THE INVERSE BURR LOG-LOGISTIC DISTRIBUTION: PROPERTIES, APPLICATIONS AND DIFFERENT METHODS OF ESTIMATION

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Abstract

Lifetime distributions have played a significant role in lifetime data analysis. Despite the numerous distributions in literature, there have been several motivations for developing new ones. In this paper, a new lifetime distribution is proposed. Some important functions of the new distribution, such as probability density, cumulative distribution, survival, hazard, and quantile are derived in closed form. Some distributional properties such as moments, moment generating function, linear representation, probability weighted moments, etc. are obtained. Some estimators such as the least square estimator (LSE), the weighted least square estimator (WLSE), the Anderson-Darling estimator (ADE) and the Cramer-von Mises estimator (CvME) are investigated for three unknown parameters. The efficiency of the estimators is checked via Monte Carlo simulation based on the bias and mean square error criteria. The usability of the new distribution is investigated with two real data sets and empirical results obtained reveal that the new distribution offers a promising fit for the data sets under study.

Keywords: Bur distribution, log-logistic distribution, parameter estimation, quantile

1. INTRODUCTION

Statistical distributions have played a significant role in lifetime data analysis. Despite the numerous distributions in literature, there have been several motivations for developing new ones. In all, the central goal has remained to develop a more flexible and tractable distribution in fitting real-world problems. In the last decades, researchers have introduced different methodologies for generating new statistical distributions which are hoped to provide a better fit than the existing distributions in lifetime data analysis. Some of these methods are the Beta-G family by [7], Marshall-Olkin extended family by [11], Transmuted-G family by [14], Kumaraswamy-G family by [5], Transformer (T-X) family by [2], Weibull-G family by [4], Odd Burr-G family by [1], Type II Topp-Leone generated family by [6], etc.

Recently, [13] introduced the Inverse Burr-G family of distributions using the idea of [15]. By considering the inverse Burr as the generator, they defined the cumulative distribution function of the inverse Burr-G family of distribution as

$$\begin{aligned} F(x,\xi) &= \alpha\,\beta \int_0^{-\log[1-G(x,\xi)]} x^{-(\alpha+1)} \left(1+x^{-\alpha}\right)^{-(\beta+1)} dx, \\ &= \left[1+\left\{-\log[1-G(x,\xi)]\right\}^{-\alpha}\right]^{-\beta}, \qquad x>0,\ \alpha,\ \beta>0, \end{aligned} \tag{1}$$

where α and β are the shape parameters and $G(x,\xi)$ is the baseline distribution which depends on a parameter vector ξ.

The corresponding density function associated with (1) is given by

$$f(x,\xi) = \alpha\,\beta\, g(x,\xi)[1-G(x,\xi)]^{-1} \left\{-\log[1-G(x,\xi)]\right\}^{-(\alpha+1)} \left[1+\left\{-\log[1-G(x,\xi)]\right\}^{-\alpha}\right]^{-(\beta+1)}. \tag{2}$$

In this paper, we employed the technique defined in (1) and consider in particular, the case where the baseline distribution $G(x,\xi)$ follows the log-logistic distribution.

The cumulative distribution function (cdf) and probability density function (pdf) of the log-logistic distribution with shape parameter $\lambda > 0$ are respectively defined as

$$G(x) = 1-\left(1+x^{\lambda}\right)^{-1}, \tag{3}$$

and

$$g(x) = \lambda\, x^{\lambda-1}\left(1+x^{\lambda}\right)^{-2}, \qquad x>0,\ \lambda>0. \tag{4}$$

Inserting (3) and (4) into (1) and (2), we define the cdf and pdf of a new statistical distribution as

$$F(x) = \left[1+\left\{\log\left(1+x^{\lambda}\right)\right\}^{-\alpha}\right]^{-\beta}, \qquad x>0,\ \alpha,\lambda,\beta>0, \tag{5}$$

and

$$f(x) = \alpha\,\beta\,\lambda\, x^{\lambda-1}\left(1+x^{\lambda}\right)^{-1} \left\{\log\left(1+x^{\lambda}\right)\right\}^{-(\alpha+1)} \left[1+\left\{\log\left(1+x^{\lambda}\right)\right\}^{-\alpha}\right]^{-(\beta+1)}. \tag{6}$$

Suppose a random variable *X* has the density function in (6), then we say that *X* follows the Inverse Burr Log-Logistic ("IBLL" for short) distribution with shape parameters α, β and λ.

The motivation of this paper is to develop a tractable distribution that spans all the various forms of the hazard rate properties and provides a consistently better fit than most available statistical distribution in the literature.

The rest sections of the paper are structured as follows. In Section 2, we discuss in detail, some basic mathematical properties of the proposed distribution. Section 3 presents some methods of estimation of the unknown parameters of the proposed distribution. The asymptotic behavior of unknown parameters through a Monte Carlo simulation study are investigated in Section 4. In Section 5, we illustrate the applicability of the proposed distribution in lifetime data analysis two data sets and compared its fit alongside with fit attain by some existing non-nested distributions. Finally, in Section 6, we gave a concluding remark.

2. MATHEMATICAL PROPERTIES OF THE IBLL DISTRIBUTION

In this Section, some of the mathematical properties of the IBLL distribution are discussed. These include survival, hazard, quantile functions, the linear representation of the distribution, moments, moment generating function, probability weighted moment, Rényi entropy and distribution of order statistics.

2.1 Survival, Hazard and Quantile Functions

The survival, hazard and quantile functions of the IBLL distribution are respectively derived from (5) and (6) as follows.

$$S(x) = 1 - \left[1+\left\{\log\left(1+x^{\lambda}\right)\right\}^{-\alpha}\right]^{-\beta}, \tag{7}$$

$$h(x) = \frac{\alpha\,\beta\,g(x,\xi)[1-G(x,\xi)]^{-1}\left\{-\log[1-G(x,\xi)]\right\}^{-(\alpha+1)}\left[1+\left\{-\log[1-G(x,\xi)]\right\}^{-\alpha}\right]^{-(\beta+1)}}{1-\left[1+\left\{\log\left(1+x^{\lambda}\right)\right\}^{-\alpha}\right]^{-\beta}}, \tag{8}$$

and

$$Q_x(p) = \left[\exp\left(p^{-1/\beta}-1\right)^{-1/\alpha}-1\right]^{1/\lambda}, \qquad 0 < p < 1. \tag{9}$$

The quantile function in (9) is derived by simply inverting the distribution function in (5). This is one of the most important properties of any distribution, as it allows for generating random numbers from the distribution for the simulation study. Substituting $p = 0.5$ in (9), we obtain the median of the IBLL distribution as

$$Q_x(0.5) = \left[\exp\left((0.5)^{-1/\beta}-1\right)^{-1/\alpha}-1\right]^{1/\lambda}. \tag{10}$$

Some graphical presentations of the pdf and hazard function of the IBLL distribution are displayed in Figures 1 and 2 respectively.

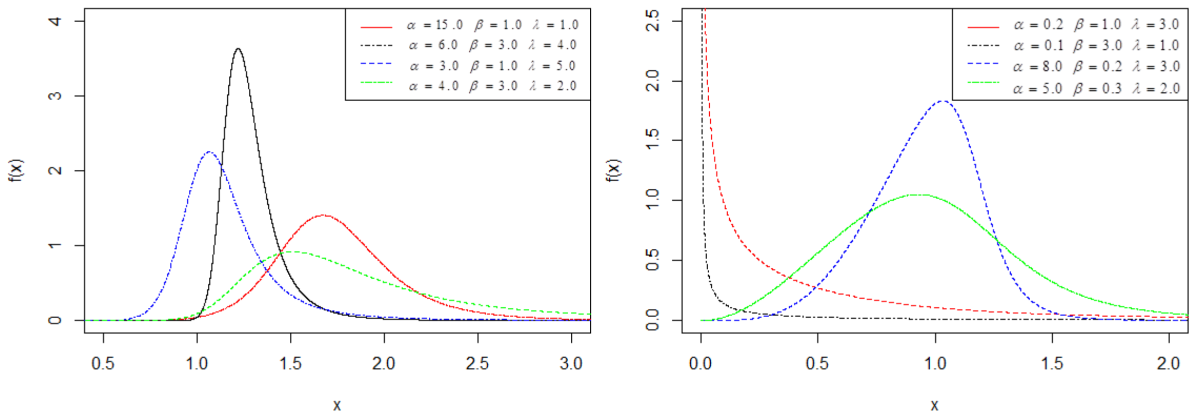

Figure 1: Density Plots of the IBLL Distribution

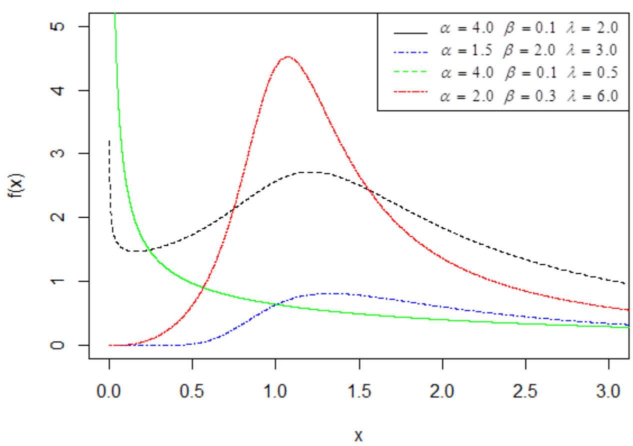

Figure 2: Hazard Plots of the IBLL Distribution

Figure 1 shows that the density plot of the IBLL distribution accommodates decreasing, left-skewed, right-skewed and symmetric shapes, while the plots displayed in Figure 2 indicates that the hazard function of the IBLL distribution exhibits a decreasing, increasing, upside down bathtub and decreasing-increasing-decreasing hazard properties.

2.2 Linear Representation

The linear representation of the density and distribution functions allow for easy derivation of some properties such as the moments, probability weighted moment, moment generating function, distribution of order statistics, etc. The following lemmas will guide us in the derivation of the linear representation of the density and distribution functions of the IBLL distribution.

Lemma 1:
For any positive real non-integer $s > 0$, consider the generalized binomial series expansion (see [12]).

$$(1+x)^{-s} = \sum_{k=0}^{\infty} \binom{s+k-1}{k} (-1)^k x^k.$$

Lemma 2:
For any real parameter $\alpha > 0$, the convergent series holds.

$$(-\log[1-y])^{\alpha-1} = y^{\alpha-1}\left[\sum_{m=0}^{\infty}\binom{\alpha-1}{m} y^m \left(\sum_{s=0}^{\infty}\frac{y^s}{s+2}\right)^m\right], \quad 0<y<1.$$

Applying the result on power series raised to a positive integer, with $a_s = (s+2)^{-1}$ that is,

$$\left(\sum_{s=0}^{\infty} a_s y^s\right)^m = \sum_{s=0}^{\infty} b_{s,m} y^s ,$$

so that,

$$(-\log[1-y])^{\alpha-1} = \sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\binom{\alpha-1}{m} b_{s,m}\, y^{\alpha+m+s-1} ,$$

where $b_{s,m} = (sa_0)^{-1}\sum_{j=0}^{s}\{m(j+1)-s\}a_j b_{s-q,m}$ *and* $b_{0,m} = a_0^m$ (see [8]).

Now, applying the above two lemmas in (5),

$$\left[1+\left(\log\left[1+x^{\lambda}\right]\right)^{-\alpha}\right]^{-\beta} = \sum_{k=0}^{\infty}\binom{\beta+k-1}{k}(-1)^k\left(\log\left[1+x^{\lambda}\right]\right)^{-\alpha k},$$

$$\left(\log\left[1+x^{\lambda}\right]\right)^{-\alpha k} = \sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\binom{-\alpha k}{m} b_{s,m}\left[1-\left(1+x^{\lambda}\right)^{-1}\right]^{m+s-\alpha k},$$

so that (5) now becomes,

$$\begin{aligned} F(x) &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\binom{\beta+k-1}{k}\binom{-\alpha k}{m}(-1)^k b_{s,m}\left[1-\left(1+x^{\lambda}\right)^{-1}\right]^{m+s-\alpha k} \\ &= \sum_{k,m,s=0}^{\infty}\psi_{m+s-\alpha k} H_{m+s-\alpha k}(x) \end{aligned} \tag{11}$$

where,

$$\psi_{m+s-\alpha k} = \sum_{k,m,s=0}^{\infty} \binom{\beta+k-1}{k}\binom{-\alpha k}{m}(-1)^k b_{s,m}$$

and $H_{m+s-\alpha k}(x) = [G(x)]^{m+s-\alpha k}$ is the distribution function of the log-logistic distribution with $m+s-\alpha k$ as the power parameter.
Differentiating (11), we obtain its associated density function as

$$f(x) = \sum_{k,m,s=0}^{\infty} \psi_{m+s-\alpha k} h_{m+s-\alpha k+1}(x). \tag{12}$$

where $h_{m+s-\alpha k}(x) = \lambda(m+s-\alpha k+1)x^{\lambda-1}(1+x^{\lambda})^{-2}\left[1-(1+x^{\lambda})^{-1}\right]^{m+s-\alpha k}$ is the density function of the log-logistic distribution with $m+s-\alpha k+1$ as the power parameter.
Other useful properties such as the moments and moment generating function can be directly obtained from (12).

2.3 The Moments and Moment Generating Function

Let *X* be a continuous random variable following a known probability distribution with density function $f(x)$, then the r^{th} ordinary moment of *X* is defined as

$$E[X^r] = \mu_r' = \int_{-\infty}^{\infty} x^r f(x)dx. \tag{13}$$

Substituting (6) into (12), the r^{th} ordinary moment of the IBLL distribution is obtained as

$$\begin{aligned} E[X^r] &= \sum_{k,m,s=0}^{\infty} \psi_{m+s-\alpha k+1} \int_{-\infty}^{\infty} x^r h_{m+s-\alpha k+1}(x)dx, \\ &= \lambda \sum_{k,m,s=0}^{\infty} \psi_{m+s-\alpha k+1}(m+s-\alpha k+1)\int_{-\infty}^{\infty} x^{r+\lambda-1}(1+x^{\lambda})^{-2}\left[1-(1+x^{\lambda})^{-1}\right]^{m+s-\alpha k} dx, \end{aligned} \tag{14}$$

using lemma 1,

$$\left[1-(1+x^{\lambda})^{-1}\right]^{m+s-\alpha k} = \sum_{q=0}^{\infty}\binom{m+s-\alpha k}{q}(-1)^q (1+x^{\lambda})^{-q},$$

Substituting this expression into (14), we have

$$E[X^r] = \lambda \sum_{k,m,s=0}^{\infty}\sum_{q=0}^{\infty}\binom{m+s-\alpha k}{q}(-1)^q(m+s-\alpha k+1)\psi_{m+s-\alpha k+1}\int_{-\infty}^{\infty} x^{r+\lambda-1}(1+x^{\lambda})^{-(q+2)}dx, \tag{15}$$

Further simplification of (15) and invoking the beta function, yields

$$E[X^r] = \sum_{k,m,s,q=0}^{\infty}\binom{m+s-\alpha k}{q}(-1)^q(m+s-\alpha k+1)\psi_{m+s-\alpha k+1} B\left[1+\frac{r}{\lambda}, q+1-\frac{r}{\lambda}\right]. \tag{16}$$

When $r=1$ in (16) we obtain the mean of *X*. The variance, skewness and kurtosis of *X* can be computed from (16), using the following mathematical relationships.

$$\text{variance}\left(\sigma^2\right) = \mu_2^{'} - \left(\mu_1^{'}\right)^2,$$

$$\text{skewness}\left(S_k\right) = \frac{\mu_3^{'} - 3\mu_2^{'}\mu_1^{'} + 2\left(\mu_1^{'}\right)^3}{\left(\mu_2^{'} - \left(\mu_1^{'}\right)^2\right)^{\frac{3}{2}}},$$

$$\text{kurtosis}\left(K_s\right) = \frac{\mu_4^{'} - 4\mu_3^{'}\mu_1^{'} + 6\mu_3^{'}\left(\mu_1^{'}\right)^2 - 3\left(\mu_1^{'}\right)^4}{\left(\mu_2^{'} - \left(\mu_1^{'}\right)^2\right)^2},$$

where $\mu_1^{'}$, $\mu_2^{'}$, $\mu_3^{'}$ and $\mu_4^{'}$ are the first four ordinary moments of the IBLL distribution.

The r^{th} incomplete moment of *X* is obtained from (16) as

$$\varphi_r(t) = \sum_{k,m,s,q=0}^{\infty} \binom{m+s-\alpha k}{q} (-1)^q (m+s-\alpha k+1)\psi_{m+s-\alpha k+1} B_z\left[1+\frac{r}{\lambda}, q+1-\frac{r}{\lambda}\right], \tag{17}$$

where $B(\alpha,\beta) = \int_0^\infty x^{\alpha-1}(1+x)^{-(\beta+\alpha)}dx$ and $B_z(\alpha,\beta) = \int_0^z x^{\alpha-1}(1+x)^{-(\beta+\alpha)}dx$ are respectively the beta function of the second kind and the incomplete beta function of the second kind.

The moment generating function of *X* is define using the Maclaurin series expansion of the exponential function as

$$M_X(t) = E\left[e^{tx}\right] = \sum_{n=0}^{\infty} \frac{t^n}{n!} E\left[X^n\right]. \tag{18}$$

Inserting (16) into (18), we define the moment generating function of IBLL distribution as

$$M_X(t) = \sum_{k,m,s,q,n=0}^{\infty} \frac{t^n}{n!} \binom{m+s-\alpha k}{q} (-1)^q (m+s-\alpha k+1)\psi_{m+s-\alpha k+1} B\left[1+\frac{n}{\lambda}, q+1-\frac{n}{\lambda}\right]. \tag{19}$$

Table 1 presents the numerical computation of the mean, variance, skewness and kurtosis of the IBLL distribution at varying values of the parameters.

Table 1: *Moments of the IBLL Distribution at varying values of the Parameters*

λ	α	β	μ	σ^2	S_k	K_s
0.5	6	6	3.2316	11.9924	0.4073	1.5944
		8	2.7794	12.6009	0.6770	1.7675
	8	6	4.3219	10.0142	-0.1784	1.7717
		8	4.1435	11.8485	-0.0735	1.5029
1.0	6	6	3.3973	2.5320	1.1591	5.3969
		8	3.6548	2.8901	0.8689	4.6926
	8	6	2.9264	1.1707	2.0539	10.003
		8	3.1257	1.3177	1.8620	8.8985

From Table 1, we observed that the IBLL distribution is positively skewed ($S_k > 0$), negatively skewed ($S_k < 0$) and approximately symmetric ($S_k \approx 0$). This result is consistent with the plots of the density function displayed in Figure 1. Also, at some selected values of the parameters, the IBLL distribution is both leptokurtic ($K_s > 3$) and platykurtic ($K_s < 3$).
Figure 3 displays the plot of the skewness and kurtosis of IBLL distribution for $\lambda = 1$.

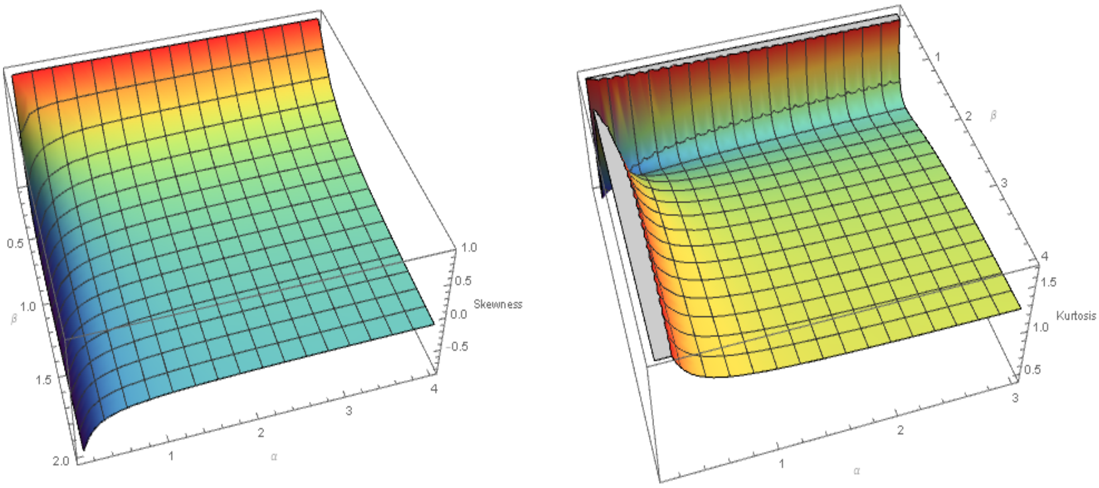

Figure 3: The Skewness and Kurtosis for IBLL Distribution $(\alpha, \beta, 1)$

2.4 The Probability Weighted Moments

The probability weighted moments (PWMs) are generally used to construct the estimator of the parameters as well as the quantiles of a known statistical distribution whose cdf is invertible. For a random variable X, [9] defined the $(q, r)^{th}$ PWMs as

$$\rho_{q,r} = E\left[X^r F(x)^q\right] = \int_{-\infty}^{\infty} x^r f(x) F(x)^q \, dx, \tag{20}$$

combining (5) and (6), we have

$$f(x)F(x)^q = \alpha\beta\lambda x^{\lambda-1}\left(1+x^\lambda\right)^{-1} \left\{\log\left(1+x^\lambda\right)\right\}^{-(\alpha+1)} \left[1+\left\{\log\left(1+x^\lambda\right)\right\}^{-\alpha}\right]^{-(\beta[q+1]+1)}, \tag{21}$$

applying the lemmas in (21), we have

$$\left[1+\left(\log\left[1+x^\lambda\right]\right)^{-\alpha}\right]^{-(\beta[q+1]+1)} = \sum_{k=0}^{\infty} \binom{\beta[q+1]+k}{k} (-1)^k \left(\log\left[1+x^\lambda\right]\right)^{-\alpha k},$$

$$\left(\log\left[1+x^\lambda\right]\right)^{-(\alpha[k+1]+1)} = \sum_{m=0}^{\infty}\sum_{s=0}^{\infty} \binom{-\alpha[k+1]+1}{m} b_{s,m} \left[1-\left(1+x^\lambda\right)^{-1}\right]^{m+s-\alpha[k+1]-1},$$

$$\left[1-\left(1+x^\lambda\right)^{-1}\right]^{m+s-\alpha[k+1]-1} = \sum_{p=0}^{\infty} \binom{m+s-\alpha[k+1]-1}{p} (-1)^p \left(1+x^\lambda\right)^{-p}.$$

Substituting these expressions into (21), we have

$$f(x)F(x)^q = \alpha\,\beta\,\lambda \sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\binom{\beta[q+1]+k}{k}\binom{-\alpha[k+1]+1}{m}\binom{m+s-\alpha[k+1]-1}{p}(-1)^{k+p}\,b_{s,m}\,x^{\lambda-1}\left(1+x^{\lambda}\right)^{-(p+1)} \quad (22)$$

By inserting (22) into (20) and further simplification, we obtain the PWMs of the IBLL distribution as

$$\rho_{q,r} = \alpha\,\beta \sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\binom{\beta[q+1]+k}{k}\binom{-\alpha[k+1]+1}{m}\binom{m+s-\alpha[k+1]-1}{p}(-1)^{k+p}\,b_{s,m}\,B\left[1+\frac{r}{\lambda},\,p-\frac{r}{\lambda}\right].$$
(23)

From (23), we remark that the PWMs of the inverse Burr log-logistic distribution can be expressed as a linear combination of the log-logistic densities.

2.5 Distribution of Order Statistics

Let $X_1, X_2, \ldots, X_n$ be random samples of size *n* from a known probability distribution. Suppose $X_{r:n}$ denotes the r^{th} order statistics, then the density function of $X_{r:n}$ is defined by

$$f_{r:n}(x) = \frac{1}{B(r, n-r+1)}\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j f(x)F(x)^{r+j-1}. \quad (24)$$

Inserting (5) and (6) into (24), we define the r^{th} order statistics of the density of IBLL distribution as follows.

$$f(x)F(x)^{r+j-1} = \alpha\,\beta\,\lambda\, x^{\lambda-1}\left(1+x^{\lambda}\right)^{-1}\left\{\log\left(1+x^{\lambda}\right)\right\}^{-(\alpha+1)}\left[1+\left\{\log\left(1+x^{\lambda}\right)\right\}^{-\alpha}\right]^{-(\beta[r+j]+1)}. \quad (25)$$

We further simplify (25) using a similar approach in (21) as

$$f(x)F(x)^{r+j-1} = \alpha\,\beta\,\lambda \sum_{m=0}^{\infty}\varpi_m\, x^{\lambda-1}\left(1+x^{\lambda}\right)^{-(p+1)}, \quad (26)$$

Substituting (26) into (24), we have

$$f_{r:n}(x) = \frac{\alpha\,\beta\,\lambda}{B[r, n-r+1]}\sum_{m=0}^{\infty}\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j\varpi_m\, x^{\lambda-1}\left(1+x^{\lambda}\right)^{-(p+1)} \quad (27)$$

where

$$\varpi_m = \sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\binom{\beta[q+1]+k}{k}\binom{-\alpha[k+1]+1}{m}\binom{m+s-\alpha[k+1]-1}{p}(-1)^{k+p}\,b_{s,m}.$$

(27) is readily the r^{th} order statistics of the density function of IBLL distribution.

An expression for the q^{th} moment of the r^{th} order statistics of the density of IBLL distribution is obtained using (27) as

$$E\left[X_{r:n}^{q}\right] = \frac{\alpha\,\beta}{B\left[r, n-r+1\right]} \sum_{m=0}^{\infty} \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^{j}\, \varpi_m\, B\left[1+\frac{r}{\lambda}, p-\frac{r}{\lambda}\right]. \tag{28}$$

Again, we show that the q^{th} moment of the r^{th} order statistics of the density of IBLL distribution can be expressed as a linear combination of the log-logistic densities.

2.6 Rényi Entropy

The entropy of a random variable *X* represents the measure of randomness associated with the random variable *X*. The Rényi entropy of *X* is defined by

$$\tau_R(\gamma) = \frac{1}{1-\gamma} \log \int_{-\infty}^{\infty} f^{\gamma}(x)\,dx, \qquad \gamma > 0,\ \gamma \neq 1. \tag{29}$$

The Rényi entropy of a random variable X following the IBLL distribution is derived by inserting (6) into (29) as

$$\tau_R(\gamma) = \frac{1}{1-\gamma} \log\left[(\alpha\beta\lambda)^{\gamma} \int_{-\infty}^{\infty} x^{\gamma(\lambda-1)} \left(1+x^{\lambda}\right)^{-\gamma} \left\{\log\left(1+x^{\lambda}\right)\right\}^{-\gamma(\alpha+1)} \left[1+\left\{\log\left(1+x^{\lambda}\right)\right\}^{-\alpha}\right]^{-\gamma(\beta+1)} \right]. \tag{30}$$

Applying the lemmas in (30), we have

$$\left[1+\left(\log\left[1+x^{\lambda}\right]\right)^{-\alpha}\right]^{-\gamma(\beta+1)} = \sum_{k=0}^{\infty} \binom{\gamma(\beta+1)+k-1}{k} (-1)^{k} \left(\log\left[1+x^{\lambda}\right]\right)^{-\alpha k},$$

$$\left(\log\left[1+x^{\lambda}\right]\right)^{-(\alpha[k+\gamma]+\gamma)} = \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} \binom{-\alpha[k+\gamma]-\gamma}{m} b_{s,m} \left[1-\left(1+x^{\lambda}\right)^{-1}\right]^{m+s-\alpha[k+\gamma]-\gamma},$$

$$\left[1-\left(1+x^{\lambda}\right)^{-1}\right]^{m+s-\alpha[k+\gamma]-\gamma} = \sum_{p=0}^{\infty} \binom{m+s-\alpha[k+\gamma]-\gamma}{p} (-1)^{p} \left(1+x^{\lambda}\right)^{-p}.$$

Substituting these expressions into (30), we have

$$\tau_R(\gamma) = \frac{1}{1-\gamma} \log\left[(\alpha\beta\lambda)^{\gamma} \sum_{k=0}^{\infty} \omega_k \int_{0}^{\infty} x^{\gamma(\lambda-1)} \left(1+x^{\lambda}\right)^{-(p+\gamma)} dx \right]. \tag{31}$$

Evaluating the integral function in (31) yields,

$$\tau_R(\gamma) = \frac{1}{1-\gamma} \log\left[(\alpha\beta)^{\gamma} \lambda^{\gamma-1} \sum_{k=0}^{\infty} \omega_k\, B\left[\frac{\gamma(\lambda-1)+1}{\lambda}, \frac{\gamma+\lambda p-1}{\lambda}\right] \right]. \tag{32}$$

where,

$$\omega_k = \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} \binom{\gamma(\beta+1)+k-1}{k} \binom{-\alpha[k+\gamma]-\gamma}{m} \binom{m+s-\alpha[k+\gamma]-\gamma}{p} (-1)^{k+p}\, b_{s,m}.$$

3. METHODS OF PARAMETER ESTIMATION

In this section, five estimators, i.e., maximum likelihood, least squares, weighted least squares, Anderson-Darling, and Cramer-von Mises, in order to estimate the unknown parameters of the IBLL distribution are investigated. Let $X_1, X_2, \ldots, X_n$ be a random sample from the $\mathrm{IBLL}(\Xi)$ distribution, $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ represent the associated order statistics and $x_{(i)}$ indicates the observed values of $X_{(i)}$ for $i = 1, 2, \ldots, n,$ where $\Xi = (\alpha, \beta, \lambda)$. The likelihood and log-likelihood functions are obtained, respectively, by,

$$L(\Xi)=\alpha^n\beta^n\lambda^n\prod_{i=1}^{n}x_i^{\lambda-1}\left(1+x_i^{\lambda}\right)^{-1}\left\{\log\left(1+x_i^{\lambda}\right)\right\}^{-(\alpha+1)}\left[1+\left\{\log\left(1+x_i^{\lambda}\right)\right\}^{-\alpha}\right]^{-(\beta+1)} \tag{33}$$

and

$$\begin{aligned}\ell(\Xi)&=n\log(\alpha)+n\log(\beta)+n\log(\lambda)+(\lambda-1)\sum_{i=1}^{n}\log(x_i)\\&-\sum_{i=1}^{n}\log\left(1+x_i^{\lambda}\right)-(\alpha+1)\sum_{i=1}^{n}\log\left(\log\left(1+x_i^{\lambda}\right)\right)-(\beta+1)\sum_{i=1}^{n}\log\left(1+\left\{\log\left(1+x_i^{\lambda}\right)\right\}^{-\alpha}\right)\end{aligned} \tag{34}$$

Then, the maximum likelihood estimator (MLE) of Ξ is obtained by

$$\hat{\Xi}_1=\underset{\Xi}{argmax}\,\ell(\Xi). \tag{35}$$

Let us give the following functions that give us the four different estimators:

$$Q_{LS}(\Xi)=\sum_{i=1}^{n}\left(\left[1+\left\{\log\left(1+x_{(i)}^{\lambda}\right)\right\}^{-\alpha}\right]^{-\beta}-\frac{i}{n+1}\right)^2, \tag{36}$$

$$Q_{WLS}(\Xi)=\sum_{i=1}^{n}\frac{(n+2)(n+1)^2}{i(n-i+1)}\left(\left[1+\left\{\log\left(1+x_{(i)}^{\lambda}\right)\right\}^{-\alpha}\right]^{-\beta}-\frac{i}{n+1}\right)^2, \tag{37}$$

$$\begin{aligned}Q_{AD}(\Xi)=-n-\sum_{i=1}^{n}\frac{2i-1}{n}\Bigg[&\log\left\{\left[1+\left\{\log\left(1+x_{(i)}^{\lambda}\right)\right\}^{-\alpha}\right]^{-\beta}\right\}\\&+\log\left\{1-\left[1+\left\{\log\left(1+x_{(i)}^{\lambda}\right)\right\}^{-\alpha}\right]^{-\beta}\right\}\Bigg]\end{aligned} \tag{38}$$

and

$$QCvM(\Xi)=\frac{1}{12n}+\sum_{i=1}^{n}\left[\left[1+\left\{\log\left(1+x_{(i)}^{\lambda}\right)\right\}^{-\alpha}\right]^{-\beta}-\frac{2i-1}{2n}\right]^2. \tag{39}$$

Then, the least square estimator (LSE), the weighted least square estimator (WLSE), the Anderson-Darling estimator (ADE) and the Cramer-von Mises estimator (CvME) of the Ξ are achieved, respectively, by

$$\hat{\Xi}_2=\underset{\Xi}{argmin}\,Q_{LS}(\Xi), \tag{40}$$

$$\hat{\Xi}_3=\underset{\Xi}{argmin}\,Q_{WLS}(\Xi), \tag{41}$$

$$\hat{\Xi}_{4_4}=\underset{\Xi}{argmin}\,Q_{AD}(\Xi) \tag{42}$$

and

$$\hat{\Xi}_5=\underset{\Xi}{argmin}\,Q_{CvM}(\Xi). \tag{43}$$

All of the maximization and minimization problems in Equations (35), (40), (41), (42), and (43) can be obtained by **optim** function in the R software.

4. SIMULATION EXPERIMENTS

In this section, the bias and mean square errors (MSEs) of the estimators are calculated with 5000 reputations based on the Monte Carlo simulation. The quantile function given in Equation (9) is used to generate data from the $\text{IBLL}(\Xi)$ distribution by taking $U(0,1)$ instead of p, where $U(0,1)$ is the standard uniform distribution. Eight parameters setting are chosen based on Table 1 as $\Xi=(6,6,0.5)(S1)$, $\Xi=(6,8,0.5)(S2)$, $\Xi=(8,6,0.5)(S3)$, $\Xi=(8,8,0.5)(S4)$, $\Xi=(6,6,1)(S5)$, $\Xi=(6,8,1)(S6)$, $\Xi=(8,6,1)(S7)$ and $\Xi=(8,8,1)(S8)$.

The sample size $n = 50,100,150,200,250,500,1000$ is selected in the simulation experiment. The simulation results are given in Tables 2-3. It can be inferred from Tables 2 and 3 that as sample the size increases, bias and MSEs for all estimators decrease and converge to zero. When the sample size increases, the bias and MSE values of the estimators converge. Although the bias and MSE of the estimators converge with each other when the sample size increases, generally the LSE in bias gives better results than the others.

Table 2: *Average bias for all estimators*

		MLE			LSE			WLSE			ADE			CvME		
Ξ	n	A	B	λ	A	β	λ	α	β	λ	α	β	λ	α	β	λ
S1	50	0.9375	4.1957	0.0704	0.5942	0.2506	0.0135	1.0350	7.8711	0.1362	1.2633	1.1027	0.0256	0.8859	0.2643	0.0002
	100	0.7669	1.2045	0.0116	0.5610	0.0525	-0.0018	0.7021	3.3303	0.0527	0.8007	1.1953	0.0228	0.6974	0.1023	-0.0068
	150	0.7121	0.2743	-0.0095	0.4901	0.0965	0.0011	0.5846	1.3907	0.0236	0.6546	0.7200	0.0108	0.5870	0.1106	-0.0029
	200	0.6581	-0.0165	-0.0167	0.4909	-0.0139	-0.0032	0.5050	0.6594	0.0090	0.5543	0.4239	0.0035	0.5754	-0.0416	-0.0077
	250	0.6475	-0.2638	-0.0238	0.5789	-0.2129	-0.0129	0.4872	0.2729	-0.0004	0.5208	0.1843	-0.0032	0.6399	-0.2247	-0.0160
	500	0.5943	-0.5326	-0.0327	0.3437	0.1976	0.0017	0.4036	-0.1782	-0.0143	0.4037	-0.1478	-0.0131	0.3731	0.1995	0.0003
	1000	0.5817	-0.6845	-0.0380	0.2675	0.0253	-0.0040	0.3801	-0.3884	-0.0218	0.3547	-0.3211	-0.0187	0.2826	0.0241	-0.0047
S2	50	1.1512	4.9371	0.0428	0.3964	0.0485	0.0064	1.1587	10.6314	0.1139	1.4767	0.3899	-0.0102	0.6971	0.1149	-0.0078
	100	0.9526	1.6496	-0.0035	0.5254	-0.1075	-0.0100	0.8302	5.5053	0.0471	0.9739	1.0317	0.0011	0.6672	-0.0897	-0.0162
	150	0.8626	0.3856	-0.0203	0.4366	-0.1390	-0.0078	0.6930	2.9162	0.0208	0.7759	0.9129	0.0003	0.5498	-0.1608	-0.0130
	200	0.8322	-0.2357	-0.0301	0.4681	-0.3456	-0.0151	0.6290	1.0993	0.0021	0.6780	0.5895	-0.0046	0.5559	-0.3655	-0.0193
	250	0.8124	-0.5348	-0.0359	0.5519	-0.5409	-0.0222	0.6234	0.4238	-0.0090	0.6521	0.2444	-0.0114	0.6099	-0.5278	-0.0246
	500	0.7788	-1.1148	-0.0479	0.3783	0.0574	-0.0067	0.5499	-0.5278	-0.0268	0.5343	-0.4227	-0.0236	0.4093	0.0538	-0.0083
	1000	0.7426	-1.2944	-0.0518	0.3472	-0.1384	-0.0121	0.5063	-0.8184	-0.0333	0.4556	-0.6417	-0.0276	0.3623	-0.1365	-0.0128
S3	50	0.6404	6.6985	0.1097	0.6121	0.0227	0.0017	1.0037	11.7552	0.1591	1.4091	0.9300	0.0156	0.9293	0.1141	-0.0063
	100	0.3057	3.7878	0.0679	0.4339	0.4604	0.0109	0.4177	7.2752	0.1038	0.6287	1.8584	0.0394	0.6244	0.5242	0.0068
	150	0.2074	2.3201	0.0453	0.4792	0.2701	0.0040	0.2730	4.4954	0.0717	0.3839	1.9209	0.0406	0.5920	0.2946	0.0011
	200	0.1603	1.6798	0.0345	0.4903	0.1043	-0.0018	0.1926	3.0110	0.0533	0.2641	1.8022	0.0377	0.5702	0.1333	-0.0036
	250	0.1602	1.1944	0.0254	0.5091	-0.0343	-0.0059	0.1992	2.0675	0.0389	0.2532	1.4024	0.0295	0.5717	-0.0091	-0.0073
	500	0.1270	0.4670	0.0100	0.1819	0.5339	0.0128	0.1055	0.8581	0.0196	0.1325	0.7466	0.0169	0.2199	0.5181	0.0112
	1000	0.1233	0.1356	0.0015	0.1057	0.5177	0.0119	0.0784	0.3467	0.0079	0.0893	0.3217	0.0071	0.1247	0.5136	0.0113
S4	50	0.7510	7.9644	0.0905	0.3044	-0.0100	0.0053	1.0016	17.0148	0.1599	1.4711	0.6111	-0.0013	0.6719	0.0439	-0.0061
	100	0.3105	5.8365	0.0692	0.3377	0.1576	0.0021	0.4839	12.2917	0.1128	0.7972	1.4888	0.0181	0.5115	0.2013	-0.0031
	150	0.3283	3.4028	0.0406	0.4856	-0.0614	-0.0064	0.3546	8.6056	0.0796	0.5630	1.8400	0.0227	0.6109	-0.0267	-0.0097
	200	0.2099	2.9491	0.0373	0.4213	-0.0560	-0.0062	0.2381	6.0182	0.0632	0.3530	2.4052	0.0327	0.5157	-0.0249	-0.0085
	250	0.2275	1.8875	0.0244	0.4184	-0.1450	-0.0077	0.2428	4.3372	0.0472	0.3183	2.2085	0.0290	0.4790	-0.0962	-0.0089
	500	0.1828	0.7228	0.0081	0.1918	0.2020	0.0015	0.1329	1.6237	0.0218	0.1614	1.3081	0.0179	0.2385	0.1780	-0.0003
	1000	0.1476	0.2229	0.0007	0.1720	0.4073	0.0044	0.0791	0.6492	0.0096	0.0887	0.6102	0.0088	0.1914	0.4067	0.0038
S5	50	0.6470	5.5044	0.2278	0.5451	0.4398	0.0442	0.9605	8.6682	0.3141	1.0958	1.4447	0.0834	0.8335	0.4091	0.0121
	100	0.3333	2.5247	0.1206	0.4569	0.2635	0.0169	0.4849	4.1211	0.1599	0.5393	1.7448	0.0941	0.6018	0.3162	0.0062
	150	0.2436	1.4603	0.0759	0.4206	0.2336	0.0163	0.3433	2.2477	0.1053	0.3757	1.3905	0.0772	0.5227	0.2319	0.0067
	200	0.1539	1.0055	0.0589	0.4435	0.0369	-0.0027	0.2277	1.4496	0.0796	0.2551	1.1010	0.0633	0.5127	0.0303	-0.0101
	250	0.1581	0.6795	0.0412	0.4363	-0.0472	-0.0088	0.2019	1.0266	0.0603	0.2300	0.8008	0.0485	0.4926	-0.0507	-0.0145
	500	0.0856	0.3212	0.0204	0.1461	0.5693	0.0383	0.0978	0.4501	0.0300	0.1111	0.4035	0.0262	0.1750	0.5688	0.0352
	1000	0.0604	0.1184	0.0067	0.0730	0.3805	0.0257	0.0541	0.1913	0.0133	0.0625	0.1720	0.0115	0.0878	0.3796	0.0242
S6	50	0.6140	7.6142	0.2150	0.3106	0.2057	0.0289	0.9251	12.6867	0.3072	1.1159	1.0707	0.0333	0.5886	0.2560	0.0003
	100	0.3562	4.3665	0.1293	0.3879	0.1270	0.0016	0.5263	7.6389	0.1810	0.6175	2.0318	0.0689	0.5437	0.1983	-0.0100
	150	0.2326	2.6693	0.0876	0.3632	-0.1900	-0.0137	0.3453	4.6530	0.1209	0.3940	2.0958	0.0704	0.4666	-0.1442	-0.0223
	200	0.1950	1.6498	0.0573	0.3831	-0.2584	-0.0208	0.2838	2.8226	0.0844	0.3148	1.6056	0.0552	0.4609	-0.2513	-0.0278
	250	0.1257	1.4433	0.0550	0.3322	-0.1799	-0.0165	0.1810	2.2602	0.0784	0.2129	1.5243	0.0583	0.3892	-0.1762	-0.0221
	500	0.1055	0.5230	0.0188	0.1714	0.4174	0.0160	0.1191	0.8336	0.0312	0.1332	0.7311	0.0268	0.2057	0.3961	0.0118
	1000	0.0716	0.2093	0.0065	0.1057	0.5346	0.0210	0.0632	0.3884	0.0160	0.0725	0.3517	0.0137	0.1214	0.5303	0.0191
S7	50	0.5456	7.2943	0.2396	0.5129	0.1875	0.0157	0.8056	12.7427	0.3507	1.2360	1.1529	0.0444	0.8404	0.3053	-0.0004
	100	0.2945	4.1175	0.1439	0.6107	0.3300	0.0101	0.4796	7.3733	0.2060	0.6888	1.8438	0.0748	0.7686	0.4205	0.0035
	150	0.1463	2.6347	0.1066	0.5751	0.2452	0.0028	0.2828	5.0317	0.1533	0.3928	1.9872	0.0843	0.6942	0.2156	-0.0055
	200	0.1257	1.8119	0.0778	0.5207	0.0253	-0.0082	0.2434	2.8158	0.1010	0.2986	1.7793	0.0742	0.6036	0.0326	-0.0128
	250	0.0829	1.4882	0.0665	0.3985	0.1005	-0.0025	0.1842	2.2312	0.0830	0.2281	1.5328	0.0649	0.4688	0.1045	-0.0066
	500	0.0838	0.6181	0.0287	0.2951	0.2163	0.0051	0.1192	0.8505	0.0375	0.1453	0.7482	0.0323	0.3361	0.1959	0.0018
	1000	0.0161	0.3224	0.0170	0.0801	0.5628	0.0267	0.0286	0.4301	0.0217	0.0454	0.3875	0.0190	0.0986	0.5609	0.0255
S8	50	0.6720	8.8634	0.2082	0.3249	-0.0013	0.0092	1.0153	17.5598	0.3332	1.3750	0.9431	0.0138	0.6687	0.0618	-0.0120
	100	0.2777	6.5039	0.1560	0.4394	0.2362	0.0066	0.5192	12.4466	0.2254	0.8599	1.3023	0.0289	0.6414	0.1311	-0.0095
	150	0.1709	4.2887	0.1125	0.4596	0.0644	-0.0083	0.2647	9.4685	0.1790	0.4694	2.0652	0.0577	0.5743	0.1148	-0.0138
	200	0.1214	3.1787	0.0867	0.4665	-0.1327	-0.0159	0.2429	6.0748	0.1277	0.3481	2.4949	0.0680	0.5472	-0.0944	-0.0200
	250	0.1023	2.5330	0.0718	0.3430	0.1276	-0.0047	0.1813	4.7362	0.1024	0.2613	2.3733	0.0639	0.4220	0.1297	-0.0094
	500	0.0812	1.0523	0.0322	0.2729	-0.0724	-0.0093	0.1200	1.6726	0.0455	0.1514	1.3147	0.0370	0.3129	-0.0768	-0.0120
	1000	0.0486	0.5072	0.0161	0.1995	0.4641	0.0092	0.0792	0.7237	0.0212	0.0927	0.6599	0.0187	0.2190	0.4620	0.0078

Table 3: *Average MSEs for all estimators*

Ξ	n	MLE			LSE			WLSE			ADE			CvME		
		A	B	λ	α	β	λ	α	β	λ	α	β	λ	α	β	λ
S1	50	8.2297	164.7673	0.0863	5.0257	11.1636	0.0284	13.5481	510.6877	0.1856	11.8089	28.9090	0.0530	5.6933	10.612	0.0253
	100	4.1556	45.3542	0.0337	3.6519	7.9128	0.0172	6.0824	245.3384	0.0766	5.8939	32.5961	0.0414	3.9543	8.3217	0.0166
	150	2.8888	16.3444	0.0184	3.2908	6.6964	0.0158	3.9830	64.0081	0.0397	3.9870	21.7640	0.0294	3.5216	6.7264	0.0152
	200	2.1614	8.9423	0.0134	2.9582	5.8082	0.0143	2.8287	29.2083	0.0246	2.8786	14.9217	0.0222	3.1087	5.6329	0.0137
	250	1.7531	5.1202	0.0097	2.8691	5.1904	0.0127	2.2584	12.0601	0.0174	2.3131	9.9242	0.0166	2.9603	5.0758	0.0123
	500	1.0220	2.4029	0.0054	1.6475	6.4392	0.0127	1.1197	3.8997	0.0078	1.1672	4.0569	0.0082	1.6769	6.4718	0.0125
	1000	0.6702	1.4077	0.0035	0.8923	3.8779	0.0075	0.6056	1.7534	0.0039	0.6217	1.8678	0.0041	0.9029	3.8582	0.0074
S2	50	8.8167	217.3302	0.0646	3.0613	10.8334	0.0187	13.7269	786.3757	0.1495	12.2312	31.7471	0.0364	3.5605	12.515	0.0171
	100	4.4242	90.7887	0.0323	2.7898	13.0720	0.0137	6.2139	478.1817	0.0770	6.0499	44.3473	0.0340	2.9681	11.4890	0.0128
	150	3.1118	41.5761	0.0211	2.3110	10.2296	0.0116	4.1584	249.6613	0.0476	4.1946	43.6199	0.0292	2.5483	9.0402	0.0112
	200	2.5062	19.6554	0.0143	2.0040	6.8737	0.0091	3.0617	69.6146	0.0271	3.1435	31.6162	0.0230	2.1611	6.7379	0.0090
	250	2.0678	14.2230	0.0117	2.0892	7.0033	0.0093	2.5515	36.3777	0.0200	2.6280	24.2292	0.0188	2.1822	6.9496	0.0091
	500	1.3012	5.4485	0.0066	1.4503	9.4806	0.0100	1.2919	8.5585	0.0084	1.3441	9.0206	0.0090	1.4850	9.4552	0.0098
	1000	0.9071	3.5782	0.0048	0.9311	7.4636	0.0074	0.7359	4.1769	0.0048	0.7322	4.5383	0.0050	0.9446	7.5026	0.0074
S3	50	12.8929	251.5244	0.0902	6.4170	9.1597	0.0163	21.4541	828.8226	0.1873	17.4964	23.7414	0.0328	7.1741	8.4168	0.0140
	100	6.2261	120.4071	0.0488	5.0548	10.9579	0.0147	9.3815	515.7800	0.1039	8.4520	36.4127	0.0343	5.6795	11.9140	0.0145
	150	4.1157	58.1565	0.0294	4.7730	9.5840	0.0120	6.2919	260.9974	0.0624	5.7657	39.1557	0.0315	4.9602	9.3776	0.0117
	200	3.1126	33.9692	0.0208	4.0023	7.0750	0.0096	4.5631	137.2141	0.0415	4.3428	36.5044	0.0271	4.0819	7.1352	0.0094
	250	2.5849	21.0839	0.0155	3.5837	5.8694	0.0088	3.8436	88.9140	0.0295	3.7146	28.4078	0.0220	3.6651	5.9658	0.0086
	500	1.3443	5.8824	0.0064	2.3211	7.6789	0.0096	1.9815	14.4027	0.0119	1.9674	11.5138	0.0109	2.3334	7.5252	0.0093
	1000	0.6629	2.3354	0.0029	1.4838	6.5431	0.0074	0.9678	4.3373	0.0049	0.9659	4.1395	0.0048	1.4899	6.4957	0.0073
S4	50	12.7339	323.5838	0.0714	4.3827	7.9723	0.0099	21.7192	1370.9550	0.1690	17.0598	29.9717	0.0248	5.0078	7.4320	0.0083
	100	6.1949	233.2069	0.0480	3.4387	11.5590	0.0093	10.0976	1050.0127	0.1098	8.5251	41.0548	0.0258	3.5890	11.5626	0.0088
	150	4.3135	114.9577	0.0294	3.6243	11.2110	0.0084	6.6965	786.8659	0.0729	5.9103	46.8700	0.0239	3.8311	11.7176	0.0083
	200	3.2894	91.4859	0.0245	2.9855	10.5398	0.0077	5.0877	423.5659	0.0518	4.6558	63.3257	0.0259	3.1638	10.8964	0.0077
	250	2.7217	49.9875	0.0172	2.7306	8.8887	0.0071	4.2953	289.5450	0.0391	4.0458	62.1376	0.0235	2.8216	9.0300	0.0070
	500	1.3877	15.3314	0.0074	1.4374	7.0680	0.0054	2.0690	49.0908	0.0146	2.0385	29.1442	0.0125	1.4884	6.9479	0.0052
	1000	0.7350	5.9916	0.0036	1.4175	9.4144	0.0061	1.0540	11.8543	0.0062	1.0497	11.2937	0.0061	1.4281	9.4462	0.0061
S5	50	8.1641	195.7027	0.4446	5.0974	12.6878	0.1316	14.0401	549.4940	0.8162	11.6743	29.8464	0.2232	5.7930	11.6472	0.1094
	100	3.5454	74.3962	0.2060	3.6045	9.2761	0.0764	5.5268	251.6307	0.3654	5.0756	40.0784	0.1980	3.9402	10.0963	0.0746
	150	2.3815	34.2574	0.1209	3.3376	7.5331	0.0704	3.6924	108.0385	0.2098	3.5101	27.8111	0.1456	3.4972	7.3340	0.0670
	200	1.6541	15.8051	0.0797	2.8503	5.8383	0.0564	2.5552	36.9380	0.1343	2.4107	19.5292	0.1090	2.9013	5.6422	0.0531
	250	1.3997	9.0560	0.0595	2.5443	5.2502	0.0516	2.1205	22.5280	0.1001	2.0613	13.3906	0.0848	2.5984	5.1582	0.0496
	500	0.6750	3.4245	0.0268	1.4751	7.3287	0.0553	0.9738	5.7074	0.0412	0.9556	5.2053	0.0389	1.4930	7.3074	0.0543
	1000	0.3345	1.4597	0.0126	0.7874	4.6437	0.0347	0.4722	2.2476	0.0187	0.4699	2.1680	0.0183	0.7923	4.6257	0.0343
S6	50	7.8179	310.9795	0.3789	3.1524	13.3794	0.0889	13.7931	949.0247	0.7276	11.1650	34.4827	0.1613	3.5131	13.2201	0.0762
	100	3.6646	177.4051	0.2185	2.7496	14.5714	0.0580	5.8716	604.6772	0.4157	5.3194	56.4762	0.1663	3.0771	16.3857	0.0592
	150	2.2539	83.5288	0.1373	2.1236	8.1668	0.0425	3.5175	351.2598	0.2552	3.2827	57.1584	0.1426	2.2514	9.5506	0.0429
	200	1.6717	41.3850	0.0887	1.9406	7.3814	0.0383	2.6789	137.1735	0.1655	2.5211	40.1409	0.1089	2.0465	7.5400	0.0381
	250	1.3400	29.7434	0.0714	1.5852	6.9718	0.0360	2.1197	89.1849	0.1257	2.0253	33.1473	0.0929	1.6268	6.9089	0.0351
	500	0.6809	8.6834	0.0308	1.1749	9.5075	0.0411	0.9909	18.2522	0.0507	0.9762	14.8899	0.0473	1.1975	9.4081	0.0403
	1000	0.3562	3.7743	0.0154	0.8043	9.7517	0.0362	0.5063	6.3992	0.0241	0.5032	6.1113	0.0234	0.8103	9.7011	0.0358
S7	50	12.9186	286.8685	0.3886	6.3044	9.3243	0.0684	20.5358	896.3676	0.7858	16.6925	24.0555	0.1260	7.0636	9.8315	0.0612
	100	6.2518	140.0123	0.2125	6.1765	11.6613	0.0624	9.7176	495.3834	0.4156	8.6520	37.3454	0.1412	6.4745	13.3001	0.0617
	150	4.1932	66.0174	0.1328	5.1175	10.5405	0.0526	6.3932	326.9165	0.2768	5.8300	40.5734	0.1309	5.2672	9.7465	0.0491
	200	3.2774	37.2524	0.0905	3.9563	6.8701	0.0395	4.9175	118.5219	0.1600	4.6726	37.1928	0.1107	4.0352	6.7307	0.0383
	250	2.6473	27.0774	0.0716	3.0137	5.7995	0.0334	3.8568	94.0344	0.1251	3.7213	30.0171	0.0914	3.0889	5.7940	0.0325
	500	1.3850	7.6326	0.0304	2.2580	5.8288	0.0312	1.9551	13.9261	0.0473	1.9263	11.9486	0.0439	2.2817	5.6618	0.0302
	1000	0.6859	2.7299	0.0132	1.4708	6.8897	0.0303	0.9413	4.6254	0.0202	0.9364	4.3075	0.0194	1.4775	6.8678	0.0300
S8	50	13.5950	362.1085	0.3166	4.4474	9.4767	0.0446	22.7560	1438.6746	0.7077	17.3753	32.9939	0.1059	4.8009	8.7280	0.0384
	100	6.4227	265.6164	0.2145	4.3368	16.3254	0.0513	10.5993	1111.1386	0.4408	8.9670	38.2959	0.0998	4.6587	14.2683	0.0449
	150	4.3203	140.1165	0.1392	3.5661	14.5328	0.0375	6.6212	864.3592	0.3178	5.7875	48.2825	0.0983	3.7190	14.6579	0.0373
	200	3.3002	97.5383	0.1041	3.2768	10.2293	0.0329	5.2374	430.0210	0.2116	4.8011	66.7278	0.1074	3.3280	10.6299	0.0326
	250	2.7079	68.5264	0.0814	2.4830	10.6151	0.0310	4.0239	343.1026	0.1598	3.7871	63.5251	0.0945	2.6327	10.5669	0.0303
	500	1.4285	19.2085	0.0353	1.6788	6.0184	0.0197	2.1088	52.8106	0.0608	2.0804	29.9620	0.0515	1.7198	6.0311	0.0195
	1000	0.7436	7.1666	0.0165	1.5655	10.8404	0.0273	1.0989	13.4083	0.0267	1.0918	12.3481	0.0255	1.5768	10.8464	0.0271

5. Real Data Analysis

In this section, two applications to the real data sets are examined to demonstrate the applicability of the IBLL distribution. We compare the IBLL distribution with Log-Logistic (LL), Burr III (BIII), Burr XII (BXII), Weibull (W) and Lindley (L) for two data sets. The pdfs of these distributions are given in Table 4. The MLEs of parameters and standard (SE) of MLEs are obtained and reported in Tables 5-6 for two datasets. To select the best distribution some criteria and goodness of-fit statistics such as the estimated log-likelihood values $\hat{\ell}(\Xi)$, Akaike information criteria (AIC), Bayesian information criteria (BIC), consistent Akaike information criteria (CAIC), Hannan–Quinn information criterion (HQIC), Kolmogorov-Smirnov (KS), Anderson-Darling (AD) and Cramer von Mises (CvM) statistics and related p values (KS-pval, AD-pval, CvM-pval) are calculated for all distributions. The fitted cdfs for two data sets are plotted in Figures 4 and 5. It is easily seen from Tables 5–6 and Figures 4–5 that the IBLL distribution gives the best modeling for both datasets, according to all criteria. The IBLL distribution can be used and be a good alternative in the literature because of its superior modeling capability.

The first data set represents the remission times (in months) of a random sample of 128 bladder cancer patients and it can be found in [10]. The first data is given by 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39,10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

The second data set represents the survival times (in days) of guinea pigs injected with different amount of tubercle bacilli and it can be consulted detail information in [3]. The second data is given by 12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83, 84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376.

Table 4: *The pdfs list of the all distributions*

Distribution	Pdf	Range of the parameters
IBLL	$f(x)=p_1\,p_2\,p_3\,x^{p_3-1}\left(1+x^{p_3}\right)^{-1}\left\{\log\left(1+x^{p_3}\right)\right\}^{-(p_1+1)}\left[1+\left\{\log\left(1+x^{p_3}\right)\right\}^{-p_1}\right]^{-(p_2+1)}$	$p_1,p_2,p_3>0$
LL	$f(x)=p_1\,x^{p_1-1}\left(1+x^{p_1}\right)^{-2}$	$p_1>0$
BIII	$f(x)=p_1p_2x^{-(p_1+1)}\left(1+x^{-p_1}\right)^{-(p_2+1)}$	$p_1,p_2>0$
BXII	$f(x)=p_1p_2x^{p_1-1}\left(1+x^{p_1}\right)^{-(p_2+1)}$	$p_1,p_2>0$
W	$f(x)=(p_1/p_2)(x/p_2)^{p_1-1}\exp\left(-(x/p_2)^{p_1}\right)$	$p_1,p_2>0$
L	$f(x)=p_1^2(x+1)/(1+p_1)\exp(-p_1x)$	$p_1>0$

Table 5: *The modelling results for first data set*

	IBLL	LL	BIII	BXII	W	L
$\hat{\ell}(\Xi)$	-409.8744	-504.8603	-426.6864	-453.5166	-414.0869	-419.5299
AIC	825.7488	1011.7206	857.3729	911.0332	832.1738	841.0598
BIC	834.3048	1014.5726	863.0769	916.7372	837.8778	843.9118
CAIC	825.9423	1011.7523	857.4689	911.1292	832.2698	841.0916
HQIC	829.2251	1012.8794	859.6905	913.3508	834.4913	842.2186
KS	0.0345	0.5260	0.1017	0.2507	0.0700	0.1164
AD	0.1184	63.3436	2.9190	13.3638	0.9577	2.7853
CvM	0.0179	13.4609	0.4508	2.7195	0.1537	0.5191
KS-pval	0.9980	0.0000	0.1413	0.0000	0.5570	0.0623
AD-pval	0.9998	0.0000	0.0302	0.0000	0.3801	0.0353
CVM-pval	0.9986	0.0000	0.0531	0.0000	0.3789	0.0355
$\hat{p}_1$	15.8105	0.7897	1.0333	2.3349	1.0478	0.1960
$\hat{p}_2$	0.4855		4.2070	0.2337	9.5607	
$\hat{p}_3$	0.2312					
SE of $\hat{p}_1$	3.2570	0.0556	0.0604	0.3541	0.0676	0.0123
SE of $\hat{p}_2$	0.1236		0.4054	0.0400	0.8529	
SE of $\hat{p}_3$	0.0182					

Table 6: *The modelling results for second data set*

	IBLL	LL	BIII	BXII	W	L
$\hat{\ell}(\Xi)$	-389.6891	-526.9707	-395.5659	-490.5493	-397.1477	-394.5197
AIC	785.3781	1055.9415	795.1318	985.0986	798.2953	791.0394
BIC	792.2081	1058.2182	799.6852	989.6519	802.8487	793.3160
CAIC	785.7311	1055.9986	795.3057	985.2725	798.4693	791.0965
HQIC	788.0972	1056.8478	796.9445	986.9113	800.1080	791.9457
KS	0.0871	0.7210	0.1512	0.4813	0.1465	0.1326
AD	0.5375	57.3053	1.4907	23.5566	2.3730	1.8706
CvM	0.0898	12.0148	0.2500	5.0353	0.4312	0.3452
KS-pval	0.6450	0.0000	0.0745	0.0000	0.0911	0.1592
AD-pval	0.7083	0.0000	0.1787	0.0000	0.0580	0.1085
CVM-pval	0.6384	0.0000	0.1884	0.0000	0.0596	0.1011
$\hat{p}_1$	29.2029	0.3533	1.4165	48.7608	1.3932	0.0198
$\hat{p}_2$	1.2525		286.9916	0.0047	110.5552	
$\hat{p}_3$	0.1294					
SE of $\hat{p}_1$	5.8536	0.0322	0.1163	18.8737	0.1184	0.0016
SE of $\hat{p}_2$	0.5107		125.6638	0.0017	9.9344	
SE of $\hat{p}_3$	0.0081					

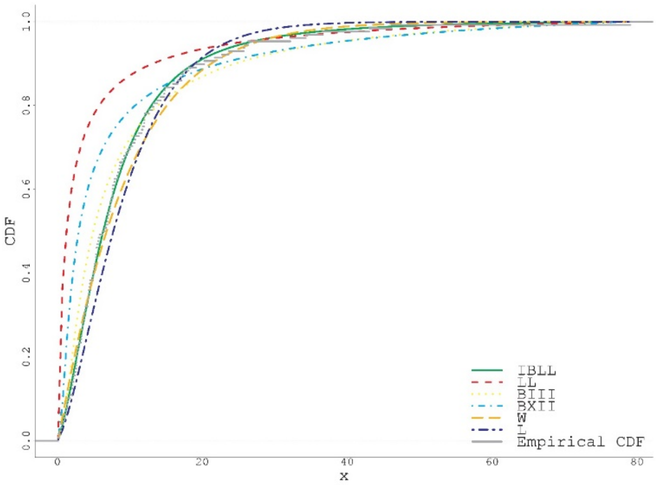

Figure 4: Empirical and fitted cdf plots for the first data

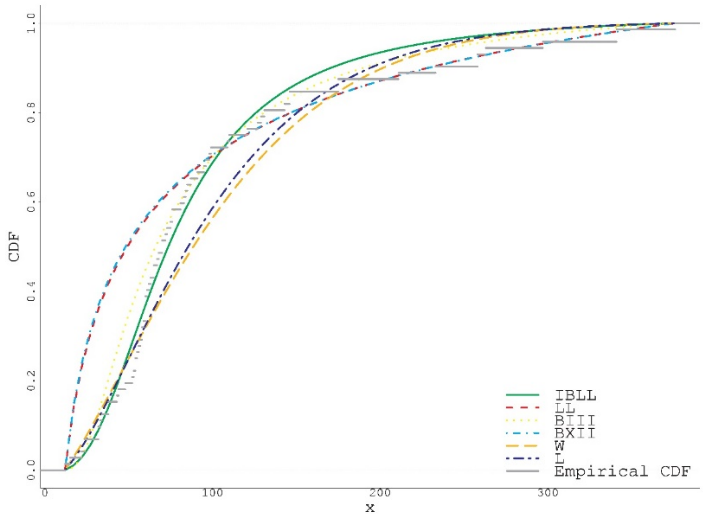

Figure 5: Empirical and fitted cdf plots for the second data

6. CONCLUSION

In this article, a new flexible distribution is introduced. The density and hazard functions of the new model are illustrated by various plots that are very flexible. Many properties related to the distribution have been obtained. Rényi entropy, which is an important measure of randomness, is examined. Some estimation techniques are used to investigate the parameter estimation problem. The performances of the estimators are examined by Monte Carlo simulation. Finally, bladder cancer and survival times of guinea pig data are modeled. As a result of the modeling, it was seen that the best distribution according to all criteria is the IBLL distribution for both data.

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