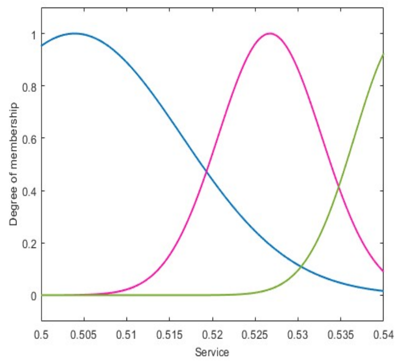
The ANFIS modal is actually applicable in a variety of fields, such as modes of transport, congestion, telecommuting, atmospheric research, etc. Artificial neural networks are used in communications networks to accomplish a variety of goals, including an increase in customers, expense reduction, shorter wait times, etc. With variations in arrival rates while on vacation, service rates, repair rates, and repair to busy rates, the current modal allows us to examine the impatience of the client while they wait for the service.

A very helpful approach for ANFIS is created by combining soft computing methods, artificial neural networks (ANNs), and fuzzy systems (FS). We are showing a simplified idea of the ANFIS architecture by using the fuzzy parameters. We can implement an ANFIS input-output function and input-output data pairs as fuzzy if-then logic. The fuzzy toolbox of MATLAB software can be utilized for contrasting the computational findings with the implementation of an ANFIS network.

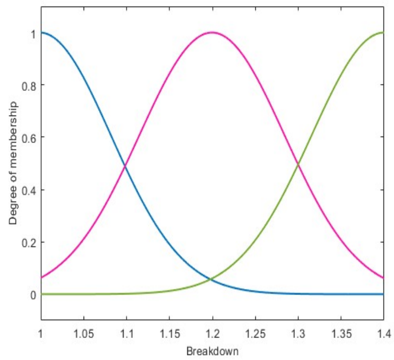
The input parameters and the membership function are assumed to be the Λ, $\phi(\varsigma)$, and τ Gaussian functions in order to produce computational results based on ANFIS. (see Fig. 3a, b, c). It is assumed that the linguistic values are low, moderate, or high. Tick marks are placed over the curves made for the results obtained analytically in Figure 1a, 1b and 1c to indicate the results produced by the ANFIS approach for the queue size. The figures show that the numerical outcomes produced using the Runge-Kutta method and the ANFIS results are nearly identical.



(a) ρ, L_q, W_q verses arrival rate Λ



(b) ρ, L_q, W_q verses Service rate $\phi(\varsigma)$



(c) ρ, L_q, W_q verses Breakdown rate τ

Figure 3: *ANFIS representation effects*

10. Cost Optimization:

The term "optimization" describes the method of determining the set of parameters for an objective function that produces the highest or lowest outcome. The continual, business-oriented activity known as "cost optimization" aims to reduce expenditures and costs while raising the organization's value. Standardizing, streamlining, and rationalizing platforms, application development, procedures, and services are all part of this process, along with establishing the most competitive possible terms and prices for all business transactions. The operating cost and profit of a system are closely tied in real-world situations. Therefore, the system's designers or managers place a lot of emphasis on reducing operational expenses per unit of time in order to enhance the system's earnings. Our objective is to identify the best cost per unit of time (TC) characteristics. In order to do this and increase the cost-effectiveness of our developed approach, we will build our competence in this field.
C_h - Holding expense for every user in the system per unit of time.
C_b - The cost for each unit of time the server is turned on and used.
C_v - The cost imposed on the server in vacation mode per unit of time.
C_r - The cost to repair the server after its failure, calculated per unit of time.
C_1 - The cost per unit over a busy time.
C_2 - Cost for each unit of time used over the vacation period.

$$TC = C_h L_q + C_v V + C_b Q + C_r R + C_1 \gamma_b + C_2 \gamma_v$$

The TC problem is solved using metaheuristic optimisation methods including PSO, ABC, and GA. In view of the importance of cost optimisation, this study was conducted using the global search optimisation algorithms particle swarm optimisation (PSO), artificial bee colony (ABC), and genetic algorithms (GA), each of which is separately described in three different subsections of this section. If the algorithm's assumptions are correct, local search techniques frequently offer the level of computer efficiency required to find the global optimal. Tables 5 to 7 display the effects of Λ, τ, and ϕ on TC* using PSO, ABC, and GA.

Table 4: ***Cost sets for optimal policy***

Cost sets	C_h	C_v	C_b	C_r	C_1	C_2
1	10	9	7	6	7	8
2	8	4	6	4	8	9
3	7	6	8	3	9	6

10.1. Particle Swarm Optimization (PSO)

One of the meta-heuristic methods used to solve optimization issues is the particle swarm optimization (PSO) technique, which has been employed successfully in a number of single objective optimization problems. Kennedy and Eberhart first proposed this algorithm. The PSO algorithm has the benefit of being simple to implement and apply for solving different function optimization problems, which can be categorized as function minimization or maximization problems.

Table 5: ***Effect of*** $\Lambda, \tau, \phi(\varsigma)$ ***on*** TC^* ***using PSO***

$g = 0.2$, $p = 0.7$, $G = 9$, $\theta = 0.95$, $d = 7$, $e = 8.6$,
$c = 0.2$, $B = 7$, $D = 2$, $\tau = 1.6$, $b = 4$

Cost sets		TC^*		
		Cost set 1	Cost set 2	Cost set 3
Λ	0.4	149.1752	133.0711	127.2781
	0.5	162.6882	143.5244	136.3697
	0.6	173.5857	152.1798	143.7058
τ	1.6	149.1752	133.0711	127.2781
	1.7	161.5959	141.8755	134.7960
	1.8	175.6139	151.7985	143.2466
$\phi(\varsigma)$	7	149.1752	133.0711	127.2781
	8	184.5033	159.0594	151.0219
	9	230.2302	192.6975	181.7546

10.2. Artificial Bee Colony(ABC)

One of Dervis Karaboga's most recent algorithms—created in 2005—is called the Artificial Bee Colony and was modeled after the cunning behaviour of honey bees. Basic process indicators like colonies and highest levels are essentially all that are used. Like PSO and differential evolutionary approaches, it is equally simple to comprehend. The search for huge areas of nectar-containing food sources, and ultimately the one with the most nectar, is the bees' main goal. This population-based search approach is the main one used by ABC. The cost of the suggested structure is decreased through a process known as ABC.

Table 6: ***Effect of*** $\Lambda, \tau, \phi(\varsigma)$ ***on*** TC^* ***using ABC***

$g = 0.2$, $p = 0.7$, $G = 9$, $\theta = 0.95$, $d = 7$, $e = 8.6$,
$c = 0.2$, $\Lambda = 0.4$, $D = 2$, $\tau = 1.6$, $b = 4$

Cost sets		TC^*		
		Cost set 1	Cost set 2	Cost set 3
Λ	0.4	108.0030	108.8585	109.6965
	0.5	112.6458	113.7397	114.0666
	0.6	115.8805	117.2631	116.9601
τ	1.6	108.0030	108.8585	109.6965
	1.7	112.4344	113.2737	114.1395
	1.8	117.4656	117.8425	118.7674
$\phi(\varsigma)$	7	108.0030	108.8585	109.6965
	8	120.7245	120.9982	122.5394
	9	138.2173	134.5400	136.4184

10.3. Genetic Algorithm (GA)

The genetic algorithm, created in the 1960s and 1970s by Bremermann, Holland, and their colleagues, is a technique for addressing optimization problems brought on by natural selection, the mechanism that promotes evolution in biology. They are frequently employed to deliver superior solutions to stochastic search issues. The full procedure serves as a representation of the criteria for choice that were used to select the people who would make the best parents for the coming human generation.

Table 7: ***Effect of*** $\Lambda, \tau, \phi(\varsigma)$ ***on*** TC^* ***using GA***

$g = 0.2$, $p = 0.7$, $G = 9$, $\theta = 0.95$, $d = 7$, $e = 8.6$,
$c = 0.2$, $\Lambda = 0.4$, $D = 2$, $B = 7$, $b = 4$

Cost sets		TC^*		
		Cost set 1	Cost set 2	Cost set 3
Λ	0.4	152.5341	131.8364	124.6592
	0.5	163.2414	141.3111	131.8853
	0.6	170.0165	148.0781	136.4720
τ	1.6	152.5341	131.8364	124.6592
	1.7	167.5959	142.6723	133.7200
	1.8	184.4744	154.7915	143.8279
$\phi(\varsigma)$	7	152.5341	131.8364	124.6592
	8	192.8582	162.3320	151.7762
	9	243.6750	200.7627	185.9493

10.4. Analogy of PSO, ABC and GA

This section compares the three approaches—particle swarm optimization (PSO), artificial bee colony (ABC), and genetic algorithm (GA)—to determine which has the least expense using the corresponding MATLAB programs. Then, one by one, the MATLAB programs for each of the aforementioned algorithms are run. We found that all three programs generated values that were nearly identical. Because of this, the three solutions are nearly comparable in terms of their optimum results and the fewest associated costs. It proves the reliability (local) and potency of these three simple techniques. Any technique can be used to calculate the optimal cost; however, PSO outperforms all others in comparison to our model. Because PSO has so many advantages, we have found that it is the best approach out of all of them. It performs well in global queries, requires a small number of arguments, is easy to configure, and is unaffected by design variable scalability. In addition to suffering sluggish convergence in a concentrated searching region, PSO has a tendency to lead to swift and early convergence in mid-optimal locations (being able to impair local search capabilities).

10.5. Convergence in PSO, ABC and GA

After employing an optimization methodology like PSO, ABC, or GA, it is crucial to comprehend whether a particle recovers to normal or not and when it will roam around in search of a better solution. As a result, convergence is a significant component of cost evaluation. A statistical analysis (Fig. 4) of the outcomes demonstrates that ABC exceeds the PSO approach. For the whole standard optimization, ABC had fewer functional evaluations overall than PSO. The findings demonstrate that PSO converges more quickly. ABC cannot be employed if a speedy result is required for time-sensitive applications.

The study shows the applicability of our concept to real-world situations. Some of the analysts' financial issues will be partially overcome once they know how much the system will cost overall. The current situation may heavily rely on the cost-benefit assessment that was produced, which serves to illustrate the logic of our strategy and aid network administrators and specialists in lowering the issue of communications services that explicitly deal with blocking.

11. Conclusion:

This paper investigates the $M^X/G(a,b)/1$ retrial queue with random failure and feedback under extended Bernoulli vacation with impatient customers. The SVT is utilized to determine indicators of efficiency for the various system stages. The efficiency of the system is then evaluated after considering the effects of various parameters. Finally, we gave a thorough explanation of the ANFIS. PSO, ABC, and GA are also used to compute the total cost. In an effort to find the best offer, these techniques compare and contrast the outcomes. The impetus for this study came from the prospective applications for the developed model, such as call centres, wireless networks, or telecommunication infrastructures, which might be powered by controlled precision test queueing systems to provide outstanding service at low prices. The simple mail transfer protocol utilizes a way to convey the messages between the mail servers. The recommended approach might be used in an email system's transfer model.

Declarations:

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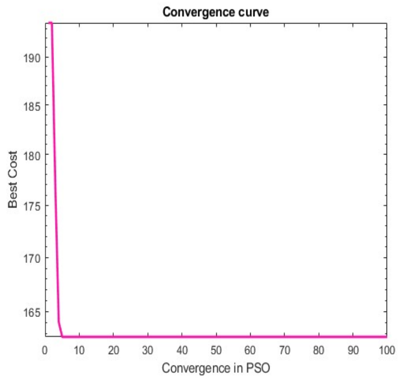
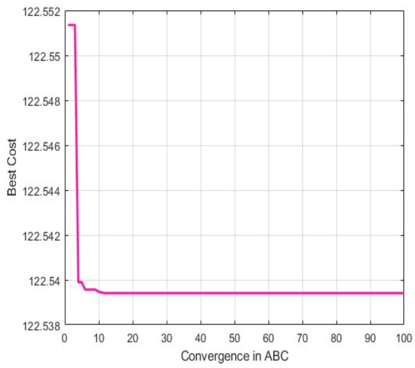
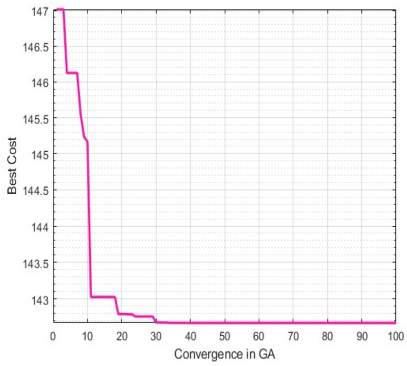


(a) TC verses Arrival rate (Λ) in PSO



(b) TC verses Service $\phi(\varsigma)$ in ABC



(c) TC verses Breakdown rate (τ) in GA

Figure 4: *Cost Optimization effects*

the public, commercial, or not-for-profit sectors.

Conflicts of interest: The authors declare no conflict of interest.

Data availability: Not applicable

Authors contribution: All the authors made substantial contributions to the conception or design of the work.

Competing Interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] Chaudhry, M. and Templeton, J. (1983). A first course in bulk queues. *John Wiley & Sons*. http://dx.doi.org/10.1016/j.asej.2016.08.025.

[2] Bailey, N.T. (1954). On queueing processes with bulk service. *J R Stat Soc Ser A Stat Soc., Series B (Methodological)*. 80-87. https: //doi.org/10.1111/j.2517-6161.1954.tb00149.x

[3] Sumitha, D. and Udaya Chandrika, K. (2012). Retrial queuing system with starting failure, single vacation and orbital search.*Int.J. Comput. Appl.* 40.13, 29-33.

[4] Radha,J., Rajadurai, P., Indhira, K. and Chandrasekaran, V.M. (2014). A batch arrival retrial queue with K-optional stages of service, Bernoulli feedback, single vacation and random breakdown. *Glob. J. Pure Appl. Math.* 10.2, 265-283.

[5] Krishna Kumar,B., Pavai Madheswari,S. and Vijayakumar,A. (2002). The M/G/1 retrial queue with feedback and starting failures. *Appl.Math. Model*. 26.11, 1057-1075.https://doi.org/10.1016/S0307-904X(02)00061-6

[6] Yang, Tao, Hui Li. (1994). The M/G/1 retrial queue with the server subject to starting failures. *Queueing Syst.* 16.1, 83-96. https://doi.org/10.1007/BF01158950

[7] Mokaddis, G.S., Metwally,S.A. and Zaki,B.M. (2007). A feedback retrial queuing system with starting failures and single vacation. *J. Appl. Sci. Eng.* 10.3, 183-192. https://doi.org/10.6180/jase.2007.10.3.01

[8] Bouchentouf, Amina Angelika, Abdelhak Guendouzi. (2021). Single server batch Arrival Bernoulli feedback queueing system with waiting server, K-variant vacations and impatient customers. *Oper. Res.Forum.* Vol.2. No.1. Springer International Publishing. https://doi.org/10.1007/s43069-021-00057-0.

[9] Ayyappan, G. and Sathiya, K. (2013). Transient analysis of batch arrival feedback retrial queue with starting failure and Bernoulli vacation. *Math. Model. Anal.* 3.8, 60-67.

[10] Ayyappan, Govindhan, Udayageetha, J. (2020). Transient Analysis of M [X1], M [X2]/G1, G2/1 retrial queueing system with priority services, working breakdown, start up/close down time, Bernoulli vacation, reneging and balking. *Pak. J. Stat. Oper.* 203-216. https://doi.org/10.18187/pjsor.v16i1.2181.

[11] Kulkarni, Vidyadhar, G., Bong Dae Choi. (1990). Retrial queues with server subject to breakdowns and repairs. *Queueing syst.* 7.2, 191-208. https://doi.org/10.1007/BF01158474

[12] Ayyappan, G. and Shyamala,S. (2013). M[X]/G/1 with Bernoulli Schedule Server Vacation Random Break Down and second optional Repair. *J. Comp., & Model.* 3, 159-175.

[13] Ke, Jau-Chuan, Tzu-Hsin Liu, Siping Su, Zhe-George Zhang. (2022). On retrial queue with customer balking and feedback subject to server breakdowns. *Commun. Stat.* 51.17, 6049-6063. https://doi.org/10.1080/03610926.2020.1852432

[14] Takacs, L. (1963). A single-server queue with feedback. *Bell Syst.Tech.J.* 2, 505-519.

[15] Rajadurai, P., Saravanarajan, M., Chandrasekaran, V.M. and Indhira, K. (2015). An M/(G1, G2)/1 Feedback Retrial Queue with Two Phase Service, Variant Vacation Policy Under Delaying Repair for Impatient Customer. *Int. J. Fuzzy Math.* Arch 6, 45-55.

[16] D'Arienzo, M.P., Dudin, A.N., Dudin,S.A. and Manzo, R. (2020). Analysis of a retrial queue with group service of impatient customers. *J Ambient Intell Humaniz Comput.* 11.6, 2591-2599. https://doi.org/10.1007/s12652-019-01318-x

[17] Nila, M. and Sumitha, D. Batch Arrival Retrial Queueing Model with Starting Failures, Customer Impatience, Multi Optional Second Phase and Orbital Search.

[18] Ayyappan, G., Thamizhselvi, P., Somasundaram, B. and Udayageetha, J. (2021). Analysis of an retrial queueing system with priority services, working breakdown, Bernoulli vacation, admission control and balking. *Int. j. stat. manag. syst.* 24.4, 685-702. https://doi.org/10.1080/09720529.2020.1744812

[19] Ayyappan, G. and Nirmala, M. (2021). Analysis of customer's impatience on bulk service queueing system with unreliable server, setup time and two types of multiple vacations. *Int. J. Ind. Syst.* 38.2, 198-222. https://doi.org/10.1504/IJISE.2021.115321

[20] Sethi, R., Jain, M., Meena, R. K. and Garg, D. (2020). Cost optimization and ANFIS computing of an unreliable M/M/1 queueing system with customers' impatience under n-policy. *Int. J. Appl. Comput. Math.* 6, 1-14. https://doi.org/10.1007/s40819-020-0802-0

[21] Miss Vaishnawi, Shweta Upadhyaya and Rakhee Kulshrestha (2022). Optimal Cost Analysis for Discrete-Time Recurrent Queue with Bernoulli Feedback and Emergency Vacation. *Int. J. Appl. Comput. Math.* 8.5, 254. https://doi.org/10.1007/s40819-022-01445-8