Transmuted Exponentiated Kumaraswamy Distribution

Jeena Joseph[1] and Meera Ravindran[2]

•

Department of Statistics
St. Thomas' College (Autonomous), Thrissur, India

sony.jeena@gmail.com, meeraravindran798@gmail.com

Abstract

In this paper, a generalization of the Exponentiated Kumaraswamy distribution referred to as the Transmuted Exponentiated Kumaraswamy distribution is proposed. The new transmuted distribution is developed using the quadratic rank transmutation map. The mathematical properties of the new distribution is provided. Explicit expressions are derived for the moments, incomplete moments, moment generating function, quantile function, entropy, mean deviation and order statistics. Survival analysis is also performed. The distribution parameters are estimated using the method of maximum likelihood. Simulation of random variables is performed in order to investigate the performance of the estimates. An analysis using real life data is conducted to demonstrate the usefulness of the proposed distribution.

Keywords: Bonferroni and Lorenz curves; Hazard function; Maximum likelihood estimation; Moments; Transmuted Exponentiated Kumaraswamy Distribution; Transmuted family.

1. Introduction

In probability theory and Statistics, a probability distribution is a mathematical function that provides the probabilities of the occurrence of various possible outcomes in an experiment.In modelling our world, probability distributions helps us, thus allowing to obtain estimates of the probability of a certain event to occur, or estimate it's variability of happening. Many distributions have been discovered suitable for many different purposes.The recognition of the proper distribution will allow a correct application of a model that would easily forecast the probability of an event.

The Kumaraswamy probability distribution was developed by Kumaraswamy [11] which is closely related to the beta distribution. It is often termed as a Beta-like distribution. But, in some situations the Kumaraswamy distribution is simpler to use and more amenable. Since it's cumulative distribution function (cdf) has a closed form, it is often preferred over the Beta distribution. Moreover, unlike the beta cdf, the cdf of Kumaraswamy distribution does not contain the incomplete Beta function, which makes it much simple to work with and the new properties of Kumaraswamy distribution such as the Kumaraswamy variables show closeness under exponentiation and under linear transformation was studied by Mitnik [16]. In numerous areas such as hydrology, electrical, civil, mechanical and financial engineering, Kumaraswamy distribution has secured appreciable interest, see Mohammed [17]. Some generalized beta distributions of the second kind having desirable application features in hydrology and meteorology was studied by Mielke and Johnson [15] and Fletcher and Ponnambalam [7]. Several authors studied more general properties of Kumaraswamy Distribution, see Silva et. al. [18], ZeinEldin et.al. [23], Dey et. al. [6], Hassan and Elgarhy [8] and Simbolan et. al. [19]. Usman et. al. [22] derived a new Weibull-Kumaraswamy distribution and studied its properties and applications. Another distribution named Kumaraswamy- Pareto distribution was derived by Bourguignon et. al. [5]. A bivariate Kumaraswamy (BVK) distribution with marginals being Kumaraswamy distributions

was introduced by Barreto-Souza and Lemonte [4]. Another new three-parameter probability model named Exponentiated Kumaraswamy distribution and its basic statistical properties and its applications using real-life datasets were studied by Lemonte et. al. [12].Also, a three-parameter weighted kumaraswamy distribution was proposed by Abd El-Monsef et. al. [1] for modeling some biological data, which could accommodate increasing and decreasing hazard rate function with bathtub shape. AL-Fattah et. al. [2] introduced the Inverted Kumaraswamy Distribution, it's properties and estimation.
The transmuted family of distributions has been receiving a high attention over the past few years. A new technique for adding a new parameter to an already existing distribution that would provide more flexibility to this distribution by Shaw and Buckley [20]. The method is named as quadratic rank transmutation map (QRTM). It includes the parent distribution as a special case and makes it more flexible to model different types of data. The generated family is also called the transmuted extended distribution. This method have been considered by several authors for different disributions, see Al-Kadim et. al. [3], Khan et. al. [10], F. Merovci [13,14] and Sherwaia et. al. [21]. Transmuted Kumaraswamy distribution and its basic statistical properties were discussed by Khan et. al. [9]. The new model was found to outperform some existing baseline distributions when applied to real-life data sets.
A random variable X is said to have an Exponentiated Kumaraswamy distribution with parameters $\alpha, \beta, \gamma > 0$ if its probability density function (pdf) is given by

$$f(x;\alpha,\beta,\gamma) = \alpha\beta\gamma x^{\alpha-1}(1-x^{\alpha})^{\beta-1}[1-(1-x^{\alpha})^{\beta}]^{\gamma-1}, \qquad 0 < x < 1, \quad \alpha,\beta,\gamma > 0 \tag{1}$$

and the respective cdf is

$$F(x;\alpha,\beta,\gamma) = [1-(1-x^{\alpha})^{\beta}]^{\gamma-1}, \;\; 0 < x < 1, \;\; \alpha,\beta,\gamma > 0 \tag{2}$$

In this paper, a generalization of the Exponentiated Kumaraswamy distribution referred to as the Transmuted Exponentiated Kumaraswamy distribution is proposed. The new transmuted distribution is obtained using the quadratic rank transmutation map introduced by Shaw and Buckley [20]. According to this method, transmutation maps consists of the functional composition of the cumulative distribution function of one distribution with the inverse cumulative distribution (quantile) function of another. A comprehensive account of the mathematical properties of the new distribution is provided.
The organization of this paper is as follows: Section 2 explains the quadratic rank transmutation method. In section 3, the pdf and cdf of our new model, Transmuted Exponentiated Kumaraswamy distribution is given and provide the graphical presentation of its pdf, cdf, survival function and hazard rate function for selected values of the parameters. Section 4 provides its statistical properties such as moments, moment generating function, characteristic function, quantile function, incomplete moments, entropy, mean deviation and order statistics. Estimation of parameters of the distribution is done using maximum likelihood estimation is also included in this section. In section 5, a simulation study is included which is done to validate the estimates and a real data analysis illustrates the practicability of the proposed distribution. Finally, the summary and conclusions are stated in section 6.

2. Transmuted Distribution

A random variable X is said to have transmuted distribution if its cumulative distribution function(cdf) satisfy the relation,

$$F(x) = G(x)[(1+\lambda) - \lambda G(x)], \quad |\lambda| < 1 \tag{3}$$

which on differentiation yields the corresponding pdf

$$f(x) = g(x)[1+\lambda - 2\lambda G(x)] \tag{4}$$

where G(x) and g(x) are the cdf and pdf of the base distribution. Observe that at λ = 0, we have the distribution of the base random variable.

3. Transmuted Exponentiated Kumaraswamy Distribution

Using (3) and (4) we have the cdf of Transmuted Exponentiated Kumaraswamy (TEKw) distribution

$$F(x;\alpha,\beta,\gamma,\lambda) = (1+\lambda)[1-(1-x^{\alpha})^{\beta}]^{\gamma} - \lambda[1-(1-x^{\alpha})^{\beta}]^{2\gamma} \tag{5}$$

with shape parameters $\alpha, \beta, \gamma > 0$ and the transmuting parameter $|\lambda| < 1$.
Hence, the pdf of TEKw distribution is given as,

$$f(x;\alpha,\beta,\gamma,\lambda) = \alpha\beta\gamma x^{\alpha-1}(1-x^{\alpha})^{\beta-1}[1-(1-x^{\alpha})^{\beta}]^{\gamma-1}[(1+\lambda)-2\lambda[1-(1-x^{\alpha})^{\beta}]^{\gamma}] \tag{6}$$

where $\alpha, \beta, \gamma > 0$ and $|\lambda| < 1$.

Note that the transmuted Exponentiated Kumaraswamy distribution is an extended model to analyze more complex data and it generalizes some of the widely used distributions.The Exponentiated Kumaraswamy distribution is clearly a special case for λ = 0.

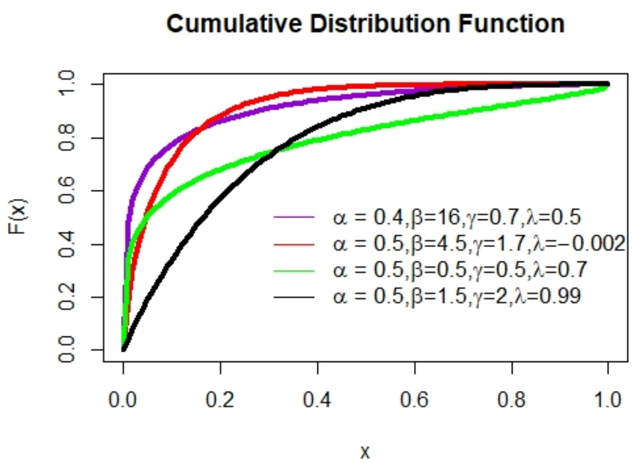

Figure 1: *Plot of the cumulative distribution function for different values of parameters.*

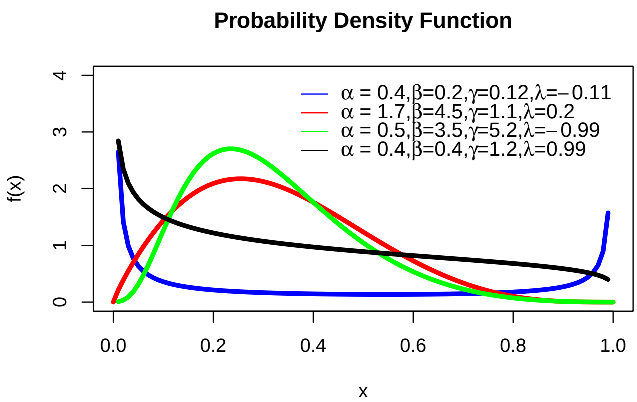

Figure 2: *Plot of the density function for different values of parameters.*

4. Statistical Properties

4.1. Moments

Let X be a random variable with its pdf given by (6), then its r^{th} raw moment is given by,

$$\begin{aligned}\mu_r^{'} &= E(X^r)\\ &= \int_0^1 x^r f(x)dx\\ &= \int_0^1 x^r \alpha\beta\gamma x^{\alpha-1}(1-x^\alpha)^{\beta-1}[1-(1-x^\alpha)^\beta]^{\gamma-1}[(1+\lambda)-2\lambda[1-(1-x^\alpha)^\beta]^\gamma]dx\end{aligned} \tag{7}$$

splitting into two parts,

$$= \alpha\beta\gamma[\int_0^1 x^{r+\alpha-1}(1-x^\alpha)^{\beta-1}(1+\lambda)[1-(1-x^\alpha)^\beta]^{\gamma-1}dx - \int_0^1 2\lambda x^{r+\alpha-1}(1-x^\alpha)^{\beta-1}[1-(1-x^\alpha)^\beta]^{2\gamma-1}dx]$$

Substituting u=x^α, du=$\alpha x^{\alpha-1}$ dx

and evaluating both parts using the series expansions

$$\begin{aligned}[1-(1-x^\alpha)^\beta]^{\gamma-1} &= \sum_{i=0}^{\infty}\frac{(-1)^i\Gamma\gamma}{\Gamma(\gamma-i)i!}(1-x^\alpha)^{\beta i}\\ &= \sum_{i=0}^{\infty}\frac{(-1)^i\Gamma\gamma}{\Gamma(\gamma-i)i!}\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(\beta j+1)}{\Gamma(\beta j+1-j)j!}x^{\alpha j}\\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}\Gamma\gamma\Gamma(\beta j+1)}{i!j!\Gamma(\gamma-i)\Gamma(\beta i-j+1)}x^{\alpha j}\\ &= Mx^{\alpha j}\end{aligned} \tag{8}$$

and,

$$\begin{aligned}[1-(1-x^\alpha)^\beta]^{2\gamma-1} &= \sum_{i=0}^{\infty}\frac{(-1)^i\Gamma 2\gamma}{\Gamma(2\gamma-i)i!}(1-x^\alpha)^{\beta i}\\ &= \sum_{i=0}^{\infty}\frac{(-1)^i\Gamma 2\gamma}{\Gamma(2\gamma-i)i!}\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(\beta i+1)}{\Gamma(\beta i+1-j)j!}x^{\alpha j}\\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}\Gamma 2\gamma\Gamma(\beta i+1)}{i!j!\Gamma(2\gamma-i)\Gamma(\beta i-j+1)}x^{\alpha j}\\ &= Nx^{\alpha j}\end{aligned} \tag{9}$$

The r^{th} moment is given by,

$$\begin{aligned}E(X^r) &= \alpha\beta\gamma[\frac{M(1+\lambda)}{\alpha}B(\frac{r}{\alpha}+j+1,\beta)-\frac{2\lambda N}{\alpha}B(\frac{r}{\alpha}+j+1,\beta)]\\ &= \frac{\alpha\beta\gamma}{\alpha}[M(1+\lambda)B(\frac{r}{\alpha}+j+1,\beta)-2\lambda NB(\frac{r}{\alpha}+j+1,\beta)]\\ &= \beta\gamma B(\frac{r}{\alpha}+j+1,\beta)[M+\lambda(M-2N)]\end{aligned} \tag{10}$$

where,
$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$ is the Beta function and,

$$
\begin{aligned}
M &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+j}\Gamma\gamma\Gamma(\beta j+1)}{i!j!\Gamma(\gamma-i)\Gamma(\beta i-j+1)} \\
N &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+j}\Gamma 2\gamma\Gamma(\beta i+1)}{i!j!\Gamma(2\gamma-i)\Gamma(\beta i-j+1)}
\end{aligned} \tag{11}
$$

Therefore, the expected value E(X) and variance Var(X) of a transmuted exponentiated Kumaraswamy random variable X are respectively, given by

$$E(X) = \beta\gamma B(\frac{1}{\alpha}+j+1,\beta)[M+\lambda(M-2N)] \tag{12}$$

and

$$V(X) = \beta\gamma B(\frac{2}{\alpha}+j+1,\beta)[M+\lambda(M-2N)] - (\beta\gamma B(\frac{1}{\alpha}+j+1,\beta)[M+\lambda(M-2N)])^2 \tag{13}$$

4.2. Moment Generating Function

The moment generating function of TEKw($\alpha,\beta,\gamma,\lambda$) is given by,

$$M_X(t) = E(e^{tx}) = \int_0^1 e^{tx} f(x)dx$$

Using the Taylor series expansion,

$$e^{tx} = \sum_{n=0}^{\infty} \frac{(tx)^n}{n!}$$

$$
\begin{aligned}
M_X(t) &= \int_0^1 \sum_{n=0}^{\infty} \frac{(tx)^n}{n!} f(x)dx \\
&= \sum_{n=0}^{\infty} \frac{(tx)^n}{n!} \int_0^1 x^n f(x)dx \\
&= \sum_{n=0}^{\infty} \frac{(tx)^n}{n!} \int_0^1 x^n \alpha\beta\gamma x^{\alpha-1}(1-x^{\alpha})^{\beta-1}[1-(1-x^{\alpha})^{\beta}]^{\gamma-1}[(1+\lambda)-2\lambda[1-(1-x^{\alpha})^{\beta}]^{\gamma}]dx
\end{aligned} \tag{14}
$$

Splitting into two parts, and evaluating, the moment generating function of TEKw($\alpha,\beta,\gamma,\lambda$) is given by,

$$M_X(t) = \beta\gamma \sum_{n=0}^{\infty} \frac{t^n}{n!} B(\frac{n}{\alpha}+j+1)[M+\lambda(M-2N)] \tag{15}$$

where $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$ is the Beta function and M and N are given by (11).

4.3. Characteristic Function

The characteristic function of TEKw($\alpha,\beta,\gamma,\lambda$) is given by,

$$\phi_X(t) = \beta\gamma \sum_{n=0}^{\infty} \frac{(it)^n}{n!} B(\frac{n}{\alpha}+j+1)[M+\lambda(M-2N)] \tag{16}$$

where $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$ is the Beta function and M and N are given by (11).

4.4. Quantile Function

$$Q(p) = [1 - [1 - [\frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda p}}{2\lambda}]^{1/\gamma}]^{1/\beta}]^{1/\alpha} \tag{17}$$

Then,
Median, 2^{nd} quartile of TEKw(α,β, γ,λ) is obtained by substituting p =1/2 in (17). If U is a standard uniform variate, we can generate random variables using the following expression.

$$X = [1 - [1 - [\frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda u}}{2\lambda}]^{1/\gamma}]^{1/\beta}]^{1/\alpha} \tag{18}$$

Then the random variable X follows TEKw(α,β,γ,λ).

4.5. Incomplete Moments

The s^{th} incomplete moment, say $\phi_s(t)$ of TEKw is,

$$\begin{aligned}
\phi_s(t) &= \int_0^t x^s f(x)dx \\
&= \int_0^t x^s \alpha\beta\gamma x^{\alpha-1}(1-x^\alpha)^{\beta-1}[1-(1-x^\alpha)^\beta]^{\gamma-1}[(1+\lambda) - 2\lambda[1-(1-x^\alpha)^\beta]^\gamma]dx \\
&= \int_0^t x^{s+\alpha-1}\alpha\beta\gamma(1-x^\alpha)^{\beta-1}[1-(1-x^\alpha)^\beta]^{\gamma-1}[(1+\lambda) - 2\lambda[1-(1-x^\alpha)^\beta]^\gamma]dx
\end{aligned}$$

After some algebra,

$$\begin{aligned}
\phi_s(t) &= \alpha\beta\gamma\Big[\frac{M(1+\lambda)}{\alpha}B(t^\alpha; \frac{s}{\alpha}+j+1, \beta) - \frac{2\lambda N}{\alpha}B(t^\alpha; \frac{s}{\alpha}+j+1, \beta)\Big] \\
&= \beta\gamma B(t^\alpha; \frac{s}{\alpha}+j+1, \beta)\big[M + \lambda(M-2N)\big]
\end{aligned} \tag{19}$$

where $B(w;a,b) = \int_0^w t^{a-1}(1-t)^{b-1}dt$ is the incomplete beta function.The first incomplete moment can be obtained by substituting s=1 in (19).

4.6. Mean Deviations

The mean deviation is a measure of amount of scatter in a random variable. Let X follow TEKw(α,β,γ,λ) with mean μ and median M.

- Mean Deviation from the mean is given by,

$$\delta_1(x) = \int_{-\infty}^{+\infty} |x-\mu| f(x)dx = 2\mu F(\mu) - 2\phi(\mu) \tag{20}$$

- Similarly, the Mean Deviation from the median is,

$$\delta_2(x) = \int_{-\infty}^{+\infty} |x-M| f(x)dx = \mu - 2\phi(M) \tag{21}$$

where F(μ) can be determined from (5) and $\phi(q) = \int_{-\infty}^{q} xf(x)dx$ is the first incomplete moment.
The mean deviations about mean and median are obtained by substituting median obtained from (17), first incomplete moment (19) with $s = 1$ and cdf (5) in (20) and (21).

Application of these equations can be made to obtain the Bonferroni curve, $B(x) = \frac{\phi_1(X)}{E(X)}$ and the Lorenz curve, $L(X) = \frac{\phi_1(X)}{F(X)E(X)}$ where $\phi_1(X)$ is the first incomplete moment from (19), F(x) is the cdf of TEKw distribution and E(X) is the mean.
These curves are very useful in economics, reliability, medicine, insurance and demography.

4.7. Entropy

The Renyi Entropy (Alfred Renyi) of a random variable X represents a measure of variation of the uncertainity which is defined by,

$$R_p(x) = \frac{1}{1-p} log \int_{-\infty}^{+\infty} f(x)^p dx$$

where p > 0 and $p \neq 1$

We have,

$$\begin{aligned}[1-(1-x^{\alpha})^{\beta}]^{(\gamma-1)p} &= \sum_{i=0}^{\infty} \frac{\Gamma(\gamma-1)p+1}{\Gamma((\gamma-1)p+1-i)}[(1-x^{\alpha})^{\beta}]^{i} \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+j}\Gamma((\gamma-1)p+1)\Gamma(\beta i+1)}{\Gamma((\gamma-1)p+1-i)\Gamma(\beta i+1-j)} x^{\alpha j} \\ &= \eta x^{\alpha j}\end{aligned} \tag{22}$$

and

$$\begin{aligned}[1-\lambda[2(1-(1-x^{\alpha})^{\beta})^{\gamma}-1]]^{p} &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!}\frac{\Gamma(p+1)}{\Gamma(p+1-i)}[\lambda[2(1-(1-x^{\alpha})^{\beta})^{\gamma}-1]^{i} \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!}\frac{\Gamma(p+1)}{\Gamma(p+1-i)}\lambda^{i}[2(1-(1-x^{\alpha})^{\beta})^{\gamma}-1]^{i}\end{aligned}$$

and,

$$\begin{aligned}[1-\lambda[2(1-(1-x^{\alpha})^{\beta})^{\gamma}-1]]^{p} &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!}\frac{\Gamma(p+1)}{\Gamma(p+1-i)}[\lambda[2(1-(1-x^{\alpha})^{\beta})^{\gamma}-1]^{i} \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!}\frac{\Gamma(p+1)}{\Gamma(p+1-i)}\lambda^{i}[2(1-(1-x^{\alpha})^{\beta})^{\gamma}-1]^{i}\end{aligned}$$

$$\begin{aligned}&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{2i+j+k+l}}{i!j!k!l!} \frac{\Gamma(p+1)\Gamma(i+1)\Gamma(\gamma j+1)\Gamma(\beta k+1)}{\Gamma(p+1-i)\Gamma(i+1-j)\Gamma(\gamma j+1-k)\Gamma(\beta k+1-l)} \lambda^{i} 2^{\gamma j} x^{\alpha l} \\ &= \theta x^{\alpha l}\end{aligned} \tag{23}$$

Evaluating the above integral, Rennyi entropy is,

$$R_p(x) = \frac{1}{1-p} log[\alpha^{p-1}(\beta\gamma)^{p}\eta\theta B(\frac{1}{\alpha}[(\alpha+1)p+1]+j+l,(\beta-1)p+1)] \tag{24}$$

where, η and θ are given by (22) and (23). The δ entropy,$\delta > 0$,$\delta \neq 1$,say $H_{\delta}(x)$ is defined as,

$$H_p(x) = \frac{1}{p-1} log[1 - \int_{-\infty}^{+\infty} f(x)^p dx]$$

where p > 0 and $p \neq 1$

Using (24),

$$H_p(x) = \frac{1}{p-1} log[1-\alpha^{p-1}(\beta\gamma)^{p}\eta\theta B(\frac{1}{\alpha}[(\alpha+1)p+1]+j+l,(\beta-1)p+1)] \tag{25}$$

where η and θ are given by (22) and (23).

The Rennyi entropy converge to the Shannon entropy when $\delta \to 1$.

4.8. Survival Function

Survival function is the probability that a system will survive beyond a given time. Mathematically, the survival function of TEKw(α,β,γ,λ)) is defined by:

$$S(x;\alpha,\beta,\gamma,\lambda) = 1 - [1-(1-x^{\alpha})^{\beta}]^{\gamma}[(1+\lambda) - \lambda[1-(1-x^{\alpha})^{\beta}]^{\gamma}] \tag{26}$$

where $\alpha, \beta, \gamma > 0$ and $|\lambda| < 1$. By choosing some arbitrary values for parameters, we provide some possible shapes for the survival function of the TEKw as shown in Figure 3:

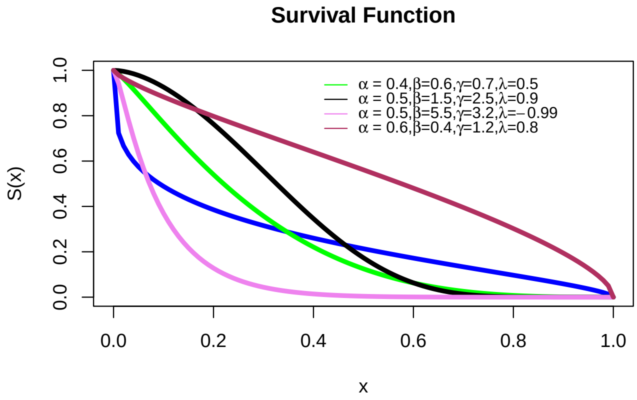

Figure 3: *Plot of the survival function for different values of parameters.*

4.9. Hazard Rate Function

The hazard rate function h(x) of TEKw($\alpha, \beta, \gamma, \lambda$) is given as,

$$h(x;\alpha,\beta,\gamma,\lambda) = \frac{\alpha\beta\gamma x^{\alpha-1}(1-x^{\alpha})^{\beta-1}[(1+\lambda) - 2\lambda[1-(1-x^{\alpha})^{\beta}]^{\gamma}]}{1-[(1+\lambda)[1-(1-x^{\alpha})^{\beta}]^{\gamma} - \gamma[1-(1-x^{\alpha})^{\beta}]^{2\gamma}]} \tag{27}$$

where $\alpha, \beta, \gamma > 0$ and $|\lambda| < 1$.

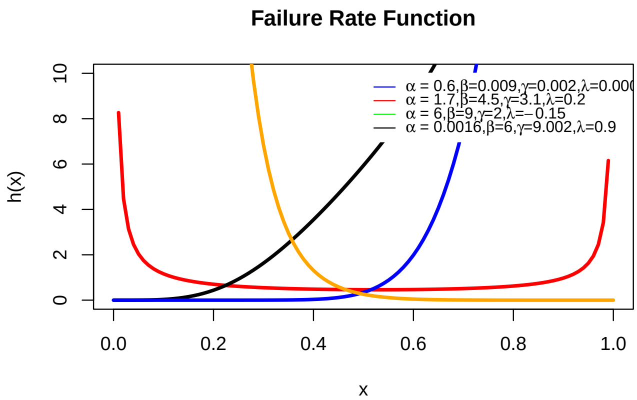

Figure 4: *Plot of the hazard rate function for different values of parameters.*

4.10. **Order Statistics**

Let $X_{(1)}, X_{(2)}, ..., X_{(n)}$ denote the order statistics of a random sample $X_1, X_2, ..., X_n$ from a population with cdf $F_X(x)$ and pdf $f_X(x)$given by (5) and (6). The pdf of k^{th} order statistic

$$
\begin{aligned}
f_{X(k)}(x) = {} & \frac{n!}{(k-1)!(n-k)!}\alpha\beta\gamma x^{\alpha-1}(1-x^{\alpha})^{\beta-1}[1-(1-x^{\alpha})^{\beta}]^{\gamma-1}[(1+\lambda)-2\lambda[1-(1-x^{\alpha})^{\beta}]^{\gamma}] \\
& \times \left\{[(1+\lambda)[1-(1-x^{\alpha})^{\beta}]^{\gamma}-\lambda[1-(1-x^{\alpha})^{\beta}]^{2}\gamma]^{k-1}\right\} \\
& \times \left\{[1-(1+\lambda)[1-(1-x^{\alpha})^{\beta}]^{\gamma}+[\lambda[1-(1-x^{\alpha})^{\beta}]^{2\gamma}]]^{n-k}\right\}
\end{aligned}
$$

4.11. Maximum Likelihood Estimation

The estimation of parameters α, β, γ and λ is done using the maximum likelihood estimation method. Let $X_1, X_2, ..., X_n$ be an observed random sample from TEKw$(\alpha, \beta, \gamma, \lambda)$ distribution with unknown parameters α,β,γ and λ. The likelihood function is,

$$L(x) = \prod_{i=1}^{n} f(x_i; \alpha, \beta, \gamma, \lambda)$$

i.e.,

$$L(x) = \prod_{i=1}^{n} \alpha\beta\gamma x_i^{\alpha-1}(1-x_i^{\alpha})^{\beta-1}[1-(1-x_i^{\alpha})^{\beta}]^{\gamma-1}[(1+\lambda)-2\lambda[1-(1-x_i^{\alpha})^{\beta}]^{\gamma}]$$

Then the log-likelihood function is given by

$$
\begin{aligned}
lnL = {} & nln\alpha + nln\beta + nln\gamma + (\alpha-1)\sum_{i=1}^{n} ln(x_i) + (\beta-1)\sum_{i=1}^{n} ln(1-x_i^{\alpha}) \\
& + (\gamma-1)\sum_{i=1}^{n} ln(1-(1-x_i^{\alpha})^{\beta}) + \sum_{i=1}^{n} ln[(1+\lambda)-2\lambda(1-(1-x_i^{\alpha})^{\beta})^{\gamma}]
\end{aligned}
\tag{28}
$$

Therefore, the MLEs of $\alpha, \beta, \gamma, \lambda$ which maximize (28) must satisfy the following normal equations;

$$
\begin{aligned}
& \frac{n}{\alpha} + \sum_{i=1}^{n} lnx_i - (\beta-1)\sum_{i=1}^{n}\frac{x_i^{\alpha}lnx_i}{1-x_i^{\alpha}} + \beta(\gamma-1)\sum_{i=1}^{n}\frac{(1-x_i^{\alpha})^{\beta-1}x_i^{\alpha}lnx_i}{[1-(1-x_i^{\alpha})^{\beta}]} \\
& -2\lambda\beta\gamma\sum_{i=1}^{n}\frac{[1-(1-x_i^{\alpha})^{\beta}]^{\gamma-1}(1-x_i^{\alpha})^{\beta-1}x_i^{\alpha}lnx_i}{[(1+\lambda)-2\lambda[1-(1-x_i^{\alpha})^{\beta}]^{\gamma}]} = 0
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
& \frac{n}{\beta} + \sum_{i=1}^{n} ln(1-x_i^{\alpha}) - (\gamma-1)\sum_{i=1}^{n}\frac{(1-x_i^{\alpha})^{\beta}ln(1-x_i^{\alpha})}{[1-(1-x_i^{\alpha})^{\beta}]} \\
& +2\lambda\gamma\sum_{i=1}^{n}\frac{[1-(1-x_i^{\alpha})^{\beta}]^{\gamma-1}(1-x_i^{\alpha})^{\beta}ln(1-x_i^{\alpha})}{[(1+\lambda)-2\lambda[1-(1-x_i^{\alpha})^{\beta}]^{\gamma}]} = 0
\end{aligned}
\tag{30}
$$

$$\frac{n}{\gamma} + \sum_{i=1}^{n} ln[1-(1-x_i^{\alpha})^{\beta}] - 2\lambda\sum_{i=1}^{n}\frac{[1-(1-x_i^{\alpha})^{\beta}]^{\gamma}ln[1-(1-x_i^{\alpha})^{\beta}]}{[(1+\lambda)-2\lambda[1-(1-x_i^{\alpha})^{\beta}]^{\gamma}]} = 0 \tag{31}$$

$$\sum_{i=1}^{n}\frac{1-2[1-(1-x_i^{\alpha})^{\beta}]^{\gamma}}{[(1+\lambda)-2\lambda[1-(1-x_i^{\alpha})^{\beta}]^{\gamma}]} = 0 \tag{32}$$

Hence, the MLEs of the parameters are obtained by solving these nonlinear system of equations. Solving these system of nonlinear equations are complicated, we can therefore use statistical software to solve the equations numerically.

5. Simulation Study and Data Analysis

5.1. Simulation Study

Considering (18), the simulation is done for two instances using different parameter values.The chosen parameter values here are,

- $\alpha = 0.1, \beta = 0.5, \gamma = 0.8, \lambda = 0.01$
- $\alpha = 0.8, \beta = 1.5, \gamma = 3, \lambda = -0.9$

As the n increases, mean square error decreases for the selected parameter values given in table 1 and 2. Moreover, the bias is close to zero as the sample size increases. Thus, as the sample size increases. the estimates become closer to the true parameter values.

Table 1: *Simulation study at α = 0.1,β = 0.5,γ = 0.8,λ= 0.01*

n	Parameter	Estimate	Bias	MSE
25	α	0.0188	-0.0812	0.0066
	β	0.7373	0.2373	0.0563
	γ	2.9549	2.155	4.6439
	λ	-0.2509	-0.2609	0.0681
50	α	0.0212	-0.0788	0.0062
	β	0.6358	0.1359	0.0185
	γ	2.1147	1.3147	1.7284
	λ	-0.1819	-0.2929	0.03681
100	α	0.1132	0.0132	0.0002
	β	0.4486	-0.0514	0.0026
	γ	0.6076	-0.1924	0.0370
	λ	0.1659	0.1559	0.02430
500	α	0.0949	-0.0051	2.55e-05
	β	0.5348	0.0348	0.0012
	γ	0.8485	0.0485	0.0024
	λ	0.1121	0.1021	0.01041
1000	α	0.1027	0.003	7.56e-06
	β	0.5112	0.0116	0.0001
	γ	0.7835	-0.0164	0.0003
	λ	0.0129	0.003	8.81e-06

Table 2: *Simulation study at α = 0.8,β = 1.5,γ = 3,λ= -0.9*

n	Parameter	Estimate	Bias	MSE
25	α	12.3475	11.5475	133.3457
	β	5.6831	4.1831	17.4990
	γ	0.1973	-2.8026	7.8549
	λ	-0.2146	0.6853	0.56964
50	α	5.3842	4.5842	21.0153
	β	2,2110	0.7110	0.50556
	γ	0.3943	-2.6056	6.7896
	λ	-0.3780	0.5219	0.27241
100	α	0.4486	-0.3513	0.1234
	β	1.3816	-0.1183	0.0140
	γ	8.1035	2.1035	4.42471
	λ	-0.4335	0.4665	0.2176
500	α	0.6669	-0.1333	0.0117
	β	1.5409	0.0409	0.0017
	γ	5.6903	2.0577	4.2344
	λ	-0.5098	0.39018	0.1522
1000	α	0.7799	0.0027	7.51e-06
	β	1.4994	0.0012	0.0001
	γ	3.0002	-0.0164	0.0021
	λ	-0.8024	0.0055	0.0004

5.2. Data Analysis

In this section, we demonstrate the usefulness of the proposed Transmuted Exponentiated Kumaraswamy TEKw$(\alpha, \beta, \gamma, \lambda)$ distribution. We fit this distribution to a real life data set and compare the results with some recent efficient models: those corresponding to the Kumaraswamy (Kw) distribution (Kumaraswamy [11]), Transmuted Kumaraswamy (TKw) distribution (Khan et.al. [9]) and Exponentiated Kumaraswamy (EKw) distribution (Lemonte et.al. [12]). The corresponding pdfs are presented below.

- The pdf of the Kw distribution is given by

$$f(x;\alpha,\beta) = \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}, 0 < x < 1$$

  where $\alpha, \beta > 0$

- The pdf of the TKw distribution is given by,

$$f(x;\alpha,\beta,\lambda) = \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}[(1-\lambda)+2\lambda(1-x^{\alpha})^{\beta}], \quad 0 < x < 1$$

  where $\alpha, \beta > 0$ and $|\lambda| < 1$.

- The pdf of EKw distribution is given by,

$$f(x;\alpha,\beta,\gamma) = \alpha\beta\gamma x^{\alpha-1}(1-x^{\alpha})^{\beta-1}[1-(1-x^{\alpha})^{\beta}]^{\gamma-1}, \qquad 0 < x < 1$$

  where $\alpha, \beta, \gamma > 0$

Shasta Reservoir capacity data is used for the purpose. The reservoir is located in California, United States. The reservoir has a height of 602 ft (183 m), a length of 3460 ft (1050 m), and a total capacity of 4.552 million acre-ft (5.615 million dam^3). The capacity of the Reservoir (after transformation) for each February from 1991 to 2010 is given by table 3, see Simbolan et. al. [19]. The analysis is carried out using R software.The parameters are estimated by maximum

Table 3: *Shasta Reservoir Capacity Data Each February from 1991 to 2010*

Year	Transformed Capacity	Year	Transformed Capacity
1991	0.338936	2001	0.768007
1992	0.431915	2002	0.843485
1993	0.759932	2003	0.787408
1994	0.724626	2004	0.849868
1995	0.757583	2005	0.69597
1996	0.811556	2006	0.842316
1997	0.785339	2007	0.828689
1998	0.78366	2008	0.580194
1999	0.815627	2009	0.430681
2000	0.847413	2010	0.742563

likelihood method. Akaike information criterion (AIC), the correct Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan information criterion (HQIC), -2 ln L, the Kolmogorov-Smirnov (K-S), Cramer-von Mises and Anderson-Darling goodness-of-fit statistic and the p-values are considered to compare the four models which are defined as follows.

$$AIC = -2l + 2k$$

$$BIC = -2l + k\ log(n)$$

$$HQIC = -2l + k\ log(log(n))$$

$$CAIC = -2l + 2kn/(n - k - 1)$$

where, l denotes the log-likelihood function, k is the number of parameters and n is the sample size.
The Kolmogorov-Smirnov test is used to decide if a sample comes from a population with a specific distribution. The test statistic is given by,

$$K - S\ statistic\ \ D_n = sup|F(x) - F_n(x)|$$

where $F_n(x)$ is the empirical distribution function.

The Cramer-von Mises criterion for testing that a sample $x_1, x_2, ..., x_n$ has been drawn from a specified continuous distribution F(x)is

$$\omega^2 = \int_{-\infty}^{+\infty} [F_n(x) - F(x)]^2 dF(x) \tag{33}$$

The Anderson-Darling is used to test if a sample of data came from a population with a specific distribution. It is a modification of the Kolmogorov-Smirnov (K-S) test and is given by,

$$AD = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)[lnF(X_i) + ln(1 - F(X_{n-i+1}))] \tag{34}$$

where, n is the sample size, F(x) is the cdf for the specified distribution, and i is the i^{th} sample, calculated when the data is sorted in ascending order.
The parameter estimates based on the reservoir capacity data for the four models considered are given by table 4.

Table 4: *The MLEs and log-likelihood (l) estimate of the model parameters for reservoir capacity data.*

Distribution	Parameter Estimates				log-likelihood
	α	β	γ	λ	
Kw	6.891239	5.215555	-	-	15.90481
TKw	6.2451572	5.4432386	-0.5010172	-	16.67048
EKw	24.0913041	64.7302856	0.2273644	-	17.97171
TEKw	24.8114998	83.6984162	0.1786838	-0.5454838	20.32666

The tables 5 gives the estimates of the model parameters, AIC , BIC, CAIC and the HQIC values.

Table 5: *AIC, BIC, CAIC and HQIC statistics of the fitted model in data set*

Distribution	-2l	AIC	BIC	CAIC	HQIC
Kw	-31.80962	-27.80962	-25.62753	-27.17804	-25.53863
TKw	-33.34096	-27.34096	-24.06783	-26.00763	-26.56991
EKw	-35.94341	-29.94341	-26.67028	-28.61008	-29.17236
TEKw	**-40.65332**	**-32.65332**	**-28.28915**	**-30.30038**	**-31.62525**

From table 4, it shows that the proposed Transmuted Exponentiated Kumaraswamy model has a maximum value of log likelihood. Table 5 shows that the proposed model has a minimum values of statistics AIC, BIC, CAIC and HQIC compared to other models. In order to compare the distributions, we had considered the Kolmogorov-Smirnov (K-S) test, Cramer-von Mises and

Table 6: *Test statistic values and corresponding p values*

Distribution	K-S Statistic (p-value)	Anderson-Darling Statistic (p-value)	Cramer-Von Statistic (p-value)
Kw	0.22384 (0.1892)	0.96123 (0.01243)	0.13848 (0.03077)
TKw	0.19077 (0.3543)	0.80139 (0.03188)	0.10962 (0.0768)
EKw	0.18032 (0.4221)	0.65474 (0.07574)	0.085587 (0.1656)
TEKw	**0.16457 (0.5365)**	**0.50999 (0.1762)**	**0.060932 (0.3527)**

Anderson-Darling goodness-of-fit statistics for the Shastha Reservoir Capacity data. From table 6, it is seen that Transmuted Exponentiated Kumaraswamy model has largest p-value based on K-S Statistic, Cramer-von Mises and Anderson-Darling statistic. As the results indicate, the proposed model performed better than other models.

6. Conclusion

In this paper, we have introduced a new generalization of the exponentiated Kumaraswamy distribution called the transmuted exponentiated Kumaraswamy distribution. The graphical representations of its density function,cumulative distribution function, hazard rate function and survival function are obtained. We derived the moments, moment generating function, characteristic function, entropy, mean deviations, quantile function, etc. of the proposed distribution. Estimation of parameters of the distribution is performed using maximum likelihood method. A simulation study is performed to validate the estimates of the model parameters. Finally, TEKw(α,β,γ,λ) distribution is applied to a real data set and compared with other distributions. It is empirically verified that the new TEKw model is a better model than the other competing models.

References

[1] Abd El-Monsef, M. M. E. and Ghoneim, S. A. E. (2015). The weighted kumaraswamy distribution. *Information : An International Interdisciplinary Journal*, 18(8): 3289-3300.

[2] Al-Fattah, A.M. and EL-Helbawy, Abeer and Al-Dayian, Gannat. (2017). Inverted Kumaraswamy distribution: Properties and estimation. *Pakistan Journal of Statistics*, 33(1): 37-61.

[3] Al-Kadim, K. A. and Mahdi, A. A. (2018). Exponentiated transmuted exponential distribution. *Journal of University of Babylon for Pure and Applied Sciences*, 26(2): 78-90.

[4] Barreto-Souza, W. and Lemonte, A. J. (2013). Bivariate Kumaraswamy distribution: properties and a new method to generate bivariate classes. *Statistics*, 47(6): 1321-1342.

[5] Bourguignon, M., Silva, R. B., Zea, L. M. and Cordeiro, G. M. (2013). The Kumaraswamy Pareto distribution.*Journal of Statistical Theory and Applications*, 12(2): 129-144.

[6] Dey, S., Mazucheli, J. and Nadarajah, S. (2018). Kumaraswamy distribution: different methods of estimation. *Computational and Applied Mathematics*, 37(2): 2094-2111.

[7] Fletcher, S. C. and Ponnambalam, K. (1996). Estimation of reservoir yield and storage distribution using moments analysis. *Journal of Hydrology*, 182(1): 259-275.

[8] Hassan, A. S. and Elgarhy, M. (2016). Kumaraswamy Weibull-generated family of distributions with applications. *Advances and Applications in Statistics*, 48(3): 205-239.

[9] Khan, M. S., King, R. and Hudson, I. L. (2016). Transmuted Kumaraswamy Distribution. *Statistics in Transition new series*, 17(2): 183-210.

[10] Khan, M. S., King, R. and Hudson, I. L. (2017). Transmuted Weibull distribution: Properties and estimation. *Communications in Statistics-Theory and Methods*, 46(11): 5394-5418.

[11] Kumaraswamy, P. (1980). A generalized probability density functions for double-bounded random processes. *Journal of Hydrology*, 46(1-2): 79-88.

[12] Lemonte, A. J., Barreto-Souza, W. and Cordeiro, G. M. (2013). The exponentiated Kumaraswamy distribution and its log-transform. *Brazilian Journal of Probability and Statistics*, 27(1): 31-53.

[13] Merovci, F. (2013). Transmuted Exponentiated Exponential Distribution. *Mathematical Sciences and Applications E-Notes*, 1(2): 112-122.

[14] Merovci, F. (2016). Transmuted Rayleigh Distribution. *Austrian Journal of statistics*, 42 (1): 21-32.

[15] Mielke, P. W., and Johnson, E. S. (1974). Some generalized beta distributions of the second kind having desirable application features in hydrology and meteorology.*Water Resources Research*, 10(2): 223-226.

[16] Mitnik, P. A. (2013). New properties of the Kumaraswamy distribution. *Communications in Statistics-Theory and Methods*, 42(5): 741-755.

[17] Mohammed, A. S. (2019). Theoretical Analysis of the Exponentiated Transmuted Kumaraswamy Distribution with Application. *Annals of Statistical Theory and Applications (ASTA)*,1: 51-60.

[18] Silva, R., Gomes-Silva, F., Ramos, M., Cordeiro, G., Marinho, P. and De Andrade, T. A. (2019). The exponentiated Kumaraswamy-G class: general properties and application. *Revista Colombiana de Estadstica*, 42(1): 1-33.

[19] Simbolon, H. G., Fithriani, I. and Nurrohmah, S. (2017). Estimation of shape parameter in Kumaraswamy distribution using Maximum Likelihood and Bayes method. *AIP Conference Proceedings*, 1862(1): 030160.

[20] Shaw, W. T. and Buckley, I. R. (2009). The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skewkurtotic-normal distribution from a rank transmutation map. *arXiv preprint* arXiv: 0901.0434.

[21] Sherwania, R. A. K., Waqas, M., Saeed, N., Farooq, M., Ali Raza, M. and Jamal, F. (2021). Transmuted Inverted Kumaraswamy Distribution: Theory and Applications. *Punjab University Journal of Mathematics*, 53(3): 29-45.

[22] Usman, A., Ishaq, A., Tasiu, M., Aliyu Y. and Idris, F.A. (2017). A new Weibull-Kumaraswamy distribution: theory and applications. *Nigerian Journal of Scientific Research*, 16(2): 158-166.

[23] ZeinEldin, R. A., Hashmi, S., Elsehety, M. and Elgarhy, M. (2020). Type II half logistic Kumaraswamy distribution with applications. *Journal of Function Spaces*, 2020: 1-15.