PHASE TYPE QUEUEING MODEL OF SERVER VACATION, REPAIR AND DEGRADING SERVICE WITH BREAKDOWN, STARTING FAILURE AND CLOSE-DOWN

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Abstract

We consider a single server phase type queueing model with server vacation, repair, breakdown, degrading service, starting failure and closedown. When the arrival rate of the customer follows the Markovian Arrival Process (MAP) and the service rate of the server follows the phase-type distribution. If no one is in the system when the server is back from the vacation, then the server will wait until the customer arrives. If the customer arrives at the moment with no starting failure, then he provides service, otherwise the server immediately goes to the repair process. Here, the service rate declining until degradation fixed. After completion of K services the degradation is addressed. During the period of service, the server may get a breakdown at any moment, and then the server immediately goes for a repair process. After completing the service, he switches to the close-down process, and then he goes on vacation. Using the Matrix-Analytic method, The stationary probability vector representing the total number of customers in the system is examined. The analysis of the busy period, the mean waiting time, and cost analysis are discussed. A few significant performance measures are attained. Finally, some numerical examples are given.

Keywords: Phase type Distribution, Markovian Arrival Process, Degrading Service, Server Vacation, Breakdown, Repair, Starting failure, Close-down, Matrix-analytic method.

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1. Introduction

The Markovian arrival process is one of the modelling techniques for studying point processes that is most flexible. In order to define arrival processes that are not fundamentally renewal processes, Neuts [13] proposed the concept of a versatile Markovian point process (VMPP). Neuts [14] first introduced and investigated the underlying Markov structure of the MAP, which fits perfectly into the framework of matrix-analytic methods and is one of its most notable properties. Qi-Ming He [16] investigated the foundations of matrix analytical methodologies in order to comprehend the idea of service and arrival process.

Chakravarthy [5] made a significant contribution to MAP. Markovian Arrival Process represents by (D_0, D_1) and the service times with representation (α, T) that follow phase type distribution and whose matrices of order m and n, respectively. He described several types of arrivals and services. The irreducible stochastic matrix $D = D_0 + D_1$ defines the generator D. If the irreducible generator D describes the Markov process, then π is the steady state probability vector, and it is defined as $\pi D = 0$ and $\pi e = 1$. Based on the Markovian arrival process, the constant $\lambda = \pi D_1 e$ represents the basic customer arrival rate per unit time.

MAP/PH/1-type queueing models with degradation and phase type vacation have been analysed by Alka et al. [6]. Degradation can be included in a service system in a number of ways. The service rate will decrease unless the degradation is addressed. In other words, the service rate will decrease as more services are provided. For vacation queueing models, we refer to Doshi's survey paper [7] and Tian and Zhang's book [20]. Li and Tian [12] investigated the M/M/1 model with working vacation and proposed an interruption in vacation, where the server returns without completing the ongoing vacation due to certain conditions. Krishna Kumar et al. [10] have analysed the several server model with server vacations under the Bernoulli schedule. Sreenivasan et al. [18] have examined the MAP/P H/1 queueing model with N-Policy, vacation interruption and working vacations.

One of the main queueing theory subfields has recently been queueing models with server breakdown. Wang et al. [21] have investigated the batch arrival queueing model with multiple vacations and the server struck with breakdown. Ayyappan and Nirmala [2] have explored the non-Markovian queueing model and the server provides service to the customers based on general bulk service rule with multiple vacations, breakdown and two-phase repair . Ayyappan and Deepa [1] have studied the batch arrival and bulk service queueing model with multiple vacations and optional repair. A single server queueing model with MAP arrival and phase type service, vacation, instantaneous feedback and breakdown has been looked into by Ayyappan and Thilagavathy [3]. In this model, they obtained stability condition and busy period analysis. Senthil Vadivu et al. [17] have performed a cost function of the bulk service queueing model of a single server with finite capacity and close-down times by using embedded Markov chain and supplementary variable techniques.

Yang et al. [22] have discussed the Markovian model of the retrial queue with multi-server and starting failure. They analyzed their model with the aid of the matrix geometric method. With respect to the stability condition, the cost analysis is built to calculate the ideal number of servers, the ideal average service rate, and the ideal average repair rate. Karpagam et al. [9] have been analysed the batch arrival and bulk service queueing system with starting failure and additional service. They obtained system performance measures and the stability condition. Ayyappan and Gowthami [4] has analysed a Phase type model with impatient customers, Setup time, vacation, feedback, Breakdown and Repair. In this article, they compute the average waiting time.

2. Description of the Model

Assume that there is a single server in a queueing model, and that customers arrive at the system according to the MAP with representation (D_0, D_1), where D_0 and D_1 are m-dimensional square matrices. Let $D = D_0 + D_1$ be the generator matrix, where D_0 governs for no arrival at the system and D_1 governs for an arrival at the system. The stationary vector of D is denoted by π, so we have $\pi D = 0$ and $\pi e = 1$. The arrival rate λ is given by $\lambda = \pi D_1 e$. The system is performed on an FCFS basis. With the notation (α, T), that is of order n, the length of the server's service is thought to be a PH-distribution, where $T^0 + Te = 0$ so that $T^0 = -Te$. The average service rate ξ is given by $\xi = [\alpha(-T)^{-1}e]^{-1}$. The service rate decreases after each service is completed. Let ξ be the first service rate and ξ_i be the i^{th} service rate such that $\xi = \xi_1 \geq \xi_2 \geq \xi_3 \geq \cdots \geq \xi_K$, where $\xi_i = \theta_i \xi$ and $0 < \theta_i \leq 1$ for all $i = 1, 2, 3, \ldots, K$. After K services are completed, the original rate of ξ is immediately applied to the degraded service rate. Because $\theta_1 = 1$, After the degradation has been corrected, the service rate for the first customer is always ξ. The server that customers use to access services could breakdown at any time and needs to be repaired.

The repair procedure is based on the PH-distribution with representation (β, S) of order n_2 and $S^0 + Se = 0$ so that $S^0 = -Se$. If no one is present in the system when the server's service is completed, the close-down process begins, and then the server goes on vacation. The vacation period is thought to be a PH-distribution with the notation (γ, V) of order n_1, where $V^0 + Ve = 0$ so that $V^0 = -Ve$. After completion of the vacation period if no customer present in the system, then the server is idle; otherwise the server starts the service. If a customer arrives while the server is idle, it may experience a starting failure with probability p or no starting failure with probability q, resulting in p+q=1. In the event of a server breakdown, the customer who is currently providing the service from the server will remain in a frozen state until the server gets rid of the repair process. After completion of the repair process, the server will serve a fresh service for the current frozen customer. The breakdown and close-down time follows an exponential distribution with the parameters σ and δ respectively. The average repair rate and vacation rate are given by ζ and η respectively.

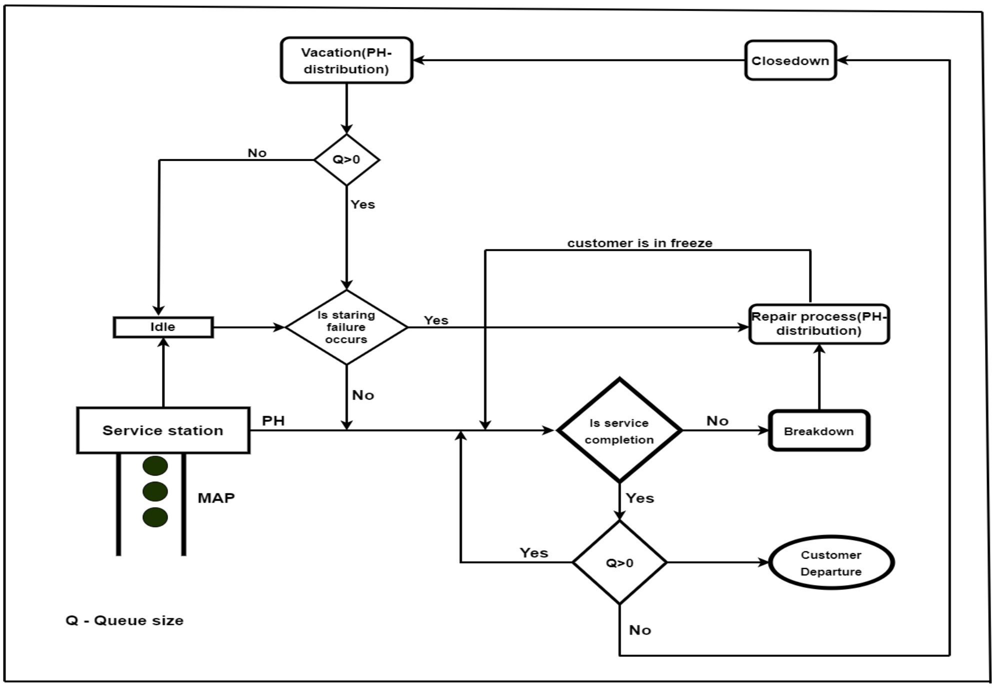

Figure 1: *Schematic Representation of the model*

3. The QBD Process of Matrix Generation

We have described our model's notation for the basis of generating the QBD process in this section as follows.

Matrix Generation Notations

- $\otimes$ - Kronecker product represents the product of any two different order matrices, can refer to the works in Steeb et al. [19].
- $\oplus$ - The Kronecker Sum represents the sum of any two of the different orders of matrices.
- I_k - An identity matrix of order k.
- $e_i'(m)$ - An m-dimensional row vector with 1 in the i^{th} position and 0 elsewhere.
- e-Each entry in a column vector of appropriate dimension is 1.
- The customer's arrival rate is denoted by λ and is defined by $\lambda = \pi D_1 e_m$
- The server's service rate is denoted by ξ and is defined by $\xi = [\alpha(-T)^{-1} e_n]^{-1}$

- The server's vacation rate is denoted by η and is defined by $\eta = [\gamma(-V)^{-1}e_{n_1}]^{-1}$
- The server's repair rate is denoted by ζ and is defined by $\zeta = [\beta(-S)^{-1}e_{n_2}]^{-1}$
- Define $\theta = (\theta_1, \theta_2, \ldots, \theta_K)^t$ and $\Delta(\theta) = \begin{pmatrix} \theta_1 & 0 & \ldots & \ldots \\ 0 & \theta_2 & \ldots & \ldots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & \ldots & \theta_K \end{pmatrix}$
- Let N(t) be the number of customers in the system at epoch t
- Let V(t) be the server's status at epoch t

$$V(t) = \begin{cases} 0, \text{ if the server is on vacation,} \\ 1, \text{ if the server is in idle,} \\ 2, \text{ if the server is in busy,} \\ 3, \text{ if the server is in repair process,} \\ 4, \text{ if the server is in closedown process.} \end{cases}$$

- I(t) is the type of service at time t
- $J_1(t)$ represents the vacation process as framed by phases.
- $J_2(t)$ represents the repair process as framed by phases.
- S(t) represents the service process as framed by phases.
- M(t) represents the arrival process as framed by phases.

Let { N(t),V(t),I(t),$J_1(t)$,$J_2(t)$,S(t),M(t):$t \geq 0$} denote the Continuous Time Markov Chain (CTMC) with state level independent Quasi-Birth and Death process, the state space of which is as follows:

$$\Omega = l(0) \cup l(q) \ ,$$

where

$$l(0) = \{(0,0,j_1,k) : 1 \leq j_1 \leq n_1, 1 \leq k \leq m\} \cup \{(0,1,k) : 1 \leq k \leq m\} \cup \{(0,4,k) : 1 \leq k \leq m\}$$

for $q \geq 1$,

$$l(q) = \{q,0,j_1,k) : 1 \leq j_1 \leq n_1, 1 \leq k \leq m\} \cup \{(q,2,l,j,k) : 1 \leq l \leq K, 1 \leq j \leq n, 1 \leq k \leq m\}$$
$$\cup\{(q,3,l,j_2,k) : 1 \leq l \leq K, 1 \leq j_2 \leq n_2, 1 \leq k \leq m\} \cup \{(q,4,k) : 1 \leq k \leq m\}.$$

The QBD process's infinitesimal matrix generation is given by

$$Q = \begin{bmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 & \ldots \\ B_{10} & A_1 & A_0 & 0 & 0 & 0 & \ldots \\ 0 & A_2 & A_1 & A_0 & 0 & 0 & \ldots \\ 0 & 0 & A_2 & A_1 & A_0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

The entries in the block matrices of Q are defined as follows,

$$B_{00} = \begin{bmatrix} V \oplus D_0 & V^0 \otimes I_m & 0 \\ 0 & D_0 & 0 \\ \gamma \otimes \delta I_m & 0 & D_0 - \delta I_m \end{bmatrix},$$

$$B_{01} = \begin{bmatrix} I_{n_1} \otimes D_1 & 0 & 0 & 0 \\ 0 & e_1'(K) \otimes \alpha \otimes qD_1 & e_1'(K) \otimes \beta \otimes pD_1 & 0 \\ 0 & 0 & 0 & D_1 \end{bmatrix}, B_{10} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \theta \otimes T^0 \otimes I_m \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A_0 = \begin{bmatrix} I_{n_1} \otimes D_1 & 0 & 0 & 0 \\ 0 & I_K \otimes I_n \otimes D_1 & 0 & 0 \\ 0 & 0 & I_K \otimes I_{n_2} \otimes D_1 & 0 \\ 0 & 0 & 0 & D_1 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} V \oplus D_0 & e_1'(K) \otimes qV^0\alpha \otimes I_m & e_1'(K) \otimes pV^0\beta \otimes I_m & 0 \\ 0 & (\Delta(\theta) \otimes T) \oplus D_0 - \sigma I_{Knm} & I_K \otimes (e_n \otimes \beta) \otimes \sigma I_m & 0 \\ 0 & I_K \otimes S^0\alpha \otimes I_m & (I_K \otimes S) \oplus D_0 & 0 \\ \gamma \otimes \delta I_m & 0 & 0 & D_0 - \delta I_m \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_{22} = \begin{bmatrix} 0 & \theta_1 T^0\alpha \otimes I_m & 0 & \dots & 0 \\ 0 & 0 & \theta_2 T^0\alpha \otimes I_m & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & \theta_{K-1} T^0\alpha \otimes I_m \\ \theta_K T^0\alpha \otimes I_m & 0 & \dots & \dots & 0 \end{bmatrix}.$$

4. Analysis of Stability Condition

We examined our model under the assumption that the system is stable.

4.1. Condition for Stableness

Let us specify the matrix A as $A = A_0 + A_1 + A_2$. It clearly demonstrates that the order of the square matrix A is $n_1m + Knm + Kn_2m + m$ and this matrix is an irreducible infinitesimal generator matrix. Let φ indicate the steady-state probability vector of A and it satisfying $\varphi A = 0$ and $\varphi e = 1$. The vector φ is partitioned by $\varphi = (\varphi_0, \varphi_1, \varphi_2, \varphi_3)$=$(\varphi_0, \varphi_{11}, \varphi_{12}, \varphi_{13}, \dots, \varphi_{1K-1}, \varphi_{1K}, \varphi_{21}, \varphi_{22}, \varphi_{23}, \dots, \varphi_{2K-1}, \varphi_{2K}, \varphi_3)$, where φ_0 is of dimension n_1m, φ_1 is of dimension Knm, φ_2 is of dimension Kn_2m, φ_3 is of dimension m. Our model's stability should satisfy the necessary and sufficient condition $\varphi A_0 e < \varphi A_2 e$ when the Markov Process is investigated using the Quasi-Birth-and-Death structure. The probability vector φ is calculated by solving the following equations

$$\begin{aligned}
&(V \oplus D)\varphi_0 + (\gamma \otimes \delta I_m)\varphi_3 = 0,\\
&(qV^0\alpha \otimes I_m)\varphi_0 + (\theta_1 T \oplus D - \sigma I_{nm})\varphi_{11} + (\theta_L T^0\alpha \otimes I_m)\varphi_{1K} + (S^0\alpha \otimes I_m)\varphi_{21} = 0,\\
&(\theta_{j-1} T^0\alpha \otimes I_m)\varphi_{1j-1} + (\theta_j T \oplus D - \sigma I_{nm})\varphi_{1j} + (S^0\alpha \otimes I_m)\varphi_{2j} = 0 \;\; for \; 2 \le j \le K,\\
&(pV^0\beta \otimes I_m)\varphi_0 + (e_n \otimes \beta \otimes \sigma I_m)\varphi_{11} + (S \oplus D)\varphi_{21} = 0,\\
&(e_n \otimes \beta \otimes \sigma I_m)\varphi_{1j} + (S \oplus D)\varphi_{2j} = 0 \;\; for \; 2 \le j \le K,\\
&(D - \delta I_n)\varphi_3 = 0.
\end{aligned}$$

subject to normalizing condition

$$\varphi_0 e_{n_1m} + \sum_{j=1}^{K} \varphi_{1j} e_{nm} + \sum_{j=1}^{K} \varphi_{2j} e_{n_2m} + \varphi_3 e_m = 1.$$

The stability condition $\varphi A_0 e < \varphi A_2 e$ is obtained after some algebraic manipulation, which turns out to be

$$\varphi_0(e_{n_1} \otimes D_1 e_m) + \sum_{j=1}^{K} \varphi_{1j}(e_n \otimes D_1 e_m) + \sum_{j=1}^{K} \varphi_{2j}(e_{n_2} \otimes D_1 e_m) + \varphi_3 D_1 e_m < \sum_{j=1}^{K} \varphi_{1j}(\theta_j T^0 \otimes e_m)$$

4.2. Analysis of Steady-State Probability Vector

Consider the steady-state probability vector x of Q and it is divided into $x = (x_0, x_1, x_2, \ldots)$. x_0 has a dimension $2m + n_1 m$ while $x_1,\ x_2, \ldots$ have a dimension $n_1 m + Knm + Kn_2 m + m$. Then x satisfied the condition $xQ = 0$ and $xe = 1$.

Furthermore, if the system is stable with the vector x, the following equation provides the remaining sub vectors except for the boundary states.

$$x_q = x_1 R^{q-1},\ q \geq 2$$

where the rate matrix R indicates the minimal non-negative solution of the matrix quadratic equation as $R^2 A_2 + RA_1 + A_0 = 0$, as referred by Neuts [15] and satisfies the relation $RA_2 e = A_0 e$.

The sub vectors of x_0 and x_1 were calculated by solving the subsequent equations.

$$x_0 B_{00} + x_1 B_{10} = 0$$

$$x_0 B_{01} + x_1(A_1 + RA_2) = 0$$

The normalizing condition is subject to

$$x_0 e_{2m+n_1 m} + x_1 (I - R)^{-1} e_{n_1 m + Knm + Kn_2 m + m} = 1$$

As a result, the rate matrix R could be mathematically calculated using crucial procedures in the Latouche algorithm for logarithmic reduction of R [11].

5. Busy period Analysis

- The time between customers entering into an empty system and the system becoming empty again after the first interval can be used to measure a busy period. This is the first passage in the transition from level 1 to 0. Thus, it is the first time returns to level 0, followed by at least one visit to a state at any other level is known as the busy cycle.
- We give an overview of the fundamental period before moving on to the busy period. The QBD process takes into account the first transition time, $q \geq 2$, from level q to level q-1.
- It is necessary to examine each of the cases q = 0, 1 that correspond to the boundary states individually. It should be noted that for each level j with $q \geq 2$, there are $(n_1 m + Lnm + Ln_2 m + m)$ states that correspond. Similarly, when the states are organised in lexicographic order, the state(q, j) at level j signifies that j^{th} state at the level q is mentioned.
- The variable $G_{jj'}(v, x)$ represents the conditional probability that the QBD process, which begins in the state (q, j) at time t=0 and visits the level q-1 but not before time x, can make changes v transition to the left and enter the state (q, j'). Let us first define the joint transform

$$\tilde{G}_{jj'}(z, s) = \sum_{v=1}^{\infty} z^v \int_0^{\infty} e^{-sx} dG_{jj'}(v, x); |z| \leq 1, Re(s) \geq 0$$

and the matrix is represented as $\tilde{G}(z, s) = \tilde{G}_{jj'}(z, s)$ [14]then the previously defined matrix $\tilde{G}(z, s)$ satisfied the equation

$$\tilde{G}(z, s) = z(sI - A_1)^{-1} A_2 + (sI - A_1)^{-1} A_0 \tilde{G}^2(z, s).$$

- The matrix $G = G_{jj'} = \tilde{G}(1,0)$, excluding the boundary states, would be used for the first passage time. If we are already familiar with the matrix R, we can use the results to discover the matrix G
$$G = -(A_1 + RA_2)^{-1} A_2.$$

  Or else, the idea of a logarithmic reduction algorithm method [11] could be used to determine the values of the G matrix.

Notations

- $G^{(1,0)}_{jj'}(v,x)$ shows that at time $t = 0$, the conditional probability has been discussed for the first time during the passage from level 1 to level 0.
- $G^{(0,0)}_{jj'}(v,x)$ shows that the conditional probability has been discussed for the return time to level 0.
- $\mathfrak{H}_{1q}$ shows the average first passage time between levels q and q-1, assuming the process is in the state (q, j) at time t=0.
- $\vec{\mathfrak{H}}_1$ identifies the column vector containing the entries $\mathfrak{H}_{1q}$.
- $\mathfrak{H}_{2q}$ shows the average number of customers expected to be served during the first passage time from level q to q-1, assuming that the state's first passage time has already begun (q, j).
- $\vec{\mathfrak{H}}_2$ identifies the column vector containing the entries $\mathfrak{H}_{2q}$.
- $\vec{\mathfrak{H}}^{(1,0)}_1$ shows the average first passage time between level 1 and level 0.
- $\vec{\mathfrak{H}}^{(1,0)}_2$ shows the expected number of services finished during the first passage time from level 1 to level 0.
- $\vec{\mathfrak{H}}^{(0,0)}_1$ shows the initial return time to level 0.
- $\vec{\mathfrak{H}}^{(0,0)}_2$ shows the expected number of services finished between the first return time and level 0.

The following equations, which are given by $\tilde{G}^{(1,0)}(z,s)$ and $\tilde{G}^{(0,0)}(z,s)$, are for the boundary levels 1 and 0 respectively.

$$\begin{aligned}
\tilde{G}^{(1,0)}(z,s) &= z(sI - A_1)^{-1}B_{10} + (sI - A_1)^{-1}A_0\tilde{G}(z,s)\tilde{G}^{(1,0)}(z,s),\\
\tilde{G}^{(0,0)}(z,s) &= (sI - B_{00})^{-1}B_{01}\tilde{G}^{(1,0)}(z,s).
\end{aligned}$$

The matrices are used to calculate the following instances because G, $\tilde{G}^{(0,0)}(1,0)$ and $\tilde{G}^{(1,0)}(1,0)$ are all stochastic in nature. We can compute the instants as follows:

$$\begin{aligned}
\vec{\mathfrak{H}}_1 &= -\frac{\partial}{\partial s}\tilde{G}(z,s)\Big|_{z=1,s=0} e = -[A_1 + A_0(I+G)]^{-1}e,\\
\vec{\mathfrak{H}}_2 &= \frac{\partial}{\partial z}\tilde{G}(z,s)\Big|_{z=1,s=0} e = -[A_1 + A_0(I+G)]^{-1}A_2 e,\\
\vec{\mathfrak{H}}^{(1,0)}_1 &= -\frac{\partial}{\partial s}\tilde{G}^{(1,0)}(z,s)\Big|_{z=1,s=0} e = -[A_1 + A_0 G]^{-1}(A_0\vec{\mathfrak{H}}_1 + e),\\
\vec{\mathfrak{H}}^{(1,0)}_2 &= \frac{\partial}{\partial z}\tilde{G}^{(1,0)}(z,s)\Big|_{z=1,s=0} e = -[A_1 + A_0 G]^{-1}(A_0\vec{\mathfrak{H}}_2 + B_{10}e),\\
\vec{\mathfrak{H}}^{(0,0)}_1 &= -\frac{\partial}{\partial s}\tilde{G}^{(0,0)}(z,s)\Big|_{z=1,s=0} e = -B_{00}^{-1}[B_{01}\vec{\mathfrak{H}}^{(1,0)}_1 + e],\\
\vec{\mathfrak{H}}^{(0,0)}_2 &= \frac{\partial}{\partial z}\tilde{G}^{(0,0)}(z,s)\Big|_{z=1,s=0} e = -B_{00}^{-1}[B_{01}\vec{\mathfrak{H}}^{(1,0)}_2].
\end{aligned}$$

6. System Performance Measures

- The average system size
$$E_{system} = \sum_{q=1}^{\infty} q x_q e$$

- Probability of the server is busy

$$P_{busy} = \sum_{q=1}^{\infty} \sum_{l=1}^{K} \sum_{j=1}^{n} \sum_{k=1}^{m} x_{q2ljk}$$

- Probability of the server is in idle

$$P_{idle} = \sum_{k=1}^{m} X_{01k}$$

- Probability of the server is on vacation

$$P_{vac} = \sum_{q=0}^{\infty} \sum_{j_1=1}^{n_1} \sum_{k=1}^{m} x_{q0j_1k}$$

- Probability of the server is breakdown

$$P_{bd} = \sum_{q=1}^{\infty} \sum_{l=1}^{K} \sum_{j_2=1}^{n_2} \sum_{k=1}^{m} x_{q3lj_2k}$$

- Probability of the server is on closedown

$$P_{cd} = \sum_{q=0}^{\infty} \sum_{k=1}^{m} x_{q4k}$$

- The average system size during vacation

$$E_{vac} = \sum_{q=1}^{\infty} \sum_{j_1=1}^{n_1} \sum_{k=1}^{m} q x_{q0j_1k} e_{n_1m}$$

- The average system size of the server is busy

$$E_{busy} = \sum_{q=1}^{\infty} \sum_{l=1}^{K} \sum_{j=1}^{n} \sum_{k=1}^{m} q x_{q2ljk} e_{Knm}$$

- The average system size during breakdown

$$E_{bd} = \sum_{q=1}^{\infty} \sum_{l=1}^{K} \sum_{j_2=1}^{n_2} \sum_{k=1}^{m} q x_{q3lj_2k} e_{Kn_2m}$$

- The average system size when the server is close-down

$$E_{cd} = \sum_{q=1}^{\infty} \sum_{k=1}^{m} q x_{q4k} e_m$$

7. Waiting Time Distribution

The first passage time analysis is used in this section to analyse the distribution of a customer's waiting time when they enter the queueing line. Let W(t) be the waiting time distribution function, which takes into account new customers joining the queue. If the server is idle when a customer arrives, they will get service immediately; otherwise, if the server is busy or on vacation, they will have to wait in a queue to receive service from the server.

Let's look at the absorption time in the state space of a Markov chain, which is given by

$$\bar{\Omega} = (*) \cup \{\bar{0}, \bar{1}, \bar{2}, \dots\}$$

where

$$\bar{0} = \{(0,0,j_1) : 1 \leq j_1 \leq n_1\} \cup \{(0,4)\}$$

and for $q \geq 1$,

$$\bar{q} = \{(q,0,j_1) : 1 \leq j_1 \leq n_1\} \cup \{(q,2,l,j) : 1 \leq l \leq K, 1 \leq j \leq n\}$$
$$\cup \{(q,3,l,j_2) : 1 \leq l \leq K, 1 \leq j_2 \leq n_2\} \cup \{(q,4)\}$$

The state space $(*)$obtained by considering the states that have the server in the idle state at the instant of arrival is as below

$$(*) = \{(0,1)\}$$

Let this Markov process's transition matrix $\bar{Q}$ be

$$\bar{Q} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots & \dots \\ J_0 & L_0 & 0 & 0 & 0 & 0 & \dots & \dots \\ J_1 & L_{10} & L & 0 & 0 & 0 & \dots & \dots \\ 0 & 0 & L_2 & L & 0 & 0 & \dots & \dots \\ 0 & 0 & 0 & L_2 & L & 0 & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \end{bmatrix}$$

where

$$J_0 = \begin{bmatrix} V^0 \\ 0 \end{bmatrix}, \quad L_0 = \begin{bmatrix} V & 0 \\ \gamma \otimes \delta & -\delta \end{bmatrix}, \quad J_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad L_{10} = \begin{bmatrix} 0 & 0 \\ 0 & \theta \otimes T^0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$L = \begin{bmatrix} V & e_1'(K) \otimes qV^0\alpha & e_1'(K) \otimes pV^0\beta & 0 \\ 0 & \Delta(\theta) \otimes T - \sigma I_{Kn} & I_K \otimes (e_n \otimes \sigma\beta) & 0 \\ 0 & I_K \otimes S^0\alpha & I_K \otimes S & 0 \\ \gamma \otimes \delta & 0 & 0 & -\delta \end{bmatrix},$$

$$L_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & L_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad L_{22} = \begin{bmatrix} 0 & \theta_1 T^0\alpha & 0 & \dots & 0 \\ 0 & 0 & \theta_2 T^0\alpha & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 0 & \theta_{K-1}T^0\alpha \\ \theta_K T^0\alpha & 0 & \dots & 0 & 0 \end{bmatrix}.$$

With the aim of determining the arriving tagged customer's waiting time distribution W(t), where($t \geq 0$). To start, we search for the system size probability vector at the arrival epoch in a steady state and it is indicated by $Z(0) = (Z_0(0), Z_1(0), Z_2(0), \dots)$. The vector $Z_0(0)$ may be further partitioned as follows $Z_0(0) = (Z_{00}, Z_{04})$. The system size probability vector at the arrival epoch in the steady state is as follows because the arrival process abides by the Markovian property:

$$Z_{00} = x_{00}\left[I_{n_1} \otimes \frac{D_1 e_m}{\lambda}\right], \quad Z_{04} = x_{04}\left[\frac{D_1 e_m}{\lambda}\right],$$

$$Z_q(0) = x_q\left[I_{n_1+Kn+Kn_2+1} \otimes \frac{D_1 e_m}{\lambda}\right], \quad for\ q \geq 1$$

where λ denotes the fundamental arrival rate of Markovian Arrival Process.
Define $Z(t) = (Z_*(t), Z_0(t), Z_1(t), \dots)$,
where
$Z_q(t),\ q \geq 1$ - vector of order $1 \times (n_1 + Kn + Kn_2 + 1)$

$Z_0(t) = (Z_{00}, Z_{04})$ - vector of order $1 \times (n_1 + 1)$
and their components give the probability that at epoch t, the CTMC generator matrix is $\bar{Q}$, will be in the appropriate level q state. Since the tagged customer's probability of being in the absorbing state at epoch t is specified by $Z_*(t)$, we get $W(t) = Z_*(t)$, where $t \geq 0$.
The differential equation $Z'(t) = Z(t)\bar{Q}$ for $t \geq 0$ becomes

$$Z'_*(t) = Z_0(t)J_0 + Z_1(t)J_1,$$
$$Z'_0(t) = Z_0(t)L_0 + Z_1(t)L_{10},$$
$$Z'_q(t) = Z_q(t)L + Z_{q+1}(t)L_2,\ q \geq 1$$

where $'$ specifies the derivative concerning t. Let's use the method suggested by Neuts et al. [15] to compute the LST for W(t). The row vector $\omega(s)$ specifies the Laplace-Stieltjes Transform (LST) of the first passage time to level 1 by starting the process at state q and using $Z_q(0),\ q \geq 1$ as the initial probability vector. Neuts et al. [15] state that we get,

$$\omega(s) = \sum_{q=1}^{\infty} Z_q(0)[(sI - L)^{-1}L_2]^{q-1} \tag{1}$$

With the restriction that the process begins at level q = 0, 1, let the LST of the time to become absorbed into the state $(*)$ be specified by $\phi(q, s)$. Similar to Neuts et al. [15], we have

$$(0, s) = [sI - L_0]^{-1}J_0, \tag{2}$$

$$\phi(1, s) = [sI - L]^{-1}L_{10}\phi(0, s) + [sI - L]^{-1}J_1. \tag{3}$$

This allows us to quickly note that the LST for the distribution of sojourn time is as follows.

$$W(s) = Z_0(0)\phi(0, s) + \omega(s)\phi(1, s) \tag{4}$$

Expected Waiting Time

The mean waiting time is given as

$$E(W) = -W'(0) = -Z_0(0)\phi'(0, 0) - \omega'(0)e_{n_1+Kn+Kn_2+1} - \omega(0)\phi'(1, 0) \tag{5}$$

The initial term of the previous equation gives the expected time to reach the absorbing state $(*)$, assuming that the system is at level 0. The final two components of the previous equation provide the expected time for accessing the absorbing state $(*)$ if the system is resting at level $q \geq 1$. By differentiating (2) and (3) and making s=0,

$$\Phi'(0, 0) = -[-L_0]^{-2}J_0 \tag{6}$$

$$\Phi'(1, 0) = -[-L]^{-2}L_{10}\Phi(0, 0) + [-L]^{-1}L_{10}\Phi'(0, 0) - [-L]^{-2}J_1 \tag{7}$$

Using (6) and the vector $Z(0) = (Z_0(0), Z_1(0), \dots)$, it is simple to calculate the first term of (5).
From (1), we get

$$\omega(0) = \sum_{q=1}^{\infty} Z_q(0)M^{q-1} \tag{8}$$

where $M = [-L]^{-1}L_2$. As M is a stochastic matrix, we get

$$\omega(0)e_{n_1+Kn+Kn_2+1} = 1 - Z_0(0)e_{n_1+1} \tag{9}$$

Equations (7) and (8), as well as the vector $Z(0) = (Z_0(0), Z_1(0), \dots)$, allow us to quickly calculate the last term of (5).
We obtain by differentiating (1) and setting s=0,

$$\omega'(0) = (-1)\sum_{q=1}^{\infty} Z_{1+q}(0)\sum_{j=0}^{q-1} M^j[-L]^{-1}M^{q-j}. \tag{10}$$

Due to the stochastic nature of matrix M,

$$(-1)\omega'(0)e_{n_1+Kn+Kn_2+1} = \sum_{q=1}^{\infty} Z_{1+q}(0) \sum_{j=0}^{q-1} M^j [-L]^{-1} e_{n_1+Kn+Kn_2+1}. \tag{11}$$

Let's assess the value of $(-1)\omega'(0)e_{n_1+Kn+Kn_2+1}$ using the technique described in Neuts et al. [15] and Kao et al. [8]. We start by building a matrix M_2 that is generalised inverse of I-M and stochastic, with $I - M + M_2$ being non-singular and M_2 being stochastic. The matrix M_2 can be viewed as $M_2 = e_{n_1+Kn+Kn_2+1}m_0$, where m_0 is the invariant probability vector of M. Additionally, using the property $MM_2 = M_2M = M_2$, we have

$$\sum_{j=0}^{q-1} M^j (I - M + M_2) = I - M^q + qM_2 \quad for\ q \geq 1. \tag{12}$$

By using (12) in (11), we obtain

$$\begin{aligned}(-1)\omega'(0)e_{n_1+Kn+Kn_2+1} = &\Big\{ x_1(I-R)^{-1}\Big[I_{n_1+Kn+Kn_2+1} \otimes \frac{D_1 e_m}{\lambda}\Big] - \omega(0) \\ &+ x_1 R(I-R)^{-2}\Big[I_{n_1+Kn+Kn_2+1} \otimes \frac{D_1 e_m}{\lambda}\Big] M_2 \Big\} \\ &\times [I - M + M_2]^{-1}[-L]^{-1} e_{n_1+Kn+Kn_2+1}.\end{aligned} \tag{13}$$

Since we have calculated all the terms in (5), we can easily calculate the average waiting time.

8. Cost Analysis

Our model's cost analysis has been created below by assuming the cost elements (per unit time) correspond to distinct measures of the system.

$$\begin{aligned}TC = &C_H E_{system} + C_{busy}P_{busy} + C_{idle}P_{idle} + C_{vac}P_{vac} + C_{bd}P_{bd} + C_{cd}P_{cd} \\ &+ \sum_{i=1}^{K} C_{1i}\theta_i \xi + \sigma C_2 + \zeta C_3 + \delta C_4\end{aligned}$$

where

- TC - Total cost per unit time
- C_H - Each customer's holding cost in the system
- C_{busy} - Cost acquired by the system during server being busy
- C_{idle} - Cost acquired due to server being idle
- C_{vac} - Cost acquired during server's vacation period
- C_{bd} - Cost acquired by the server during breakdown time
- C_{cd} - Cost acquired by the server during close-down process
- C_{1i} - Cost acquired by the server for offering i^{th} type service, $i = 1, 2, \ldots, K$
- C_2 - Cost acquired when the server caused by breakdowns
- C_3 - Cost acquired in carrying out the repair process
- C_4 - Cost acquired in carrying out the close-down process

9. Numerical

In this section, we are using numerical and graphical representations to analyze model behavior. The mean value of the subsequent five different MAP representations is 1, which is the same for all the various arrival processes. In published studies, these five sets of arrival values have been used as input data (see Chakravarthy [5]).

- **Arrival in Erlang(ERL-A):**

$$D_0 = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$$

- **Arrival in Exponential(EXP-A):**

$$D_0 = \begin{bmatrix} -1 \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 1 \end{bmatrix}$$

- **Arrival in Hyper-exponential(HEX-A):**

$$D_0 = \begin{bmatrix} -1.90 & 0 \\ 0 & -0.19 \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 1.710 & 0.190 \\ 0.171 & 0.019 \end{bmatrix}$$

- **Arrival in MAP-Negative Correlation(MAPNC-A):**

$$D_0 = \begin{bmatrix} -1.25 & 1.25 & 0 \\ 0 & -1.25 & 0 \\ 0 & 0 & -2.5 \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.0125 & 0 & 1.2375 \\ 2.4750 & 0 & 0.0250 \end{bmatrix}$$

- **Arrival in MAP-Positive Correlation(MAPPC-A):**

$$D_0 = \begin{bmatrix} -1.25 & 1.25 & 0 \\ 0 & -1.25 & 0 \\ 0 & 0 & -2.5 \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1.2375 & 0 & 0.0125 \\ 0.0250 & 0 & 2.4750 \end{bmatrix}.$$

Let's think about the service, repair, and vacation processes as three phase type distributions. In the literature, these sets of service, vacation, and repair values have been used as input data [5].

- **Service in Erlang(ERL-S):**

$$\alpha = (1, 0), \qquad T = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}$$

- **Repair in Erlang(ERL-R):**

$$\beta = (1, 0), \qquad S = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}$$

- **Vacation in Erlang(ERL-V):**

$$\gamma = (1, 0), \qquad V = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}$$

- **Service in Exponential(EXP-S):**

$$\alpha = [-1], \qquad T = [1]$$

- **Repair in Exponential(EXP-R):**

$$\beta = [-1], \qquad S = [1]$$

- **Vacation in Exponential(EXP-V):**

$$\gamma = [-1], \qquad V = [1]$$

- **Service in Hyper-exponential(HEX-S):**

$$\alpha = (0.8, 0.2), \qquad T = \begin{bmatrix} -2.8 & 0 \\ 0 & -0.28 \end{bmatrix}$$

- **Repair in Hyper-exponential(HEX-R):**

$$\beta = (0.8, 0.2), \qquad S = \begin{bmatrix} -2.8 & 0 \\ 0 & -0.28 \end{bmatrix}$$

- **Vacation in Hyper-exponential(HEX-V):**

$$\gamma = (0.8, 0.2), \qquad V = \begin{bmatrix} -2.8 & 0 \\ 0 & -0.28 \end{bmatrix}$$

9.1. Illustration 1

We investigated the consequence of the repair rate (ζ) on the average system size(E_{system}). We fix $\lambda = 2,\ \xi = 6,\ \eta = 10,\ \sigma = 1,\ \delta = 5,\ K = 10,\ \theta^t = [1, 0.97, 0.93, 0.9, 0.87, 0.83, 0.8, 0.75, 0.7, 0.6]$, $p = 0.6,\ q = 0.4$.

Table 1: *Repair rate (ζ) vs E_{system} - ERL-S*

ERL-S					
ζ	ERL-A	EXP-A	HEX-A	MAPNC-A	MAPPC-A
4	1.240013	1.428884	2.424636	1.337204	10.61745
5	1.100282	1.263127	2.098389	1.18343	8.801677
6	1.015791	1.16274	1.90369	1.090236	7.692035
7	0.959243	1.095599	1.775197	1.027872	6.946641
8	0.918743	1.047606	1.684428	0.98328	6.412992
9	0.888298	1.011625	1.617078	0.949845	6.012954
10	0.864568	0.983665	1.565214	0.923863	5.702437
11	0.845543	0.961321	1.524099	0.903103	5.454723

Table 2: *Repair rate (ζ) vs E_{system} - EXP-S*

EXP-S					
ζ	ERL-A	EXP-A	HEX-A	MAPNC-A	MAPPC-A
4	1.300513	1.477625	2.385188	1.39259	8.815389
5	1.14504	1.300952	2.072725	1.226593	7.266135
6	1.05166	1.194414	1.885579	1.126538	6.322526
7	0.989618	1.1235	1.761902	1.059955	5.692052
8	0.945487	1.073041	1.674513	1.012588	5.243369
9	0.912517	1.035369	1.609694	0.977232	4.909026
10	0.886957	1.006202	1.559813	0.949868	4.650976
11	0.866562	0.982972	1.520303	0.928081	4.446203

Table 3: *Repair rate (ζ) vs E_{system} - HEX-S*

HEX-S					
ζ	ERL-A	EXP-A	HEX-A	MAPNC-A	MAPPC-A
4	1.558351	1.660665	2.248675	1.59979	4.230083
5	1.325099	1.423599	1.946641	1.369357	3.457511
6	1.188053	1.283658	1.765584	1.233688	3.008707
7	1.099269	1.19259	1.646429	1.145591	2.721063
8	1.037696	1.12918	1.562789	1.084356	2.523394
9	0.992794	1.082778	1.501231	1.039609	2.380326
10	0.958759	1.047505	1.454251	1.005634	2.272559
11	0.932162	1.019875	1.417352	0.979044	2.188777

With the help of tables 1, 2 and 3, we can determine that increasing the repair rate reduces the average system size in various arrangement of services and arrivals of ERL-A, EXP-A, HEX-A, MAPNC-A and MAPPC-A. The positive correlation arrival decreases rapidly compared to all other arrivals.

9.2. Illustration 2

We investigated the consequence of the vacation rate (η) on the average waiting time E(W). We fix $\lambda = 2,\ \xi = 6,\ \sigma = 1,\ \zeta = 4,\ \delta = 5,\ K = 5,\ \theta^t = [1, 0.9, 0.8, 0.7, 0.6],\ p = 0.4,\ q = 0.6$.

Table 4: *Vacation rate (η) vs E(W) - ERL-S*

ERL-S					
η	ERL-A	EXP-A	HEX-A	MAPNC-A	MAPPC-A
10	0.513310803	0.652469261	1.316678245	0.597597919	6.305631788
11	0.502141873	0.641535425	1.304722472	0.586775589	6.294830579
12	0.493006347	0.632560036	1.294845632	0.577894144	6.285940644
13	0.485401455	0.625064256	1.286552181	0.570478428	6.27849855
14	0.478976132	0.618712658	1.279491771	0.564195772	6.272178893
15	0.4734782	0.613263522	1.273409875	0.55880656	6.266746629
16	0.46872206	0.608538394	1.268117234	0.554133955	6.262027841
17	0.46456821	0.604402741	1.263470256	0.550044688	6.257891127
18	0.460909846	0.600753272	1.259358071	0.546436465	6.254235397
19	0.457663866	0.597509402	1.255693735	0.543229496	6.250981623

Table 5: *Vacation rate (η) vs E(W) - EXP-S*

EXP-S					
η	ERL-A	EXP-A	HEX-A	MAPNC-A	MAPPC-A
10	0.557188985	0.69195187	1.314692759	0.63940605	5.40504825
11	0.545405475	0.680428504	1.302062559	0.627990745	5.393620067
12	0.535811233	0.671007926	1.291662266	0.618660721	5.384251971
13	0.527857453	0.663169128	1.282954639	0.610898654	5.376437568
14	0.521162512	0.656548669	1.27556107	0.604343883	5.369822843
15	0.515453367	0.650885673	1.269207338	0.598737649	5.364153128
16	0.510529823	0.645988247	1.263690113	0.593889693	5.359240685
17	0.506241951	0.641712219	1.258855539	0.589657117	5.354944174
18	0.502475357	0.637947256	1.254585115	0.585930588	5.351155205
19	0.49914132	0.634607531	1.250786132	0.58262508	5.347789301

Table 6: *Vacation rate (η) vs E(W) - HEX-S*

HEX-S					
η	ERL-A	EXP-A	HEX-A	MAPNC-A	MAPPC-A
10	0.765770257	0.873794399	1.367546979	0.829456953	2.90799282
11	0.749707796	0.858258947	1.350700199	0.814080592	2.891938923
12	0.736875156	0.84579312	1.337039635	0.801751095	2.879049871
13	0.726421715	0.835595592	1.325762738	0.791670904	2.868502654
14	0.717764748	0.827116374	1.316311418	0.783293156	2.859731284
15	0.710493246	0.819966703	1.30828648	0.77623166	2.852335034
16	0.704309955	0.813864754	1.301395404	0.770206698	2.846023015
17	0.698995159	0.808601698	1.29541931	0.765011173	2.840579451
18	0.694383327	0.804019799	1.290191394	0.760488778	2.835841216
19	0.690347624	0.799997883	1.285582472	0.756519514	2.831682906

With the help of tables 4, 5 and 6, we can determine that increasing the vacation rate reduces the average waiting time in various arrangement of services and arrivals of ERL-A, EXP-A, HEX-A, MAPNC-A and MAPPC-A.

9.3. Illustration 3

We examined the consequence of the vacation rate(η) on the Total cost(TC) of the system. We fix $\lambda = 2,\ \xi = 6,\ \zeta = 4,\ \sigma = 1,\ \delta = 5,\ K = 5,\ \theta^t = [1, 0.9, 0.8, 0.7, 0.6],\ p = 0.6,\ q = 0.4,\ C_H = 10,$ $C_{vac} = 2,\ C_{idle} = 1,\ C_{busy} = 4,\ C_{bd} = 2,\ C_{cd} = 2, C_{11} = 3,\ C_{12} = 2.9,\ C_{13} = 2.7,\ C_{14} = 2.5,$ $C_{15} = 2.2,\ C_2 = 1,\ C_3 = 2,\ C_4 = 2.$

Table 7: *Vacation rate (η) vs TC - ERL-S*

ERL-S					
η	ERL-A	EXP-A	HEX-A	MAPNC-A	MAPPC-A
10	99.68315738	102.0191734	114.5649299	100.8662306	220.3575308
11	99.58654131	101.9327141	114.4864353	100.7816754	220.2740852
12	99.50790684	101.8624352	114.422411	100.7129863	220.2062508
13	99.44275241	101.8042608	114.369262	100.6561557	220.150093
14	99.38794402	101.7553616	114.3244797	100.6084052	220.1028821
15	99.34123751	101.7137163	114.2862631	100.5677517	220.0626686
16	99.30098652	101.677845	114.253288	100.5327444	220.0280254
17	99.26595761	101.6466406	114.2245603	100.5022988	219.9978848
18	99.23520954	101.6192592	114.1993197	100.4755886	219.971433
19	99.20801222	101.595047	114.1769756	100.4519741	219.9480397

Table 8: *Vacation rate (η) vs TC - EXP-S*

EXP-S					
η	ERL-A	EXP-A	HEX-A	MAPNC-A	MAPPC-A
10	100.2339903	102.3592967	113.45068	101.3175747	196.2037767
11	100.1280443	102.2634003	113.3608307	101.2234308	196.1105677
12	100.0424323	102.1859477	113.28795	101.1474223	196.0352852
13	99.97195226	102.122204	113.2277544	101.0848836	195.973326
14	99.91300867	102.0689027	113.1772692	101.0325998	195.9215153
15	99.86304362	102.0237237	113.134369	100.9882889	195.8775984
16	99.8201921	101.9849774	113.0974976	100.9502904	195.8399333
17	99.78306479	101.9514061	113.0654919	100.9173687	195.8072978
18	99.75060709	101.9220556	113.0374655	100.8885871	195.7787647
19	99.72200511	101.8961899	113.0127327	100.8632232	195.7536187

Table 9: *Vacation rate (η) vs TC - HEX-S*

HEX-S					
η	ERL-A	EXP-A	HEX-A	MAPNC-A	MAPPC-A
10	102.3258123	103.4719568	110.0500597	102.7836354	135.054583
11	102.1616111	103.3188195	109.8998592	102.6325837	134.9021191
12	102.0324917	103.1981907	109.7806956	102.5136915	134.7822937
13	101.9288319	103.1011839	109.684308	102.4181401	134.6861809
14	101.8441297	103.0217915	109.6050482	102.3399749	134.6077393
15	101.7738585	102.9558273	109.5389409	102.2750516	134.5427562
16	101.7147844	102.9002977	109.4831181	102.2204095	134.4882179
17	101.6645447	102.8530131	109.4354652	102.1738855	134.4419202
18	101.621379	102.8123398	109.3943937	102.1338671	134.402219
19	101.5839523	102.7770377	109.3586901	102.0991312	134.3678669

With the help of tables 7, 8 and 9, we can determine that increasing the vacation rate reduces the total cost of the system in various arrangement of services and arrivals of ERL-A, EXP-A, HEX-A, MAPNC-A and MAPPC-A.

9.4. Illustration 4

We investigated the consequence of the breakdown rate (σ) on the average system size(E_{system}). We fix $\lambda = 2,\ \xi = 6,\ \eta = 10,\ \zeta = 4,\ \delta = 5,\ K = 10,\ \theta^t = [1, 0.97, 0.93, 0.9, 0.87, 0.83, 0.8, 0.75, 0.7, 0.6],\ p = 0.6,\ q = 0.4$.

With the help of figures 2, 3, 4, 5 and 6, we analyze the breakdown rate versus the average system size with the combination of arrival and service time groupings. The breakdown rate increases then the corresponding average system size is also increases rapidly in Erlang services and, increases gradually in Exponential services and slowly in Hyper-exponential services but in case of MAP positive correlation arrival increases rapidly than compared to all other arrivals.

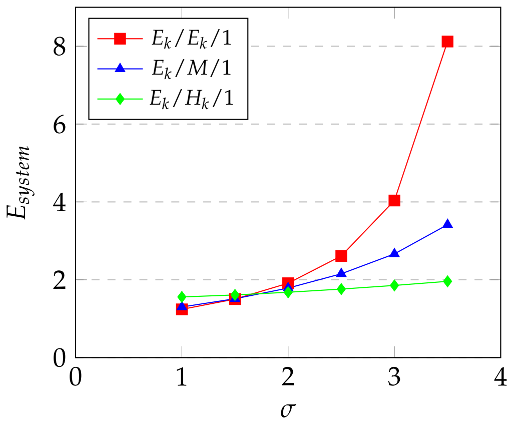

Figure 2: *Breakdown rate(σ) vs E_{system} - ERL-A*

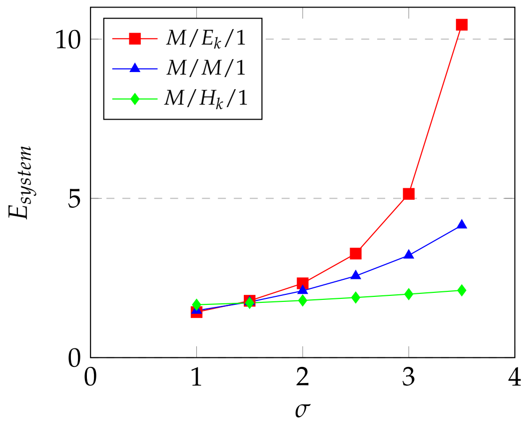

Figure 3: *Breakdown rate(σ) vs E_{system} - EXP-A*

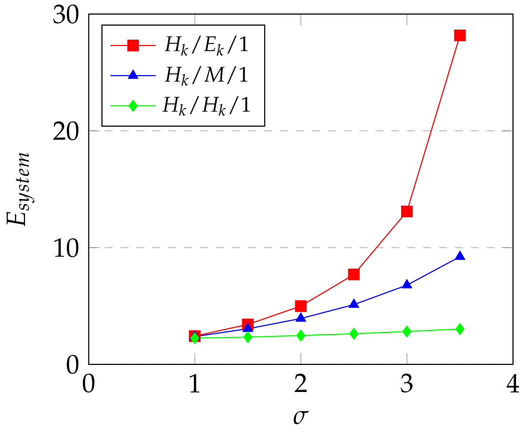

Figure 4: *Breakdown rate(σ) vs E_{system} - HEX-A*

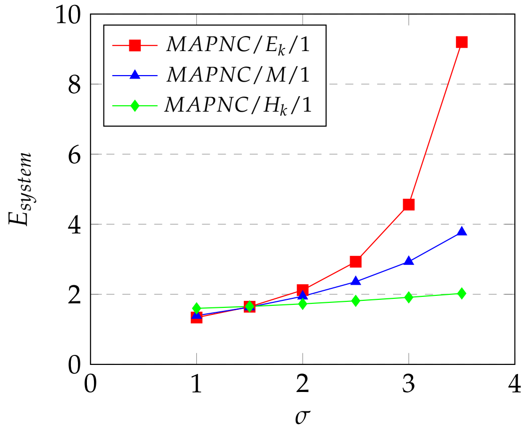

Figure 5: *Breakdown rate(σ) vs E_{system} - MAPNC-A*

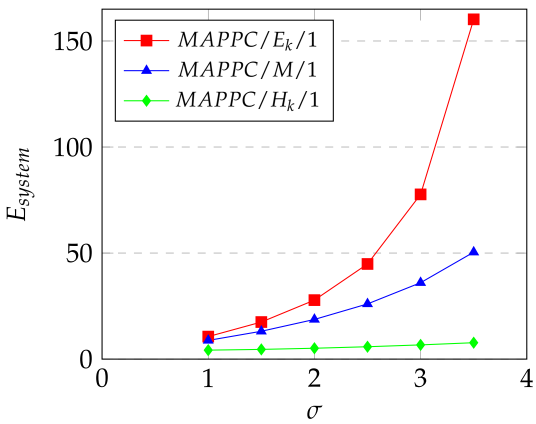

Figure 6: *Breakdown rate(σ) vs E_{system} - MAPPC-A*

9.5. Illustration 5

We have examined both the vacation rate(η) and repair rate(ζ) against the average system size(E_{system}). We fix $\lambda = 2,\ \xi = 6,\ \sigma = 1,\ \delta = 5,\ K = 10,\ \theta^t = [1, 0.97, 0.93, 0.9, 0.87, 0.83, 0.8, 0.75, 0.7, 0.6],\ p = 0.6,\ q = 0.4.$

With the help of figures 7 to 11, we analyze the both vacation rate and repair rate versus the average system size with the combination of arrival and service time groupings. Both the vacation rate and repair rate increases then the corresponding average system size is decreases rapidly in MAP positive correlation compared to all other arrivals.

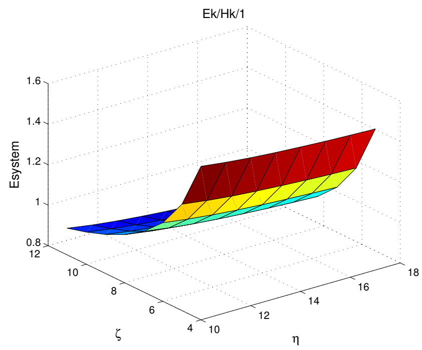

Figure 7: *$E_k/H_k/1$ - Vacation rate(η) and Repair rate(ζ) vs E_{system}*

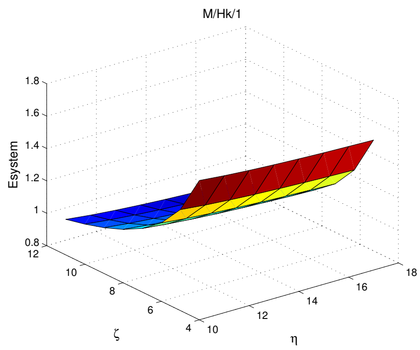

Figure 8: *$M/H_k/1$ - Vacation rate(η) and Repair rate(ζ) vs E_{system}*

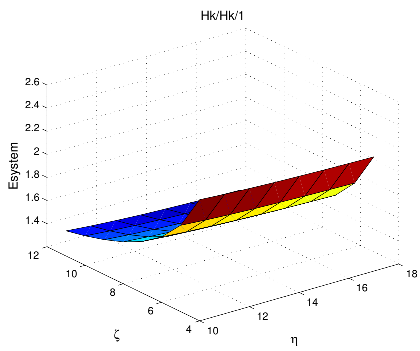

Figure 9: *$H_k/H_k/1$ - Vacation rate(η) and Repair rate(ζ) vs E_{system}*

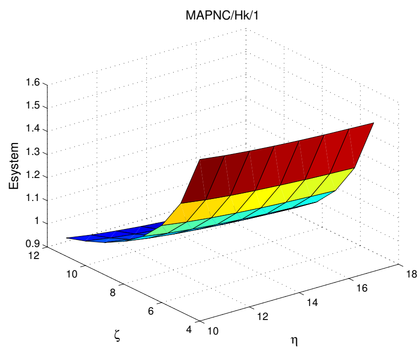

Figure 10: *$MAPNC/H_k/1$ - Vacation rate(η) and Repair rate(ζ) vs E_{system}*

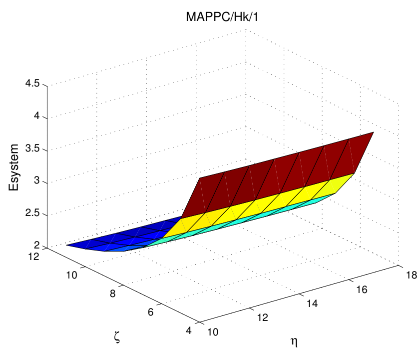

Figure 11: *$MAPPC/H_k/1$ - Vacation rate(η) and Repair rate(ζ) vs E_{system}*

10. Conclusion

In our paper, customers arrive in a Markovian Arrival Process and the service process follows a phase-type distribution with degrading service, server breakdown, vacation process in phase type distribution, repair process in phase type distribution, starting failure and close-down. We also perform the busy period analysis, waiting time distribution and cost analysis in our work. Using numerical values of arrival and service times, we tabulated the repair rate versus expected system size and the vacation rate versus the expected waiting time numerically. We compared the breakdown rate to the expected system size, as well as the vacation and repair rates to the expected system size, as shown by the graphical demonstrations.

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