HUNTSBERGER TYPE SHRINKAGE ENTROPY ESTIMATOR FOR VARIANCE OF NORMAL DISTRIBUTION UNDER LINEX LOSS FUNCTION

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Abstract

The aim of the paper is to develop a better estimator for the entropy function of variance of the normal distribution. The present paper proposes a Huntsberger type shrinkage estimator of the entropy function for the variance of normal distribution. This Huntsberger type shrinkage entropy estimator is based on test statistic, which eliminates arbitrariness of choice of shrinkage factor. For the proposed estimator risk expressions under LINEX loss function have been calculated. Numerical computations and graphical analysis is carried out for risk and relative risks for the proposed estimators. It is also compared with the existing best estimator for distinct degrees of asymmetry and different levels of significance. Based on the criteria of relative risk, it is found that the proposed Huntsberger type shrinkage estimator is better than the existing estimator for the entropy function of variance of normal distribution for smaller values of level of significance and degrees of freedom..

Keywords: Normal distribution, entropy function, shrinkage estimation, LINEX loss function, level of significance, relative risk.

1. Introduction

Normal distribution plays a vital role in theory of statistics. Its testimation and estimating its parameters have been acknowledged and refined by researchers. Pandey et al. [9] proposed some shrinkage testimators of variance under the mean square error criterion. Parsian and Farsipour [10], Mishra and Meulen [7], Ahmadi et al. [1], Singh et al. [14], Prakash et al [12], Prakash and Pandey [11] and others have studied the estimation methods under LINEX loss function in distinct contexts. The concept of entropy was introduced by Shannon [13] and is given as

$$H(f) = E[-\ln(f(X))], \tag{1}$$

where X is a random variable having probability density function f and distribution function F. For sharply peaked distribution entropy is very low and is much higher when the probability is spread out. Many authors worked on the estimation entropy for different life distributions. Misra et al. [8] proposed an entropy estimator for a multivariate normal distribution while Jeevanand and Abdul- Sathar [4] also obtained estimators for the residual entropy function of exponential distribution from censored samples. Lazo and Rathee [6] and Kayal and Kumar [5] also worked in this direction.

Suppose the random variable X has the probability distribution $f(x,\theta)$ where interest is to estimate entropy function as a function of θ. Thomson [17] proposed a shrinkage type estimator $k\hat{\theta}+(1-k)\theta_0$, where k is constant and is designed to shrink the usual estimator $\hat{\theta}$ of the parameter θ towards a natural origin θ_0 and Huntsberger [3] introduced weighted shrinkage estimator of the form

$$\tilde{\theta}_f = \phi(\hat{\theta})\hat{\theta} + (1-\phi(\hat{\theta}))\theta_0 ,$$

where $\phi(.)$, $0 \le \phi(.) \le 1$, represents a weighted function specifying the degree of belief in θ_0. In this paper, we shall concentrate on obtaining Huntsberger type shrinkage estimation of entropy function with respect to asymmetric loss function for a random sample $x_1, x_2, \dots, x_m$ of size m from a normal distribution.

The form of normal density we consider is

$$f(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} ; -\infty < x < \infty,\ -\infty < x < \infty,\ \sigma > 0 \tag{2}$$

For the normal distribution, the entropy function can be obtained as

$$H(f) = \frac{1}{2} + \frac{1}{2}\ln(2\pi) + \frac{1}{2}\ln(\sigma^2) \tag{3}$$

Since $H(f)$ is linear function of $\frac{1}{2}\ln(\sigma^2)$, estimating $H(f)$ is correspondent to estimating $\frac{1}{2}\ln(\sigma^2)$. We shall write $I(\sigma^2) = \frac{1}{2}\ln(\sigma^2)$ so that $H(f) = \frac{1}{2} + \frac{1}{2}\ln(2\pi) + I(\sigma^2)$. Now we shall discuss estimation of $I(\sigma^2)$. Since $I(\sigma^2)$ is continous function of σ^2, the MLE of $I(\sigma^2)$ is obtained by replacing σ^2 by its MLE $\hat{\sigma}^2$ in $I(\sigma^2)$. Then, the MLE of entropy function for the exponential distribution is

$$\begin{aligned} H(f) &= \frac{1}{2} + \frac{1}{2}\ln(2\pi) + \frac{1}{2}\ln(\hat{\sigma}^2) \\ &= \frac{1}{2} + \frac{1}{2}\ln(2\pi) + \frac{1}{2}\ln(s^2) \end{aligned} \tag{4}$$

where $s^2 = \frac{1}{m}\sum_{i=1}^{m}\left(x_i - \bar{x}\right)^2$ is MLE of σ^2, when μ is unknown.

It can be shown that s^2 has distribution as

$$f(s^2) = \frac{1}{2^{\frac{m-1}{2}}\Gamma\left(\frac{m-1}{2}\right)} e^{-\frac{ms^2}{2\sigma^2}} \left(s^2\right)^{\frac{m-1}{2}-1} \left(\frac{m}{\sigma^2}\right)^{\frac{m-1}{2}} \tag{5}$$

Although the SELF (squared error loss function) is commonly used for estimating various statistics parameters, it may not be convenient in actual situations, particularly in insurance claims, estimating any health statistics parameter, over-estimation and under-estimaton have distinct impacts. An analysis of various superior properties of asymmetric loss function over squared error loss function has been presented by several authors. Basu and Ebrahimi [2] derived Bayes estimators for mean lifetime and reliability function for the exponential model using asymmetric loss function. Srivastava and Tanna [16] as well as Srivastava and Shah [15] also derived estimators and studied their properties under asymmetric loss function.

A fruitful asymmetric loss function, LINEX loss function was recommended by Varian [18] as:

$$L(\Delta)=b(e^{a\Delta}-a\Delta-1),\ b>0,\ a\neq 0. \tag{6}$$

The magnitude and sign of 'a' shows the degree of asymmetry and direction respectively. When overestimation is more severe than underestimation then positive values of 'a' are taken, whereas in reverse situations its negative values are usually preferred and 'b' is constant of proportionality.

2. The Shrinkage Estimator

From a normal population with mean μ and variance σ^2, a random sample $x_1, x_2,\ldots,x_m$ of size m is taken. Let the initial guess value for σ^2 is presumed to be $\sigma_0{}^2$, available from the past knowledge or some other reliable origins. It is noted that MLE of σ^2 is s^2 having variance of $s^2=\frac{2(n-1)\sigma^4}{n^2}$.

Here, we test the null hypothesis $H_0: \sigma^2=\sigma_0{}^2$ against the alternative $H_1: \sigma^2\neq\sigma_0{}^2$ using the test statistic $\frac{ms^2}{\sigma_0^2}$, where $s^2=\frac{1}{m}\sum_{i=1}^{m}(x_i-\bar{x})^2$ which follows χ^2- distribution with degree of freedom $m-1$. If $\chi_1{}^2\leq\frac{ms^2}{\sigma_0^2}\leq\chi_2{}^2$ then H_0 may be approved at $\alpha\%$ level of significance, where lower and upper α^{th} percentile values of χ^2 distribution are $\chi_1{}^2$ and $\chi_2{}^2$ respectively with degree of freedom $m-1$. Then by taking shrinkage factor $k=\frac{ms^2}{\sigma_0^2\chi^2}$, which is negatively associated with $\chi^2(=(\chi_2{}^2-\chi_1{}^2))$, a shrinkage entropy estimator may be considered. If data does not hold, H_0 it may be dropped and in this case it is recommended to use $\frac{1}{2}\ln(s^2)$, the MLE of $\frac{1}{2}\ln(\sigma^2)$.

Thus, the proposed shrinkage entropy estimator $\tilde{I}_1(\sigma^2)$ of $\frac{1}{2}\ln(\sigma^2)$ is as under:

$$\tilde{I}_1(\sigma^2)=\begin{cases} k(\frac{1}{2}\ln(s^2))+(1-k)(\frac{1}{2}\ln(\sigma_0^2)), & \text{if} \quad \chi_1{}^2\leq\frac{ms^2}{\sigma_0^2}\leq\chi_2{}^2 \\ \frac{1}{2}\ln(s^2) \quad , & \text{Otherwise} \end{cases} \tag{7}$$

3. Risk of Estimator

Risk of estimator $\tilde{I}_1(\sigma^2)$ under LLF is obtained as under:

$$R_{LLF}(\tilde{I}_1(\sigma^2))=E(\tilde{I}_1(\sigma^2)/LLF)$$

$$=\int_{\frac{\sigma_0^2\chi_1{}^2}{m}}^{\frac{\sigma_0^2\chi_2{}^2}{m}}\left(\begin{array}{l}\exp(a((\frac{ms^2}{2\sigma_0^2\chi^2})\ln(s^2)+(1-(\frac{ms^2}{\sigma_0^2\chi^2}))\frac{\ln(\sigma_0^2)}{2}-\frac{\ln(\sigma^2)}{2})) \\ -a((\frac{ms^2}{2\sigma_0^2\chi^2})\ln(s^2)+(1-(\frac{ms^2}{\sigma_0^2\chi^2}))\frac{\ln(\sigma_0^2)}{2}-\frac{\ln(\sigma^2)}{2})-1\end{array}\right)f(s^2)ds^2$$

$$+\int_0^\infty\left(\exp(\frac{a}{2}(\ln(s^2)-\ln(\sigma^2)))-\frac{a}{2}(\ln(s^2)-\ln(\sigma^2))-1\right)f(s^2)ds^2$$

$$-\int_{\frac{\sigma_0^2\chi_1{}^2}{m}}^{\frac{\sigma_0^2\chi_2{}^2}{m}}\left(\exp(\frac{a}{2}(\ln(s^2)-\ln(\sigma^2)))-\frac{a}{2}(\ln(s^2)-\ln(\sigma^2))-1\right)f(s^2)ds^2$$

Straight forward integration gives

$$R_{LLF}(\hat{I}_1(\sigma^2)) = I_1 + I_2 + I_3 + I_4 - \frac{au\log(\frac{2}{m\phi})}{\phi\chi^2}\left[I(r_2', u+1) - I(r_1', u+1)\right] + v\ln(\frac{2}{m\phi})\left[I(r_2', u) - I(r_1', u)\right]$$
$$-\frac{2^v\Gamma(u+v)}{m^v\Gamma(u)}\left[I(r_2', u+v) - I(r_1', u+v)\right] - v\ln(\frac{2}{m}) - 1 + \frac{2^v\Gamma(u+v)}{m^v\Gamma(u)}, \tag{8}$$

where $r_1' = \frac{\phi\chi_1^2}{2}$, $r_2' = \frac{\phi\chi_2^2}{2}$, $\phi = \frac{\sigma_0^2}{\sigma^2}$, $u = \frac{m-1}{2}$, $v = \frac{a}{2}$ and $I(x, n)$ is the cumulative distribution function of gamma distribution given as

$$I(x, n) = \int_0^x \frac{e^{-t}t^{n-1}}{\Gamma(n)}dt$$

and

$$I_1 = \int_{r_1'}^{r_2'} \phi^v \left(\frac{2t}{m\phi}\right)^{\frac{at}{\phi\chi^2}} \frac{e^{-t}t^{u-1}}{\Gamma(u)}dt, \quad I_2 = \frac{-au}{\phi\chi^2}\int_{r_1'}^{r_2'} (\log t)\frac{e^{-t}t^u}{\Gamma(u+1)}dt, \quad I_3 = v\int_{r_1'}^{r_2'} (\log t)\frac{e^{-t}t^{u-1}}{\Gamma(u)}dt$$

and

$$I_4 = -v\int_0^\infty (\log t)\frac{e^{-t}t^{u-1}}{\Gamma(u)}dt$$

4. Relative Risk

A common way of analyzing risk of considered estimator, is to examine its work relative to the best possible estimator $\hat{I}(\sigma^2)$ in this case. With this motto, we calculate risk of $\hat{I}(\sigma^2)$ as:

$$R_{LLF}\left(\hat{I}(\sigma^2)\right) = E\left(\hat{I}(\sigma^2) \setminus LLF\right)$$
$$= \int_0^\infty (\exp(a(\frac{1}{2}\ln(s^2) - \frac{1}{2}\ln(\sigma^2))) - a(\frac{1}{2}\ln(s^2) - \frac{1}{2}\ln(\sigma^2)) - 1)f(s^2)ds^2$$
$$= \int_0^\infty (\exp(\frac{a}{2}\ln(\frac{s^2}{\sigma^2})) - \frac{a}{2}\ln(\frac{s^2}{\sigma^2}) - 1)f(s^2)ds^2$$

Now, taking the transformation $t = \frac{ms^2}{2\sigma^2}$ and then solving the integral, we get

$$R_{LLF}\left(\hat{I}(\sigma^2)\right) = \frac{2^v\Gamma(u+v)}{m^v\Gamma(u)} - v\ln(\frac{2}{m}) - 1 - v\psi(u), \tag{9}$$

where

$$\psi(n) = \frac{\frac{d}{dn}\Gamma(n)}{\Gamma(n)}$$

Now, we determine relative risk of $\tilde{I}_1(\sigma^2)$ under LLF as

$$RR_{LLF}(\tilde{I}_1(\sigma^2)) = \frac{R_{LLF}(\hat{I}(\sigma^2))}{R_{LLF}(\tilde{I}_1(\sigma^2))} \tag{10}$$

Using (8) and (9) the expression given in (10) can be obtained. It is observed that relative risk given above is a function of m, a, ϕ and α.

5. Numerical Computations And Graphical Analysis

To examine the performance of $\tilde{I}_1(s^2)$, a few values of these parameters have been taken as α = 0.01, 0.05, 0.1, m=5, 8, 11, ϕ = 0.2(0.2)2 and a=-2, -1, 1, 1.5, 1.75 i.e. both positive and negative values, as a is the best essential component that determines the seriousness of over/under estimation in the actual cases. Several tables and graphs for relative risk calculation are represented in Table 1, Figure 1 to Figure 6. However our recommendations depending on all these analyses are as follows:

i. For $m = 5$, $\alpha = 1\%$ and for considered values of 'a', $\tilde{I}_1(\sigma^2)$ gives better results than the existing estimator for all values of 'a' and for the whole scale of ϕ i.e. $0.2 \le \phi \le 2$.

ii. Further if we switch α to 5%, the same type of behavior noticed for relative risk (RR). However, magnitude of RR is smaller as computed to $\alpha = 1\%$ values.

iii. Taking $\alpha = 10\%$ in order to observe the pattern for higher values of α and it is found that $\tilde{I}_1(\sigma^2)$ still gives the better results as compared to the existing estimator whereas magnitude of relative risks values become lower but even though it remains above unity.

iv. After comparing these relative risks, a lower value of α is preferred. Similarly, as 'm' raises there is a fall in RR values for distinct values of α and a. However the best result of $\tilde{I}_1(\sigma^2)$ is observed at $\alpha = 1\%$ for $a = 1$ and $\alpha = 1\%$ for $a = 1.75$.

It is therefore suggested to take up a smaller values of $\alpha\,(=1\%)$ as well as m (=5 or 8) for better results for a, in particular a(= 1 & 1.75).

***Table 1**: Relative risk of estimator $\tilde{I}_1(\sigma^2)$ under LLF*

α=0.01		ϕ									
m	a	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
	-2	1.4182	2.3334	3.5356	4.3068	4.2010	3.6270	3.0222	2.5295	2.1539	1.8697
	-1	1.2002	1.9735	3.5429	5.7055	6.5919	5.5134	4.0969	3.0576	2.3684	1.9069
5	1	1.0382	1.4929	2.6592	5.1746	8.1241	7.0627	4.4323	2.7966	1.9057	1.3929
	1.5	1.0226	1.4256	2.4849	4.8294	7.8372	6.9390	4.2635	2.623	1.7512	1.2593
	1.75	1.0166	1.3967	2.4065	4.6584	7.6447	6.8323	4.1657	2.5355	1.6775	1.1972
	-2	0.9844	1.3487	2.3739	4.2491	5.384	4.3037	2.9433	2.0695	1.5473	1.2225
	-1	0.9594	1.2222	2.0956	4.0243	5.9406	4.8704	3.0967	2.0288	1.4387	1.0928
8	1	0.9493	1.0861	1.7061	3.2695	5.6045	4.9869	2.9165	1.7353	1.1407	0.8166
	1.5	0.9502	1.0664	1.6375	3.0932	5.3591	4.8641	2.8201	1.6510	1.0697	0.7563
	1.75	0.9509	1.0580	1.6064	3.0095	5.2275	4.7898	2.7692	1.6097	1.0358	0.7279
	-2	0.9615	1.0777	1.7002	3.2243	4.8735	3.9475	2.4526	1.5938	1.1345	0.872
	-1	0.9634	1.0316	1.5541	2.9483	4.8576	4.1132	2.4478	1.5165	1.0407	0.7778
11	1	0.9715	0.983	1.3555	2.4472	4.3008	3.9708	2.2651	1.3114	0.8494	0.6054
	1.5	0.9736	0.9764	1.3197	2.3413	4.117	3.8711	2.2019	1.2593	0.8058	0.5681
	1.75	0.9745	0.9738	1.3034	2.2913	4.0237	3.8159	2.1695	1.2339	0.7849	0.5504
α= 0.05											

	-2	1.4071	1.8922	2.2377	2.3004	2.1503	1.9251	1.7058	1.5193	1.3679	1.2466
	-1	1.2576	1.7852	2.4289	2.8421	2.7907	2.4451	2.0534	1.7221	1.4656	1.2712
5	1	1.1105	1.4803	2.1447	2.9731	3.3998	3.0605	2.3984	1.8212	1.4072	1.1213
	1.5	1.0933	1.4267	2.05	2.8825	3.3938	3.1	2.4085	1.7949	1.3598	1.0644
	1.75	1.0864	1.4029	2.0043	2.8294	3.3726	3.1043	2.4045	1.777	1.3339	1.0353
	-2	1.0733	1.3764	1.9092	2.4083	2.4372	2.0694	1.6564	1.3379	1.1146	0.96
	-1	1.0421	1.2813	1.7875	2.4168	2.6348	2.2633	1.7518	1.3543	1.0839	0.9028
8	1	1.0163	1.1634	1.5537	2.1943	2.6789	2.4327	1.8167	1.3106	0.9792	0.768
	1.5	1.0134	1.1445	1.5057	2.1206	2.6324	2.4297	1.8097	1.2887	0.9488	0.734
	1.75	1.0123	1.1362	1.4832	2.0833	2.6063	2.4223	1.8035	1.2768	0.9334	0.7174
	-2	1.0267	1.1768	1.5449	2.0735	2.2885	1.9517	1.4932	1.1549	0.9373	0.7999
	-1	1.0187	1.1351	1.4549	1.9908	2.3257	2.0333	1.5248	1.1407	0.8974	0.7464
11	1	1.0111	1.0846	1.3150	1.78	2.2286	2.0732	1.5306	1.0868	0.8091	0.6410
	1.5	1.01	1.0764	1.2877	1.7272	2.1801	2.0596	1.5219	1.07	0.7869	0.6161
	1.75	1.0095	1.0728	1.2749	1.7013	2.1535	2.0498	1.5162	1.0612	0.7759	0.6040

5.1. Graphs of Relative Risk for $\tilde{I}_1(\sigma^2)$

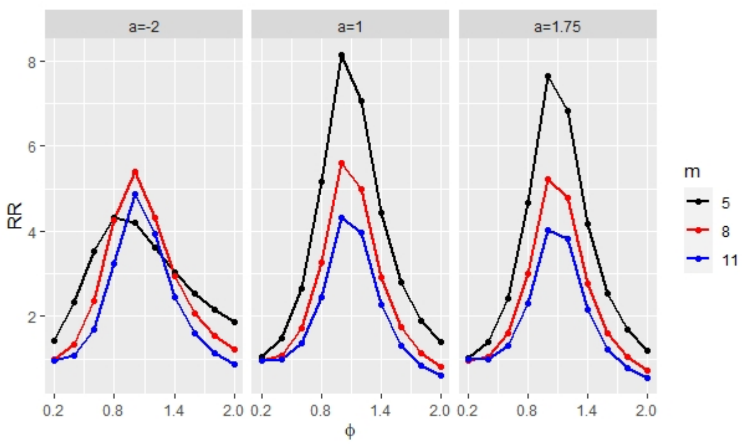

Figure 1: *For α=0.01*

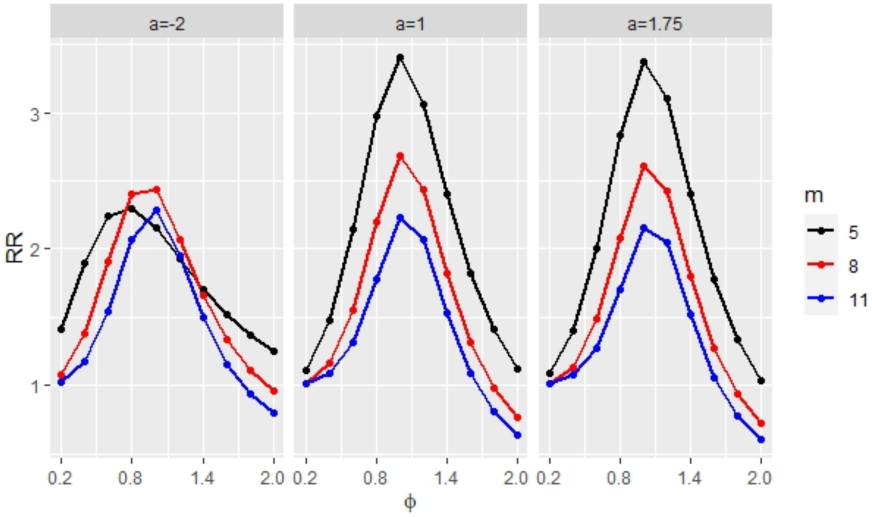

Figure 2: *For α=0.05*

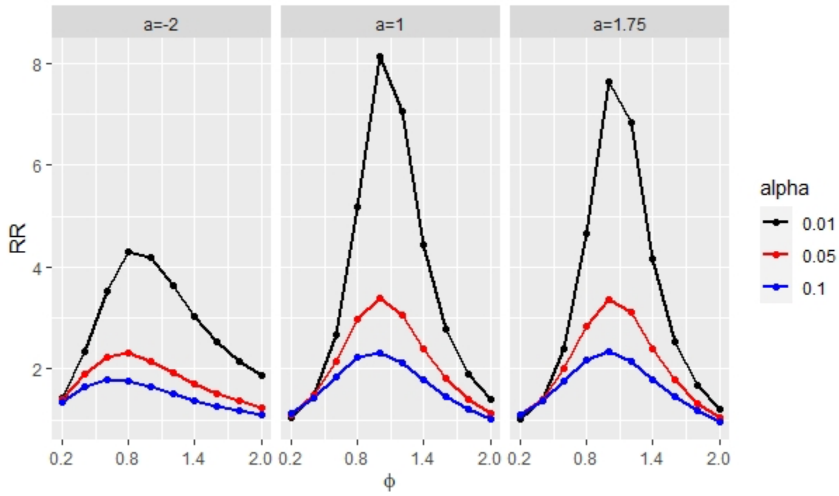

Figure 3: For m=5

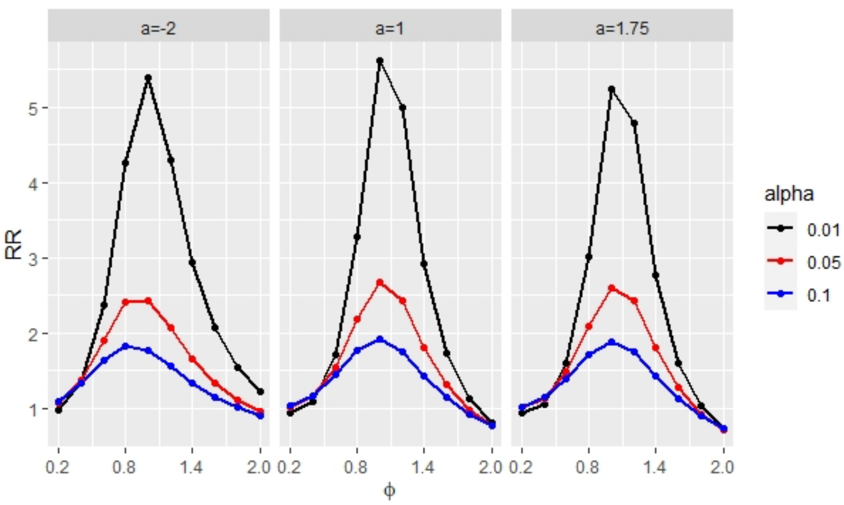

Figure 4: *For m=8*

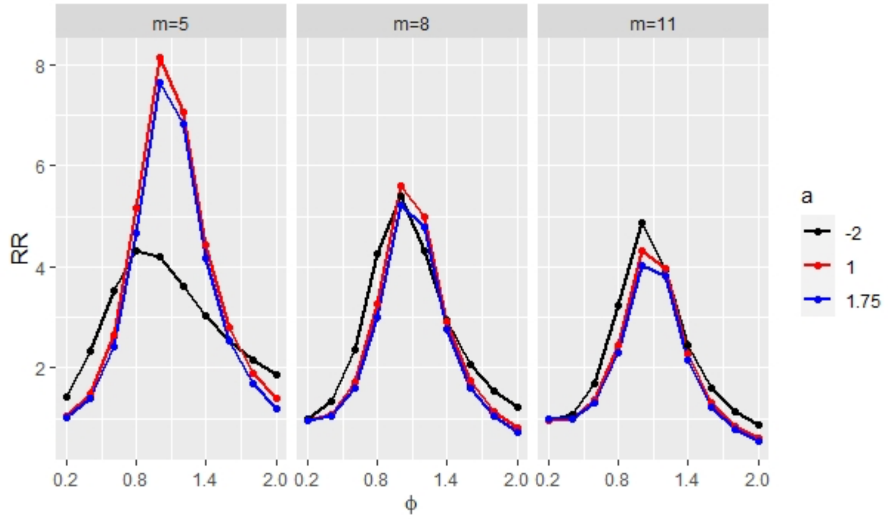

Figure 5: *For α=0.01*

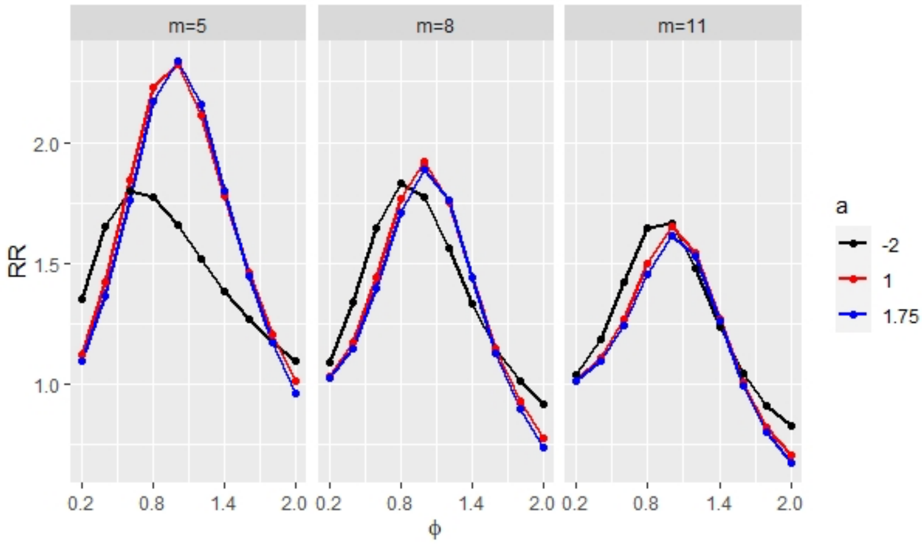

Figure 6: *For α=0.1*

6. Conclusion

In this paper, a Huntsberger Type shrinkage entropy estimator for normal distribution have been proposed and its properties have been investigated under LINEX Loss function. On the basis of relative risk, it is concluded that the proposed estimator gives better results for smaller values of level of significance and degrees of freedom.

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