REPRESENTATION OF CERTAIN LIFETIME MODELS VIA SEQUENCES OF SPECIAL NUMBERS

Christian Genest and Nikolai Kolev

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McGill University and Universidade de São Paulo
christian.genest@mcgill.ca and kolev.ime@gmail.com

Abstract

Stimulated by the work of Rządkowski et al. (2015, J. Nonlinear Math. Phys., 22 (2015), 374–380), the authors derive representations for certain classes of univariate and bivariate lifetime distributions in terms of sequences of Bell, Bernoulli, and Stirling numbers of the second kind, their generalizations, and associated polynomials. Gould–Hopper polynomials are used in the bivariate case, leading to representations for large classes of distributions satisfying a law of uniform seniority for dependent lives formulated by Genest and Kolev (Scan. Act. J., 2021-8 (2021), 726–743).

Keywords: Bell numbers, Bernoulli numbers, Gould–Hopper polynomials, law of uniform seniority, lifetime distributions, Stirling numbers of the second kind, survival functions.

1. Introduction

A random variable X is said to have a Gumbel distribution with parameters $a \in \mathbb{R}$ and $b \in (0, \infty)$ if and only if, for every real $x \in \mathbb{R}$, one has

$$\Pr(X \leq x) = \exp\{-e^{-(x-a)/b}\}. \tag{1}$$

Also known as the log-Weibull and double exponential distribution, this model belongs to the class of generalized extreme-value or Fisher–Tippett distributions. It was identified by Fisher and Tippett [6] as one of the three possible limit distributions of properly normalized maxima of a sequence of mutually independent and identically distributed random variables.

In a note published in 2015, Rządkowski et al. [18] pointed out that Gumbel distributions are related to Stirling numbers of the second kind and Bernoulli numbers. Limiting the discussion to the case $a = 0$ and $b = 1$ for simplicity, these authors showed that the cumulative distribution function of the standard Gumbel distribution, defined for every real $x \in \mathbb{R}$, by $G(x) = \exp(-e^{-x})$, is such that, for every integer $n \in \mathbb{N} = \{1, 2, \ldots\}$,

$$\int \{G^{(n)}(x)\}^2 dx = \frac{(-1)^n}{2n} B_{2n}(1 - 2^{2n}),$$

where $G^{(n)}$ denotes the nth derivative of G and B_n is the nth Bell polynomial defined in terms of the Stirling numbers $S(n,k)$ of the second kind by setting, for every real $x \in \mathbb{R}$,

$$B_n(x) = \sum_{k=1}^{n} S(n,k)\, x^k.$$

Recall that for arbitrary integers $n \in \mathbb{N}$ and $k \in \{1, \ldots, n\}$, $S(n,k)$ represents the number of ways in which one can partition a set of n elements into k non-empty and non-overlapping subsets. The nth Bell number $B_n = B_n(1) = S(n,1) + \cdots + S(n,n)$ is then the number of partitions of a set with n elements. For convenience, one also sets $B_0 = S(0,0) = 1$ and $S(n,0) = 0$ for every integer $n \in \mathbb{N}$. For additional information, the reader is referred to the book by Comtet [3].

The purpose of this note is to extend the observation of Rządkowski et al. [18] to a large class of lifetime distributions routinely used in actuarial practice, and to show how the approach can be extended to the bivariate case. The starting point is the fact that the exponential generating function of the Bell numbers is such that, for every real $x \in \mathbb{R}$,

$$B(x) = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \exp(e^x - 1), \tag{2}$$

and hence both $e^x = 1 + \ln\{B(x)\}$ and $x = \ln[1 + \ln\{B(x)\}]$.

As shown in Section 2, these relations can be used to express some of the most common univariate lifetime distributions in terms of Bell numbers. A bivariate extension is then considered in Section 3 using a special case of the Appell polynomials introduced by Gould and Hopper [10], and recently discussed in the multivariate case by Ricci et al. [17].

Specifically, for any integers $m, n \in \mathbb{N}$ with $m \geq 2$, the Gould–Hopper generalization of the classical Hermite polynomial of order $m-1$ is defined, for every pair $(x, y) \in \mathbb{R}^2$, by

$$H_n^{[m-1]}(x,y) = \sum_{k=0}^{\lfloor \frac{n}{m-1} \rfloor} \frac{n!}{k!(n-mk+k)!} \, x^{n-mk+k} y^k,$$

where in general, $\lfloor x \rfloor$ refers to the integer part of x. As shown by Gould and Hopper [10], one has, for every pair $(x, y) \in \mathbb{R}^2$ and $\gamma \in (0, \infty)$,

$$\sum_{n=0}^{\infty} H_n^{[m-1]}(x,y) \frac{\gamma^n}{n!} = \exp(\gamma x + \gamma^{m-1} y). \tag{3}$$

Relation (3) will be used in Section 3 to derive the Gould–Hopper polynomial expansion of continuous bivariate models satisfying a law of uniform seniority for dependent lives recently introduced and characterized in [7]. A few concluding comments are given in Section 4.

2. Expansions for univariate lifetime distributions

Representations in terms of Bell, Bernoulli, and Stirling numbers of the second kind are given below for various classes of univariate lifetime distributions. The classical Gompertz law of mortality is considered in Section 2.1, and its extension to the Gompertz–Makeham model is the object of Section 2.2. The little known Teissier and Chiang–Conforti distributions are treated in Section 2.3.

2.1. Bell-number expansion of Gompertz's law

Gompertz's law is possibly the oldest documented demographic model. Proposed by the British actuary Benjamin Gompertz (1779–1865) in the early 19th century [9], this model states that for any fixed scale parameter $\lambda \in (0, \infty)$ and shape parameter $\theta \in [0, \infty)$, the survival probability of a random lifetime T is given, at any age $t \in [0, \infty)$, by

$$\Pr(T > t) = \exp\{-\theta(e^{\lambda t} - 1)\}. \tag{4}$$

As is readily seen, Gompertz's law is the same as the Gumbel distribution for the negative of age, restricted to negative values. More specifically, if $T = -X$ in Eq. (1) and $t = -x$ is restricted to positive values for age, then $\Pr(T > t)$ is of the form (4) with $\lambda = 1/b$, $\theta = e^{a/b}$, save for the normalizing constant e^{θ} which ensures that $\Pr(T > 0) = 1$.

Given the result of Rządkowski et al. [18], it may thus be suspected that a similar result holds for Gompertz's law. The following simple result confirms this suspicion.

Proposition 1. *The Bell number representation of Gompertz's survival function* (4) *is given, for every real* $t \in [0, \infty)$, *by* $\Pr(T > t) = \{B(\lambda t)\}^{-\theta}$.

Indeed, it follows at once from Eq. (2) that, for every real $t \in [0, \infty)$, one has

$$\{B(\lambda t)\}^{-\theta} = \exp\{-\theta(e^{\lambda t} - 1)\}, \tag{5}$$

which is precisely the survival function (4) of Gompertz's law. The hazard rate (or force of mortality) implied by Gompertz's law is given, at any age $t \in (0, \infty)$, by $-\{\Pr(T > t)\}^{-1} d\Pr(T > t)/dt = \theta\lambda \exp(\lambda t)$. It corresponds to an exponential increase in death rate with age which is echoed in the exponential increase of the Bell numbers: the first ten are 1, 1, 2, 5, 15, 52, 203, 877, 4140 and 21147. Similarly, the convexity of this hazard rate is mirrored by the convexity of the sequence of Bell numbers, namely the fact that for every integer $n \in \mathbb{N}$, one has $B_n \leq (B_{n-1} + B_{n+1})/2$; see Exercise 1 on p. 291 in Comtet [3].

If desired, connections between Gompertz's law and other famous sequences of numbers could be found just as easily through their corresponding exponential generating function. For instance, a representation of the survival function (4) could be established in terms of

(i) Stirling numbers of the second kind $S_2(k, n)$, given that for every integer $k \in \mathbb{N}$ and real $x \in \mathbb{R}$, one has
$$(e^x - 1)^k/k! = \sum_{n=k}^{\infty} S(k, n) \frac{x^n}{n!};$$

(ii) Bernoulli numbers, $\mathcal{B}_n$, given that $\mathcal{B}_0 = 1$ and that for every real $x \in \mathbb{R}$, one has
$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n \frac{x^n}{n!}.$$
In particular, $\mathcal{B}_1 = -0.5$, $\mathcal{B}_2 = 1/6$, and $\mathcal{B}_{2k+1} = 0$ for every integer $k \in \mathbb{N}$.

Further note that if λ and θ are replaced by $-\lambda$ and $-\theta$, respectively, Eq. (5) transforms into the survival function of the negative Gompertz distribution defined, at every real $t \in [0, \infty)$, by $\exp\{\theta(e^{-\lambda t} - 1)\}$. When $\lambda = 1$ in the latter expression, one gets the Laplace transform of a Poisson distribution with parameter $\theta \in (0, \infty)$; see Chapter 10 of the book by Marshall and Olkin [15].

2.2. r-Bell number expansion of the Gompertz–Makeham law

The Gompertz–Makeham law is an extension of the Gompertz model which was proposed by another British actuary, William Matthew Makeham (1826–1891). The latter proposed to add an age-independent term to the exponentially age-dependent term of Gompertz's model. The resulting construction is one of the most effective theories to describe human mortality.

Stated differently, let X and Y be independent random variables, where X follows Gompertz's law with survival function (4) and Y has an exponential distribution with mean $\xi\lambda\theta$ for some $\xi \in (0, \infty)$. Then the random variable $Z = \min(X, Y)$ has the Gompertz–Makeham distribution with survival function given, at any age $t \in [0, \infty)$, by

$$\Pr(Z > t) = \exp\{-\xi\lambda\theta t - \theta(e^{\lambda t} - 1)\}. \tag{6}$$

Refer to Chapter 10 of the book by Marshall and Olkin [15] for details and historical notes. This model is still used today to describe the age dynamics of human mortality and construct life tables, with remarkable accuracy, for individuals between 30 and 80 years of age.

Now consider the r-Stirling number of the second kind with integer-valued parameters $n \geq k \geq r$, denoted $S(n, k, r)$. This number is a count of the partitions of the set $\{1, \ldots, n\}$ into k non-empty non-overlapping subsets, such that the integers $1, \ldots, r$ are in distinct subsets. By convention, one sets $S(n,k,0) = S(n,k)$. See Broder [1] for basic facts about r-Stirling numbers of the first and second kind.

Mezo [16] defined the nth r-Bell number, denoted $B_{n,r}$, by $B_{n,r} = S(n,0,r) + \cdots + S(n,n,r)$ so that by convention, one has $B_{n,0} = B_n$. More generally, the nth r-Bell number is a count of the partitions of a set with $n + r$ elements whose first r elements are in distinct subsets of the partition. The first few r-Bell numbers are given in Table A134980 of Sloane [19]. For example, the first seven 6-Bell numbers are 1, 7, 50, 365, 2727, 20878, and 163967. Recent results about r-Bell numbers are reported by Corcino et al. [4].

Mezo [16] showed that the exponential generating function of the nth r-Bell number $B_{n,r}$ is given, at every real $x \in \mathbb{R}$, by

$$B_r(x) = \sum_{n=0}^{\infty} B_{n,r} \frac{x^n}{n!} = \exp(e^x - 1 + rx).$$

The following result is then immediate.

Proposition 2. *Suppose that the random variable Z has a Gompertz–Makeham distribution with scale parameter $\lambda \in (0,\infty)$ and shape parameter $\theta \in (0,\infty)$. Further suppose that the parameter ξ is integer-valued with $\xi \in \{r,\ldots,n\}$. Then the survival function of Z given in Eq.* (6) *is such that, for every real $t \in (0,\infty)$, one has* $\Pr(Z > t) = \{B_\xi(\lambda t)\}^{-\theta}$.

A connection between the Gompertz–Makeham survival function and r-Stirling numbers of the second kind can also be deduced from the fact that, for every real $x \in (0,\infty)$,

$$\frac{1}{k!}\left(e^x - \sum_{\ell=0}^{r-1} \frac{x^\ell}{\ell!}\right)^k = \sum_{n=kr}^{\infty} S(n,k,r)\, \frac{x^n}{n!}.$$

2.3. Two other Bell-number representations of distributions

In 1934, the French biologist Georges Teissier (1900–1972) introduced a model to describe the mortality of several domestic animal species protected from accidents and disease, i.e., dying from "pure aging" [20]. Based on data collected on several species, this author found that animal mortality does not follow Gompertz's law, as it does for humans.

The survival function of Teissier's distribution is defined, at any age $t \in [0,\infty)$, by $\exp(t + 1 - e^t)$. This model was later rediscovered by Laurent [13], who considered a one-parameter extension. A two-parameter version called the scaled Teissier distribution with parameters $\lambda \in (0,\infty)$ and $\theta \in (0,\infty)$ such that $\lambda\theta \le 1$ has survival function is given, for every real $t \in (0,\infty)$, by

$$S_{\mathrm{T}}(t) = \exp\{\lambda t - (e^{\lambda t} - 1)/(\lambda\theta)\}. \tag{7}$$

See Kolev et al. [12] for more details on the history of the Teissier distribution.

Taking into account Eq. (5) and the identity $\ln\{B(\lambda t)\} + 1 = e^{\lambda t}$, valid for all real $t \in [0,\infty)$, one can get the following Bell-number representation of Teissier's scaled model.

Proposition 3. *The Bell-number representations of the Teissier survival functions* (7) *is given, for every real $t \in [0,\infty)$, by* $S_{\mathrm{T}}(t) = [\ln\{B(\lambda t)\} + 1]/\{B(\lambda t)\}^{1/(\lambda\theta)}$.

As a final example, Chiang and Conforti [2] designed a stochastic model assuming that mortality intensity is a function of the accumulated effect of a person's continuous exposure to toxic material in the environment (absorbing coefficient) and their biological reaction to the toxin absorbed (discharging coefficient). Given parameters $\alpha \in (0,\infty)$ and $\theta \in (1,\infty)$, the survival function of the Chiang–Conforti model is given, for every real $t \in [0,\infty)$, by

$$S_{\mathrm{CC}}(x) = \exp\{-\lambda x - \theta(e^{-\lambda x} - 1)\}. \tag{8}$$

Note in passing that an equivalent form of this model had appeared earlier, with a different parametrization, in the work of Ghurye [8].

From Eq. (2), one has, for every real $t \in [0,\infty)$, $\ln\{B(-\lambda t)\} + 1 = e^{-\lambda t}$ and $\{B(-\lambda t)\}^{-\theta} = \exp\{-\theta(e^{-\lambda t} - 1)\}$, leading to the following representation of the Chiang–Conforti model.

Proposition 4. *The Bell-number representation of the Chang–Conforti survival function* (8) *is given, for every real* $t \in [0, \infty)$, *by* $S_{\rm CC}(t) = [\ln\{B(-\lambda t)\} + 1]/\{B(-\lambda t)\}^{\theta}$.

3. A bivariate extension

A common characteristic of the univariate models considered in Section 2 is that they have what Laurent [13] termed an exponential structure. Namely, in all cases considered, a survival function S could be expressed, for every real $t \in [0, \infty)$, in the form

$$S(t) = \exp\{-\Lambda(t)\} \tag{9}$$

for a map Λ which involved a linear and an exponential term. This map is called the cumulative hazard rate in survival analysis.

A similar approach can actually be used in higher dimensions. To make this point, consider the b-BLUS class of bivariate survival functions introduced in [7] as a model for dependent lives based on an extension of the actuarial law of uniform seniority [5].

Specifically, a pair (X, Y) of continuous random variables is said to belong to the b-BLUS class if there exist real-valued parameters $\alpha, \beta \in (0, \infty)$ and a continuous, strictly decreasing univariate survival function $\psi : [0, \infty) \to [0, 1]$ such that, for every pair $(x, y) \in (0, \infty)^2$, $\Pr(X > x, Y > y) = \psi(\alpha x + \beta y)$.

It will be shown below how various models from the b-BLUS class can be expressed in terms of Bell numbers, Bernoulli numbers, and Stirling numbers of the second kind with the help of the Gould–Hopper polynomials specified by Eq. (3).

3.1. Gould–Hopper expansion of Gompertz's bivariate law

Recall that a random pair (X, Y) has a bivariate Gompertz distribution with parameters $\alpha, \beta \in (0, \infty)$ and $\theta \in [1, \infty)$ whenever, for every pair $(x, y) \in [0, \infty)^2$, one has

$$S_{\rm BG}(x, y) = \Pr(X > x, Y > y) = \exp\{-\theta(e^{\alpha x+\beta y} - 1)\}. \tag{10}$$

The proof of the following result is straightforward.

Proposition 5. *The Gould–Hopper polynomial expansion of the bivariate Gompertz joint survival function* (10) *with real-valued parameters* $\theta \in (0, \infty)$, $\alpha = \gamma \in (0, \infty)$, *and* $\beta = \gamma^{m-1}$ *for some integer* $m \in \mathbb{N}$ *is given, for every pair* $(x, y) \in [0, \infty)^2$, *by*

$$S_{\rm BG}(x, y) = \exp[-\theta\{\mathcal{H}^{[m-1]}(x, y; \gamma) - 1\}],$$

where $\mathcal{H}^{[m-1]}$ *stands for the parametric function implicitly defined in Eq.* (3).

3.2. Gould–Hopper expansion of two other bivariate models

In [7], bivariate versions of the univariate Teissier and Chang–Conforti distributions (7) and (8) are defined as follows.

Definition 1. *A random pair* (X, Y) *is said to have a bivariate Teissier distribution with real-valued parameters* $\alpha \in (0, \infty)$, $\beta \in (0, \infty)$, *and* $\theta \in [(3 + \sqrt{5})/2, \infty)$ *if and only if, for every pair* $(x, y) \in (0, \infty)^2$, *one has*

$$S_{\rm BT}(x, y) = \Pr(X > x, Y > y) = \exp\{\alpha x + \beta y - \theta(e^{\alpha x+\beta y} - 1)\}. \tag{11}$$

Definition 2. *A random pair* (X, Y) *is said to have a bivariate Chang–Conforti distribution with real-valued parameters* $\alpha \in (0, \infty)$, $\beta \in (0, \infty)$, *and* $\theta \in [0, (3 - \sqrt{5})/2]$ *if and only if, for every pair* $(x, y) \in (0, \infty)^2$, *one has*

$$S_{\rm BCC}(x, y) = \Pr(X > x, Y > y) = \exp\{-\alpha x - \beta y - \theta(e^{-\alpha x-\beta y} - 1)\}. \tag{12}$$

The following results are then immediate from relation (3).

Proposition 6. *The Gould–Hopper polynomial expansion of the bivariate Teissier distribution with survival function* (11) *and real-valued parameters* $\alpha = \gamma \in (0, \infty)$, $\theta \in [(3+\sqrt{5})/2, \infty)$, *and* $\beta = \gamma^{m-1}$ *for some integer* $m \in \mathbb{N}$, *is given, for every pair* $(x, y) \in [0, \infty)^2$, *by*

$$S_{\mathrm{BT}}(x, y) = \exp\{\ln[\mathcal{H}^{[m-1]}(x, y; \gamma)] - \theta[\mathcal{H}^{[m-1]}(x, y; \gamma) - 1]\}$$

where $\mathcal{H}^{[m-1]}$ *stands for the parametric function implicitly defined in Eq.* (3).

Proposition 7. *The Gould–Hopper polynomial expansion of the bivariate Chang–Conforti distribution with survival function* (12) *and real-valued parameters* $\alpha = \gamma \in (0, \infty)$, $\theta \in [0, (3-\sqrt{5})/2]$, *and* $\beta = \gamma^{m-1}$ *for some integer* $m \in \mathbb{N}$ *is given, for every pair* $(x, y) \in [0, \infty)^2$, *by*

$$S_{\mathrm{BCC}}(x, y) = \exp\{-\ln[\mathcal{H}^{[m-1]}(x, y; \gamma)] - \theta[\mathcal{H}^{[m-1]}(-x, -y; \gamma) - 1]\},$$

where $\mathcal{H}^{[m-1]}$ *stands for the parametric function implicitly defined in Eq.* (3).

3.3. Two b-BLUS models with positive or negative dependence

The b-BLUS property discussed in [7] can model lives that are either positively or negatively dependent. The following two examples illustrate each one of these two cases. Given real-valued parameters $\alpha \in (0, \infty)$, $\beta \in (0, \infty)$, $\delta \in (0, \infty)$ and $\rho \in [1, \infty)$, consider the joint survival functions of random pairs (X, Y) defined, for every pair $(x, y) \in [0, \infty)^2$, by

$$S_1(x, y) = \Pr(X > x, Y > y) = (1 + \alpha x + \beta y)^{-1/\delta}$$

and

$$S_2(x, y) = \Pr(X > x, Y > y) = (1 - \alpha x - \beta y)^{\rho}\mathbf{1}_{(0,1)}(\alpha x + \beta y),$$

where in general, $\mathbf{1}_A$ refers to the indicator function of the set A.

In these two cases, a direct application of identity (3) with $\alpha = \gamma$ and $\beta = \gamma^{m-1}$ for some integer $m \in \mathbb{N}$ leads to the following Gould–Hopper polynomial expansions:

$$S_1(x, y) = \left\{1 + \ln \mathcal{H}^{[m-1]}(x, y; \gamma)\right\}^{-1/\delta} \quad \text{and} \quad S_2(x, y) = \left\{1 + \ln \mathcal{H}^{[m-1]}(x, y; \gamma)\right\}^{\rho}.$$

Here again, $\mathcal{H}^{[m-1]}$ stands for the parametric function implicitly defined in Eq. (3).

4. Final remarks

The initial motivation for this note was the intriguing observation of Rządkowski et al. [18] to the effect that Gumbel's distribution is related to Stirling numbers of the second kind and Bernoulli numbers. It was shown that upon expressing a survival function S through its cumulative hazard rate Λ as in Eq. (9), new relations involving Bell numbers, Bernoulli numbers, and Stirling numbers of the second kind can be obtained via Eq. (2) for a large class of survival models, at the cost of suitable restrictions on the parameters. Such representations may or may not be elegant, insightful or useful for computational purposes.

In the bivariate case, the survival function of any continuous random pair (X, Y) can be written, for every pair $(x, y) \in [0, \infty)^2$, in the form

$$\Pr(X > x, Y > y) = \exp\left\{-\int_{\mathcal{C}} R(z)\, dz\right\}, \tag{13}$$

where R is the hazard gradient vector and $\mathcal{C}$ is any sufficiently smooth continuous path beginning at $(0, 0)$ and terminating at (x, y). Relation (13) holds so long as S is absolutely continuous along the path of integration; see Marshall [14]. Therefore, the expressions derived in Section 3 for distributions in the b-BLUS class in terms of the Gould–Hopper polynomial are just as natural as those presented by Rządkowski et al. [18] and in Section 2.

Of course, the Gould–Hopper polynomials are not always the right tool. For instance, consider a random pair (X, Y) with Gumbel's bivariate exponential distribution [11] with real-valued parameters $\alpha \in (0, \infty)$, $\beta \in (0, \infty)$, and $\delta \in [0, \alpha\beta]$, whose survival function is defined, for every pair $(x, y) \in [0, \infty)^2$, by

$$\Pr(X > x, Y > y) = \exp(-\alpha x - \beta y - \delta xy) = \exp(-\alpha x - \beta y) \exp(xy)^{-\delta}.$$

The first factor can be expressed via the Gould–Hopper polynomial via Eq. (3), but for the second factor one must resort to the map defined, for every pair $(x, y) \in [0, \infty)^2$, by $W(x, y) = ye^{xy}/(e^y - 1)$, i.e., the generating function of the classical Bernoulli polynomials; see Comtet [3] for details. Letting B denote the Bell polynomial, one then finds, for every pair $(x, y) \in [0, \infty)^2$,

$$\exp(xy) = W(x, y)(e^y - 1)/y = W(x, y) \ln\{B(y)\}/\ln[1 + \ln\{B(y)\}].$$

Representations of multivariate continuous survival functions might be derived from recent results for Appell polynomials due to Ricci et al. [17]. This may be the object of future work.

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