Applications and Some Characteristics of Inverse Power Cauchy Distribution

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Abstract

Based on the power Cauchy distribution, we have purposed a new distribution called inverse power Cauchy distribution that offers greater modeling flexibility for lifetime data. Real-world data can be efficiently analyzed using the suggested model because it is analytically sound. Its density function can take on a number of different shapes, including reversed-J, symmetrical, and right-skewed. Depending on the different values of the parameters, it can adapt to different hazard forms, such as an upside-down bathtub, a monotonically increasing or decreasing curve, and others. Its moments, quantile, reliability, hazard, order statistics with density function, moment generating function, and entropy are all given with various explicit forms. The observed information matrix is created when the new model's parameters are calculated through maximum likelihood technique. A simulation study is conducted to investigate the behaviour of maximum likelihood estimators. The proposed model gets a superior fit compared to certain well-known distributions, according to the test of goodness-of-fit we conducted. The significance of the purposed distribution is demonstrated empirically using two real-world data sets.

Keywords: Entropy, Maximum likelihood estimation, Moment, order statistics, Power Cauchy distribution

1. Introduction

Many families of probability models have been developed in the recent decades. New distributions are frequently produced from modifications of a random variable X of parent distribution by: power (e.g., Weibull is obtained from the exponential); linear; non-linear (e.g., log-logistic from logistic); log (e.g., log gamma, log-normal, logistic) and inverse transformation (e.g., inverse Lindley, inverse Exponential models); T-X family framework proposed by Alzaatreh et al. [3]; the compounding of some important lifetime and discrete distributions (e.g. the Poisson-X family distribution) by Tahir et al. [28]. In most cases, a given mixture of baseline models or linear combination defines a class of probability distributions where baseline model is a particular case.

In contrast to the Gaussian distribution, the Cauchy distribution has a significantly thicker tail and is symmetric, unimodal, and bell-shaped. It can be used to analyze data that contains outliers. The ratio of two independent normal variates can be used to generate the Cauchy distribution. It is a widely used distribution that has applications in a variety of disciplines, including biology, applied mathematics, engineering, econometrics, physics, clinical trials, stochastic modeling of decreasing failure rate survival data, queuing theory, and reliability. With location parameter $\theta > 0$ and non-negative scale parameter $\beta > 0$, the cumulative distribution function (CDF) of the Cauchy distribution is,

$$F_X(x;\beta,\theta) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{x-\theta}{\beta}\right) \; ; x \in \Re,\ \beta ¿ 0. \tag{1}$$

and probability density function (PDF) of Equation (1) is

$$f_X(x;\beta,\theta) = \left\{\beta\pi\left[1+\left(\frac{x-\theta}{\beta}\right)^2\right]\right\}^{-1} \; ; x \in \Re,\ \beta ¿ 0 \tag{2}$$

The centre limit theorem (CLT) is invalid because there is no finite moment generating function for the Cauchy distribution. Due to the lack of a closed-form solution, the MLEs of its parameters are also not the good. These factors make it unlikely or unreasonable to use this distribution to model real-life data. Therefore, the Cauchy distribution must be modified to address the aforementioned shortcomings, some of which are listed below. The generalized form of Cauchy distribution has introduced by Rider [25] whose PDF is given by

$$f_x(x) = \frac{\Gamma(m)}{\beta\,\Gamma(1/2)\,\Gamma(m-1/2)}\left[1+\left(\frac{x-\theta}{\beta}\right)^2\right]^{-m};\ \mathrm{m} \geqslant 1,\ \beta > 0 - \infty < x < \infty. \tag{3}$$

A truncated Cauchy distribution was suggested by Nadarajah and Kotz [21] to address the issue of the lack of MLEs and moments of Cauchy distribution and having PDF,

$$f(x) = \frac{1}{\pi\beta} + \left[1+\left(\frac{x-\theta}{\beta}\right)^2\right]^{-1}\left[\tan^{-1}\left(\frac{B-\theta}{\beta}\right) - \tan^{-1}\left(\frac{A-\theta}{\beta}\right)\right]^{-1};\quad -\infty < A \leqslant x \leqslant B, \theta \in \Re, \beta > 0 \tag{4}$$

Additionally, a relationship between the Cauchy and the hyperbolic secant distribution has been established by Manoukian and Nadeau [17] and Kravchuk [15]. A Modified form of Cauchy distribution has introduced by Ohakwe and Osu [22] and another generalization of the Cauchy distribution is presented by Eugene et al. [11] and Alshawarbeh et al. [[1], [2]]. Further, the Half- Cauchy (HC) distribution has used by using Marshall and Olkin [18] and defined a generators, Beta-G by Eugene et al. [11], and Kumaraswamy-G by Cordeiro and de Castro [9]. Similar to this, some half-Cauchy families have been proposed, including the Marshall-Olkin- HC, beta- HC, and Kumaraswamy- HC families by Jacob and Jayakumar [14], Cordeiro and Lemonte [10], and Ghosh [12] respectively, and the truncated form of Cauchy power-exponential model by Chaudhary et al. [8] and exponentiated form of PC distribution has presented by Sapkota [27].

Extensive study has recently been conducted to develop models that fit survival data, which can be negatively or positively skewed and can have the unimodal hazard function. Power Cauchy (PC) distribution, a two-parameter model that is a sub-model of the modified Beta family that does well with the survival data, was introduced by Rooks et al. [26]. Positively skewed data can be employed with the PC distribution's PDF, which has a somewhat larger right tail than the other recognized humped-shaped two-parameter sub-model of the transformed beta family Rooks et al. [26]. The PC distribution's CDF and PDF are,

$$F(x) = 2\pi^{-1}\tan^{-1}(\lambda x)^{\alpha};\ x > 0,\ \alpha, \lambda > 0. \tag{5}$$

and

$$f(x) = 2\pi^{-1}(\lambda x)(\lambda x)^{\alpha-1}\left[1+(\lambda x)^{2\alpha}\right]^{-1}; x > 0, \alpha, \lambda > 0. \tag{6}$$

respectively.
The hazard function of PC distribution is

$$h(x) = \frac{2\pi^{-1}(\lambda x)(\lambda x)^{\alpha-1}\left[1+(\lambda x)^{2\alpha}\right]^{-1}}{1-2\pi^{-1}\tan^{-1}(\lambda x)^{\alpha}};\ x > 0,\ \alpha, \lambda > 0 \tag{7}$$

Also using PC distribution Weibull PC has been defined by Tahir et al. [29] and Burr XII-power Cauchy distribution by Bhatti et al. [4] using the T-X family technique. The fundamental goal of this work is to present a more adaptable model, demonstrate its applicability, and improve the fitting to the real-life data. In this study, a novel model known as the inverse power Cauchy (IPC) distribution was created using the inversion method. It is the inverse of PC distribution. We have also demonstrated some of the intended model's mathematical and statistical characteristics. The structure of this paper's contents is as follows. In section 2, we presented the IPC distribution and a few distributional features, including the graph of the density, the survival and hazard rate function, the quantile function, random number generation, and skewness and kurtosis. Section 3 presents some of the IPC distribution's crucial characteristics. We go over the procedure for estimating the model parameters in section 4. In section 5, a simulation experiment is done to examine how maximum likelihood estimators behave. We conduct a goodness-of-fit test and a model adequacy test in section 6 using two real data sets. We present some conclusions in section 7.

2. Inverse Power Cauchy Distribution

We have introduced new IPC distribution and visualized some graphs of its PDF and HRF in this section. The CDF of IPC distribution with shape parameter α and scale parameter λ can be expressed by using Equation (5) as

$$F(x) = 1 - 2\pi^{-1}\tan^{-1}\left[\left(\frac{\lambda}{x}\right)^{\alpha}\right];\ x > 0,\ \alpha, \lambda > 0. \tag{8}$$

And its corresponding PDF is obtained as,

$$f(x) = 2\pi^{-1}\alpha\lambda^{\alpha}x^{-(\alpha+1)}\left[1 + \left(\frac{\lambda}{x}\right)^{2\alpha}\right]^{-1};\ x > 0,\ \alpha, \lambda > 0. \tag{9}$$

2.1. Survival and hazard rate function (HRF)

The survival and HRF of $X \sim IPC(\alpha, \lambda)$ are

$$S(x) = 2\pi^{-1}\tan^{-1}\left[\left(\frac{\lambda}{x}\right)^{\alpha}\right];\ x > 0,\ \alpha, \lambda > 0 \tag{10}$$

and

$$h(x) = \frac{\alpha\lambda^{\alpha}x^{-(\alpha+1)}\left[1 + (\lambda/x)^{2\alpha}\right]^{-1}}{\tan^{-1}\{(\lambda/x)^{\alpha}\}};\ x > 0,\ \alpha, \lambda > 0. \tag{11}$$

respectively.

A special case of the IPC distribution: If $\alpha = 1$ and $\lambda = 1$ in Equation (9) the IPC distribution tends to two times the standard Cauchy distribution.

2.2. Cumulative hazard function and Failure rate average (FRA)

The cumulative hazard function of IPC distribution can be expressed as

$$\begin{aligned} H(x) &= \int_{-\infty}^{x} h(x)dx \\ &= -\log\left[1 - F(x)\right] \\ &= -\log\left[2\pi^{-1}\tan^{-1}(\lambda/x)^{\alpha}\right] \end{aligned} \tag{12}$$

and FRA function is

$$FRA\,(x) = \frac{H(x)}{x} = -\frac{1}{x}\log\left[2\pi^{-1}\tan^{-1}(\lambda/x)^{\alpha}\right];x > 0 \tag{13}$$

here H(x) is the cumulative hazard rate function. The IPC distribution exhibits unimodal, decreasing, right skewed and symmetrical shapes of PDF and decreasing, increasing, and inverted bathtub hazard rate shapes in Figure 1.

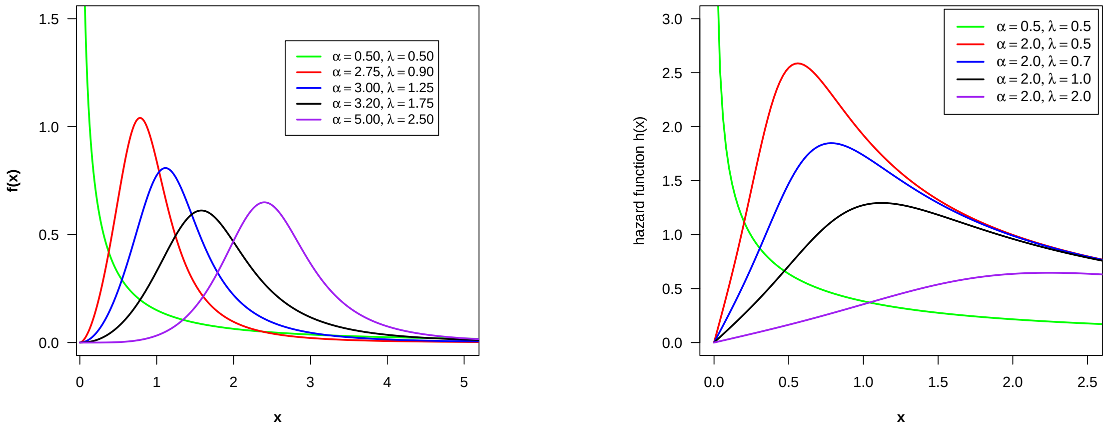

Figure 1: *Various shapes of PDF and HRF function of IPC distribution.*

3. SOME PROPERTIES OF IPC DISTRIBUTION

3.1. The Quantile Function

The quantile function can be expressed as

$$Q(p) = \lambda \left[\tan \left\{ \frac{(1-p)\,\pi}{2} \right\} \right]^{-\frac{1}{\alpha}} ; 0 < p < 1. \tag{14}$$

The median, lower quartile, and upper quartile can be calculated by using Equation (14) as follows

$$median = \lambda \left[\tan \left\{ \frac{\pi}{4} \right\} \right]^{-\frac{1}{\alpha}}$$

lower quartile

$$Q_1 = \lambda \left[\tan \left\{ \frac{3\pi}{8} \right\} \right]^{-\frac{1}{\alpha}}$$

and upper quartile

$$Q_3 = \lambda \left[\tan \left\{ \frac{\pi}{8} \right\} \right]^{-\frac{1}{\alpha}}$$

Random deviate generation
The random deviate can be generated from $IPC(\alpha, \lambda)$ by

$$x = \lambda \left[\tan \left\{ \frac{(1-u)\,\pi}{2} \right\} \right]^{-\frac{1}{\alpha}} ; 0 < u < 1 \tag{15}$$

Where u has the $U(0,1)$ uniform distribution.

3.2. Mode of IPC distribution

The probability distribution of the provided PDF's mode is its most frequent value. For computing the mode, the following requirements must be met: $\frac{df(x)}{dx} = 0$ and $\frac{d^2 f(x)}{dx^2} < 0$ respectively. Since $f(x) > 0$, by resolving the equation, the model value of the suggested distribution is determined as

$$2\alpha\lambda^{2\alpha} - (1+\alpha)\left(x^{2\alpha} + \lambda^{2\alpha}\right) = 0$$

Hence we obtained the mode of the IPC distribution is

$$x = \left\{ \frac{2\alpha\lambda^{2\alpha}}{(1+\alpha)} - \lambda^{2\alpha} \right\}^{1/2\alpha}$$

3.3. Skewness and Kurtosis

The quantile based Skewness and Kurtosis can be computed using the following expressions,

- Coefficient of Bowley's skewness can be computed by using

$$Skewness\,(B) = \frac{Q\,(0.75) - 2Q\,(0.5) + Q\,(0.25)}{Q\,(0.75) - Q\,(0.25)}$$

and

- The coefficient of kurtosis based on octiles which was defined by Moors [20] is

$$M_K = \frac{Q\,(0.875) + Q\,(0.375) - Q\,(0.625) - Q\,(0.125)}{Q\,(3/4) - Q\,(1/4)}$$

In table 1 we have displayed the distributional nature of the IPC distribution. Nine sets of random samples of equal size 100 are generated from Equation (15) for different values of parameters $\alpha\ and\ \lambda$ We have computed the values of central tendencies and dispersions like mean, mode, median, skewness, and kurtosis of the proposed distribution. From the table, we observed that the median is increased as the values of the parameters are increases while the mean and mode increases first and then decrease. Similarly, skewness is positive at first and becomes negative when the values of the parameters increase. The proposed distribution is leptokurtic initially, changing progressively to platykurtic when the parameter values are increased, according to the measure of kurtosis.

Table 1: *The mean, median, mode, skewness, and kurtosis for different values of the parameters*

Parameters		**Mean**	**Median**	**Mode**	**Skewness**	**Kurtosis**
alpha	**lambda**					
1	0.1	0.2907	0.095	0.2041	9.2321	89.1813
1	1.0	2.9069	0.9504	2.0408	9.2321	89.1813
2	1.1	1.2948	1.0723	1.5714	5.2338	38.0526
5	1.2	1.1874	1.1878	1.384	1.2998	5.4178
10	1.3	1.2765	1.2934	1.3961	0.317	2.1062
15	1.4	1.3791	1.3953	1.4682	0.0238	1.6876
20	1.5	1.4815	1.4962	1.5545	-0.1187	1.5904
25	1.6	1.5834	1.5967	1.6463	-0.2032	1.5666
30	1.8	1.6848	1.6979	1.7049	-0.2592	1.5649

4. Expansion and Properties of IPC distribution

The PDF and CDF of IPC distribution can be transform into linear form by using the binomial expansion as

$$(1+a)^{-c} = \sum_{k=0}^{\infty} (-1)^k \begin{pmatrix} c+k-1 \\ k \end{pmatrix} \mathrm{a}^k, \ \text{for} \ |a| < 1, \ \mathrm{c} > 0. \tag{16}$$

Applying Equation (16) the PDF of IPC distribution can be expressed as

$$f\,(x) = \frac{2\alpha}{\pi} \sum_{k=0}^{\infty} \eta_k \ x^{-\alpha(1+2k)-1} \tag{17}$$

where $\eta_k = (-1)^k \lambda^{\alpha(1+2k)}$ also the CDF

$$\begin{aligned}[F(x)]^h &= \left[1 - 2\pi^{-1}\tan^{-1}\left\{\left(\frac{\lambda}{x}\right)^{\alpha}\right\}\right]^h \\ &= \sum_{j=0}^{\infty} (-1)^j \binom{h}{j} \left[2\pi^{-1}\tan^{-1}\left\{\left(\frac{\lambda}{x}\right)^{\alpha}\right\}\right]^j \\ &= \sum_{j=0}^{\infty} (-2\pi^{-1})^j \binom{h}{j} \left[\tan^{-1}\left\{\left(\frac{\lambda}{x}\right)^{\alpha}\right\}\right]^j\end{aligned} \tag{18}$$

4.1. Moments of IPC distribution

Let X be a random variable that follows $IPC(\alpha, \lambda)$ then raw moments about the origin can be calculated as

$$\begin{aligned}\mu_r' = E(X^r) &= \int_0^{\infty} x^r 2\pi^{-1}\alpha\lambda^{\alpha}x^{-(\alpha+1)}\left[1 + \left(\frac{\lambda}{x}\right)^{2\alpha}\right]^{-1} dx \\ &= 2\pi^{-1}\alpha\lambda^{\alpha}\int_0^{\infty} x^{r+\alpha-1}\frac{1}{x^{2\alpha}+\lambda^{2\alpha}}dx \\ &= \frac{\lambda^{\alpha}}{\pi}\sum_{k=0}^{\frac{r-\alpha}{2\alpha}} \binom{\frac{r-\alpha}{2\alpha}}{k} \left(-\lambda^{2\alpha}\right)^k \int_{\lambda^{2\alpha}}^{\infty} z^{\frac{r-\alpha}{2\alpha}-k-1}dz\end{aligned} \tag{19}$$

Where $z = x^{2\alpha} + \lambda^{2\alpha}$

Case I: if $\frac{r-\alpha}{2\alpha} - k = 0$, then $\mu_r' = \frac{\lambda^{\alpha}}{\pi}\sum_{k=0}^{\frac{r-\alpha}{2\alpha}} \binom{\frac{r-\alpha}{2\alpha}}{k} \left(-\lambda^{2\alpha}\right)\left(1 - \lambda^{r-\alpha-2k\alpha}\right)$.

Case II: if $\frac{r-\alpha}{2\alpha} - k < 0$, then $\mu_r' = \frac{\lambda^{\alpha}}{\pi}\sum_{k=0}^{\frac{r-\alpha}{2\alpha}} \binom{\frac{r-\alpha}{2\alpha}}{k} \left(-\lambda^{2\alpha}\right)\left(-\lambda^{r-\alpha-2k\alpha}\right)$.

Case III: if $\frac{r-\alpha}{2\alpha} - k > 0$ then μ_r' is undefined.

4.2. Moment Generating Function

A moment-generating function is an important tool for studying random variables. Let X be a random variable that follows IPC distribution and the moment generating function can be defined as $M_X(\delta) = E(e^{\delta x}) = \sum_{r=0}^{\infty} \frac{\delta^r}{r!}E(X^r)$. By using equation (19) we can write

$$M_X(\delta) = \sum_{r=0}^{\infty}\sum_{k=0}^{\frac{r-\alpha}{2\alpha}} \frac{\delta^r}{r!}\frac{\lambda^{\alpha}}{\pi} \binom{\frac{r-\alpha}{2\alpha}}{k} \left(-\lambda^{2\alpha}\right)^k \int_{\lambda^{2\alpha}}^{\infty} z^{\frac{r-\alpha}{2\alpha}-k-1}dz \tag{20}$$

Where $z = x^{2\alpha} + \lambda^{2\alpha}$, hence the moment generating function of IPC distribution can be expressed as

$$M_X(\delta) = \sum_{r=0}^{\infty}\sum_{k=0}^{\frac{r-\alpha}{2\alpha}} \frac{\delta^r}{r!}\frac{\lambda^{\alpha}}{\pi} \binom{\frac{r-\alpha}{2\alpha}}{k} \left(-\lambda^{r-\alpha}\right) \quad for\ \frac{r-\alpha}{2\alpha} - k < 1. \tag{21}$$

4.3. Residual life function

In reliability studies, the additional lifetime given that an event or a component or a system has survived until time t is called the residual life function (RLF) of the event or component, or a system. The r^{th} moment of the residual life of random variable X of IPC distribution can be defined as

$$D_r(t) = \frac{1}{R(t)}\int_t^{\infty} (x-t)^r f(x)dx. \tag{22}$$

Using the binomial expansion $(x-t)^k = \sum_{r=0}^{k} (-1)^r \binom{k}{r} x^{k-r} t^r$ in Equation (22) and using Equation (17) we get

$$D_r(t) = \frac{1}{R(t)} \sum_{j=0}^{\infty} \sum_{r=0}^{k} (-1)^r \binom{k}{r} t^r \, \eta_j \int_t^{\infty} x^{-\alpha(1+2j)-k+r-1} dx \tag{23}$$

Where $\eta_j = \frac{2}{\pi}(-1)^k \alpha \lambda^{\alpha(1+2k)}$. Hence RLF for the time t is

$$D_r(t) = \frac{1}{R(t)} \sum_{j=0}^{\infty} \sum_{r=0}^{k} \theta_{jr} t^{r-(k+\alpha+2\alpha j)} \; for \; r < k + \alpha + 2\alpha j \tag{24}$$

here $\theta_{jr} = (-1)^r \binom{k}{r} t^r \eta_j$.

4.4. Entropy

The entropy of a random variable T is a measure of the variance of an uncertainty and has density function $f(t)$. The Renyi entropy is defined as,

$$\begin{aligned} I_R(\rho) &= (1-\rho)^{-1} \log \left[\int f(t)^\rho dt \right]; \quad \text{where } \rho > 0, \rho \neq 1 \\ &= (1-\rho)^{-1} \log \left[\int_0^{\infty} \left\{ 2\pi^{-1} \alpha \lambda^{\alpha} t^{-(\alpha+1)} \left[1 + \left(\frac{\lambda}{t} \right)^{2\alpha} \right]^{-1} \right\}^{\rho} dt \right] \\ &= (1-\rho)^{-1} \log \left[(2\alpha)^{\rho-1} \pi^{-1} \sum_{k=1}^{\frac{\alpha\rho-2\alpha-\rho+1}{2\alpha}} \lambda^{\alpha\rho+2\alpha k} (-1)^k \int_{\lambda^{2\alpha}} z^{\frac{1-\alpha\rho-2\alpha-\rho}{2\alpha}-k-\rho} dz \right] \end{aligned} \tag{25}$$

Where $z = t^{2\alpha} + \lambda^{2\alpha}$

Case I: when $\frac{1-\alpha\rho+2\alpha-\rho}{2\alpha} < k + \rho$ then

$$I_R(\rho) = (1-\rho)^{-1} \log \left[(2\alpha)^{\rho-1} \pi^{-1} \lambda^{\alpha\rho} \sum_{k=1}^{\frac{\alpha\rho-2\alpha-\rho+1}{2\alpha}} (-1)^k \left\{ -\lambda^{1-\alpha(\rho+2k+2\rho+2)-\rho} \right\} \right]$$

Case II: when $\frac{1-\alpha\rho+2\alpha-\rho}{2\alpha} > k + \rho$, then the integral is divergent.

4.5. Order Statistics (OS)

Numerous applications of probability theory and applied statistics can make use of OS. So, for the suggested distribution, we have shown some OS features. Let $X_1, ..., X_n$ be n iid random variates, each with $F(x)$. Suppose represents the r^{th} OS and denote PDF of r^{th} OS for $X_1, ..., X_n$ be n iid random variables from CDF $F(x)$ and can be defined as

$$\begin{aligned} f_{r:n}(x) &= \frac{n!}{(r-1)!\,(n-r)!} f(x) \left[F(x) \right]^{r-1} \left[1 - F(x) \right]^{n-r} \\ &= \frac{n!}{(r-1)!\,(n-r)!} f(x) \sum_{j=1}^{n-r} \binom{n-r}{j} \left[F(x) \right]^{j+r-1} \\ &= \frac{n!}{(r-1)!\,(n-r)!} 2\pi^{-1} \alpha \lambda^{\alpha} x^{-(\alpha+1)} \left[1 + \left(\frac{\lambda}{x} \right)^{2\alpha} \right]^{-1} \sum_{j=1}^{n-r} \binom{n-r}{j} \times \\ &\left[1 - 2\pi^{-1} \tan^{-1} \left[\left(\frac{\lambda}{x} \right)^{\alpha} \right] \right]^{j+r-1} \end{aligned}$$

Hence the PDF of r^{th} order statistic can be expressed as

$$f_{r:n}(x) = C\sum_{k=0}^{\infty}\sum_{j=1}^{n-r}\omega_{jk}\left[\tan^{-1}\left(\frac{\lambda}{x}\right)\right]^{\alpha k} \tag{26}$$

where $C = \frac{n!}{(r-1)!(n-r)!}2\pi^{-1}\alpha\lambda^{\alpha}x^{-(\alpha+1)}\left[1+\left(\frac{\lambda}{x}\right)^{2\alpha}\right]^{-1}$ and $\omega_{jk} = (-1)^k\left(2\pi^{-1}\right)^k\begin{pmatrix} n-r \\ j \end{pmatrix}\begin{pmatrix} j+r-1 \\ k \end{pmatrix}$.

5. PARAMETER ESTIMATION OF IPC DISTRIBUTION

In this section, we talk about the maximum likelihood estimation (MLE) method and their asymptotic properties to get approximate confidence intervals based on MLEs. Let $\underset{\sim}{x} = (x_1, \ldots, x_n)$ be a random sample drawn from $IPC(\alpha,\ \lambda)$ distribution, and log-likelihood function $l(\alpha, \lambda/\underset{\sim}{x})$ can be written as,

$$l(\alpha, \lambda/\underset{\sim}{x}) = n\ln(2/\pi) + n\ln(\alpha) + n\alpha\ln(\lambda) - (\alpha+1)\sum_{i=1}^{n}\ln(x_i) - \sum_{i=1}^{n}\ln\left\{1+\left(\frac{\lambda}{x_i}\right)^{2\alpha}\right\} \tag{27}$$

Differentiating Equation (27) with respect to α and λ we get,

$$\frac{\partial l}{\partial\alpha} = \frac{n}{\alpha} + n\ln(\lambda) + \sum_{i=1}^{n}\ln(x_i) - 2\sum_{i=1}^{n}\frac{\ln(\lambda/x_i)(\lambda/x_i)^{2\alpha}}{1+(\lambda/x_i)^{2\alpha}} \tag{28}$$

$$\frac{\partial\ell}{\partial\lambda} = \frac{\alpha}{\lambda} - \frac{2\alpha(\lambda/x)^{2x}}{\lambda\left\{1+(\lambda/x)^{2x}\right\}} \tag{29}$$

By setting Equations (28) and (29) to zero and solving them simultaneously we get the maximum likelihood estimate $\hat{\alpha}$ *and* $\hat{\lambda}$ of the model parameters. For the parameters α *and* λ the $100(1-\tau)\%$ confidence intervals can be calculated as the customary asymptotic normality of the maximum likelihood estimators $\text{var}(\hat{\alpha})$ *and* $\text{var}(\hat{\lambda})$ estimated from the inverse of the matrix of second derivatives of the log-likelihood function Casella & Berger [5] locally at $\hat{\alpha}$ *and* $\hat{\lambda}$.

$$\frac{\partial^2\ell}{\partial\alpha^2} = -\frac{n}{\alpha^2} - 4\sum_{i=1}^{n}\frac{\left\{\ln(\lambda/x_i)\right\}^2(\lambda/x_i)^{2\alpha}}{\left[1+(\lambda/x_i)^{2\alpha}\right]^2}$$

$$\frac{\partial^2\ell}{\partial\lambda^2} = -\frac{n\alpha}{\lambda^2} + \frac{2\alpha}{\lambda^2}\sum_{i=1}^{n}\frac{(\lambda/x_i)^{2\alpha}\left\{1-2\alpha+(\lambda/x_i)^{2\alpha}\right\}}{\left[1+(\lambda/x_i)^{2\alpha}\right]^2}$$

$$\frac{\partial^2\ell}{\partial\alpha\partial\lambda} = \frac{n}{\lambda} + \frac{2}{\lambda}\sum_{i=1}^{n}\frac{(\lambda/x_i)^{2\alpha}\left\{1+2\alpha\ln(\lambda/x_i)+(\lambda/x_i)^{2\alpha}\right\}}{\left[1+(\lambda/x_i)^{2\alpha}\right]^2}$$

$$\frac{\partial^2\ell}{\partial\lambda\partial\alpha} = \frac{n}{\lambda} + \frac{2}{\lambda}\sum_{i=1}^{n}\frac{(\lambda/x_i)^{2\alpha}\left\{1+2\alpha\ln(\lambda/x_i)+(\lambda/x_i)^{2\alpha}\right\}}{\left[1+(\lambda/x_i)^{2\alpha}\right]^2}$$

Let $\underset{\sim}{\delta} = (\alpha, \lambda)$ represent the parameter vector and the corresponding MLE of $\underset{\sim}{\delta}$ as $\hat{\underset{\sim}{\delta}} = (\hat{\alpha}, \hat{\lambda})$, then, $\hat{\underset{\sim}{\delta}} - \underset{\sim}{\delta} \rightarrow N_2\left[0, \left(I\left(\underset{\sim}{\delta}\right)\right)^{-1}\right]$ where $I\left(\underset{\sim}{\delta}\right)$ is the matrix of Fisher's information (FIM) given by,

$$I\left(\underset{\sim}{\delta}\right) = -\begin{pmatrix} E\left(\frac{\partial^2 l}{\partial\alpha^2}\right) & E\left(\frac{\partial^2 l}{\partial\alpha\partial\lambda}\right) \\ E\left(\frac{\partial^2 l}{\partial\lambda\partial\alpha}\right) & E\left(\frac{\partial^2 l}{\partial\lambda^2}\right) \end{pmatrix}$$

Because we don't know $\underline{\delta}$, it is pointless for practical purposes that the MLEs have asymptotic variance $\left(I\left(\underline{\delta}\right)\right)^{-1}$. So, using the estimated values of the parameters, we approximate the asymptotic variance. The general procedure is to use the observed FIM $O\left(\hat{\underline{\delta}}\right)$ as an estimate of $I\left(\underline{\delta}\right)$ given by

$$O\left(\hat{\underline{\delta}}\right) = -\begin{pmatrix} \frac{\partial^2 l}{\partial \alpha^2} & \frac{\partial^2 l}{\partial \alpha \partial \lambda} \\ \frac{\partial^2 l}{\partial \lambda \partial \alpha} & \frac{\partial^2 l}{\partial \lambda^2} \end{pmatrix}_{|(\hat{\alpha},\hat{\lambda})} = -H\left(\underline{\delta}\right)_{|\left(\underline{\delta}=\hat{\underline{\delta}}\right)}$$

here H is the Hessian matrix. The observed information matrix is given maximum likelihood using the Newton-Raphson algorithm. Consequently, the variance-covariance matrix is calculated as follows:

$$\left[-H\left(\underline{\delta}\right)_{|\left(\underline{\delta}=\hat{\underline{\delta}}\right)}\right]^{-1} = \begin{pmatrix} \mathrm{var}(\hat{\alpha}) & \mathrm{cov}(\hat{\alpha},\hat{\lambda}) \\ \mathrm{cov}(\hat{\lambda},\hat{\alpha}) & \mathrm{var}(\hat{\lambda}) \end{pmatrix}$$

Confidence intervals for α and λ can be constructed using the asymptotic normality of MLEs, $\hat{\alpha} \pm Z_{\tau/2}\sqrt{\mathrm{var}(\hat{\alpha})}$ and $\hat{\lambda} \pm Z_{\tau/2}\sqrt{\mathrm{var}(\hat{\lambda})}$, where $Z_{\tau/2}$ is the upper percentile of standard normal variate.

6. SIMULATION STUDY

A simulation study is carried out to investigate the capability of the ML estimators for estimating the parameters of the model $IPC(\alpha,\ \lambda)$. Using the random deviate function defined in Equation (15) we have generated $N = 10000$ independent samples of different sizes $n = (50, 100, 200, 250, 500)$ for three different sets of the parameter values. The estimated value of the parameters $\hat{\alpha}$ *and* $\hat{\lambda}$, absolute average bias (Bias) and mean square errors (MSEs) of the ML estimators are reported in Table 2. From Table 2 we have observed that the ML estimators tend to the actual values, Biases are comes close to zero, and MSEs are decreased as we expected under asymptotic theory when the size of the samples is increased.

Table 2: *The estimated values, Biases, and MSEs are based on 10000 simulations of IPC distribution.*

	Actual values		MLEs		Bias		MSEs	
n	α	λ	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\lambda}$
50	0.5	0.25	0.5142	0.2731	0.0142	0.0231	0.0048	0.014
	0.75	0.5	0.7705	0.5185	0.0205	0.0185	0.0106	0.0203
	1	0.75	1.0274	0.7659	0.0274	0.0159	0.0189	0.0248
100	0.5	0.25	0.5066	0.2604	0.0066	0.0104	0.0022	0.0057
	0.75	0.5	0.7597	0.5086	0.0097	0.0086	0.0049	0.0095
	1	0.75	1.0122	0.7567	0.0122	0.0067	0.0088	0.0119
200	0.5	0.25	0.5033	0.2557	0.0033	0.0057	0.0011	0.0028
	0.75	0.5	0.7548	0.505	0.0048	0.005	0.0024	0.0045
	1	0.75	1.0065	0.7544	0.0065	0.0044	0.0042	0.0058
250	0.5	0.25	0.5031	0.2537	0.0031	0.0037	0.00083	0.0021
	0.75	0.5	0.7535	0.5037	0.0035	0.0037	0.0019	0.0037
	1	0.75	1.0053	0.7539	0.0053	0.0039	0.0033	0.0045
500	0.5	0.25	0.5016	0.2519	0.0016	0.0019	0.0004	0.001
	0.75	0.5	0.752	0.5018	0.002	0.0018	0.0009	0.0018
	1	0.75	1.0029	0.7515	0.0029	0.0015	0.0016	0.0023

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7. APPLICATIONS TO REAL DATASET

Using two real datasets, we have examined the applicability of IPC distribution in this section.

Data set 1

The data set below is from an accelerated life test of 59 conductors Lawless [16] where failure times are measured in hours and there are no censored observations.

"6.545, 9.289, 7.543, 6.956, 6.492, 5.459, 8.120, 4.706, 8.687, 2.997, 8.591, 6.129, 11.038, 5.381, 6.958, 4.288, 6.522, 4.137, 7.459, 7.495, 6.573, 6.538, 5.589, 6.087, 5.807, 6.725, 8.532, 9.663, 6.369, 7.024, 8.336, 9.218, 7.945, 6.869, 6.352, 4.700, 6.948, 9.254, 5.009, 7.489, 7.398, 6.033, 10.092, 7.496, 4.531, 7.974, 8.799, 7.683, 7.224, 7.365, 6.923, 5.640, 5.434, 7.937, 6.515, 6.476, 6.071, 10.491, 5.923".

Data set 2

The second data set of 72 observations represents the coating weight (gm/m2) of Iron Sheets by a chemical method on the top center side (TCS) Rao and Mbwambo [24].

"36.8, 47.2, 35.6, 36.7, 55.8, 58.7, 42.3, 37.8, 55.4, 45.2, 31.8, 48.3, 45.3, 48.5, 52.8, 45.4, 49.8, 48.2, 54.5 ,50.1 ,48.4, 44.2, 41.2, 47.2, 39.1, 40.7, 40.3, 41.2,30.4, 42.8, 38.9, 34.0, 33.2, 56.8, 52.6, 40.5, 40.6, 45.8, 58.9, 28.7, 37.3, 36.8, 40.2, 58.2, 59.2, 42.8, 46.3, 61.2, 58.4, 38.5, 34.2, 41.3, 42.6, 43.1 ,42.3, 54.2,44.9, 42.8, 47.1, 38.9, 42.8, 29.4, 32.7, 40.1, 33.2, 31.6, 36.2, 33.6, 32.9, 34.5, 33.7, 39.9"

7.1. Estimation of the Parameters of IPC distribution

By using the **maxLik()** function Henningsen and Toomet [13] in R programming software R Core Team [23], we have computed the MLEs (see McElreath [19]) by maximizing the likelihood function for IPC distribution along with some models with their standard errors (SE) and presented in Tables 3 and 4.

Table 3: *MLEs and SE (parenthesis) for the distributions under study (data set 1)*

Distribution	Estimated parameters				
	$\hat{k}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$
IPC	-	6.3604(0.7692)	-	6.8773(0.1997)	-
WPC	-	2.7753(2.2584)	2.1901(1.7000)	6.3793(0.7396)	-
BurrXII-PC	1.480(5.005)	4.547(4.182)	4.016(15.330)	-	7.361(8.356)
LHC	-	-	-	4.495(1.117)	18.374(4.324)
NLHC	-	-	-	4.448(2.530)	1.787 (1.062)
HC	-	6.8049(0.9046)	-	-	-
Cauchy	-	-	-	6.8049(0.9046)	-

Table 4: *MLEs and SE (parenthesis) for the distributions under study (data set 2)*

Distribution	Estimated parameters				
	$\hat{k}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$
IPC	-	7.5515(0.8957)	-	42.1573(1.1228)	-
WPC	-	3.368(1.901)	2.129(1.152)	40.00(2.521)	-
BurrIX-PC	0.865(0.3027)	1.4109(0.8936)	11.658(3.871)	-	27.3261(4.1769)
LHC	-	-	-	4.2178(0.4471)	106.0269(4.4036)
NLHC	-	-	-	114.9316(4.42782)	0.5731(0.071)
HC	-	42.323(4.194)	-	-	-
Cauchy	-	-	-	42.323(4.194)	-

7.2. Adequacy test of the IPC model

To evaluate the goodness-of-fit and adequacy of the proposed model we have considered some well-known models like three-parameter Weibull power Cauchy (WPC) by Tahir et al. [29], four-parameter BurrXII-power Cauchy (BurrXII-PC) by Bhatti et al. [4], two parameters Lindley half Cauchy (LHC) by Chaudhary and Kumar [6], new Lindley half Cauchy (NLHC) by Chaudhary and Kumar [7], single parameter half Cauchy (HC) and Cauchy distributions. For the adequacy test of the model we have compared the IPC model with underlying models using the criterions Akaike information criterion (AIC), Corrected Akaike Information criterion (CAIC), Bayesian information criterion (BIC), and Hannan-Quinn information criterion (HQIC) respectively, and calculate as

$$AIC = -2l(\hat{\alpha}, \hat{\lambda}) + 2d$$

$$BIC = -2l(\hat{\alpha}, \hat{\lambda}) + d\log(n)$$

$$CAIC = AIC + \frac{2d\,(d+1)}{n-d-1}$$

$$HQIC = -2l(\hat{\alpha}, \hat{\lambda}) + 2d\log[\log(n)]$$

where d is the number of parameters associated with the concern model and n is the sample size. The results are reported in Tables 5 and 6 and noticed that a two-parameter simple IPC model can perform as complex models with three and four parameters WPC and BurrXII-PC respectively and our model is better than LHC, NLHC, HC, and Cauchy models. To evaluate the fit attained by the IPC model we have also presented the quantile-quantile (Q-Q) plot and Kolmogorov-Smirnov (KS) plot (Figures 2 and 3) and also verified that the IPC model fits the real data under study very nicely.

Table 5: *Model selection statistics (data set 1)*

Model	AIC	BIC	CAIC	HQIC	LL
IPC	228.1826	232.3376	228.3968	229.8045	-112.091
WPC	228.3944	234.627	228.8308	230.8274	-111.197
BurrXII-PC	230.6105	238.9206	231.3512	233.8544	-111.305
LHC	344.665	348.8201	344.8793	346.287	-170.333
NLHC	345.139	349.2941	345.3533	346.761	-170.57
HC	366.3982	368.4757	366.4684	367.2092	-182.199
Cauchy	448.1896	450.2671	448.2597	449.0005	-223.095

Table 6: *Model selection statistics (data set 2)*

Model	AIC	BIC	CAIC	HQIC	LL
IPC	513.6304	518.1838	513.8043	515.4431	-254.815
WPC	513.1188	519.9488	513.4718	515.8379	-253.559
BurrIX-PC	515.0942	524.2009	515.6913	518.7196	-253.547
LHC	681.5617	686.115	681.7356	683.3744	-338.781
NLHC	689.2971	693.8504	689.471	691.1098	-342.649
HC	627.2789	629.5556	627.336	628.1852	-312.639
Cauchy	808.5337	810.8104	808.5909	809.4401	-403.267

7.3. Test of goodness of fit of the proposed model

Further, we have compared the proposed distribution with competitive distributions by computing the Kolmogorov-Smirnov (KS), the Cramer-Von Mises (A^2) and the Anderson-Darling (W)

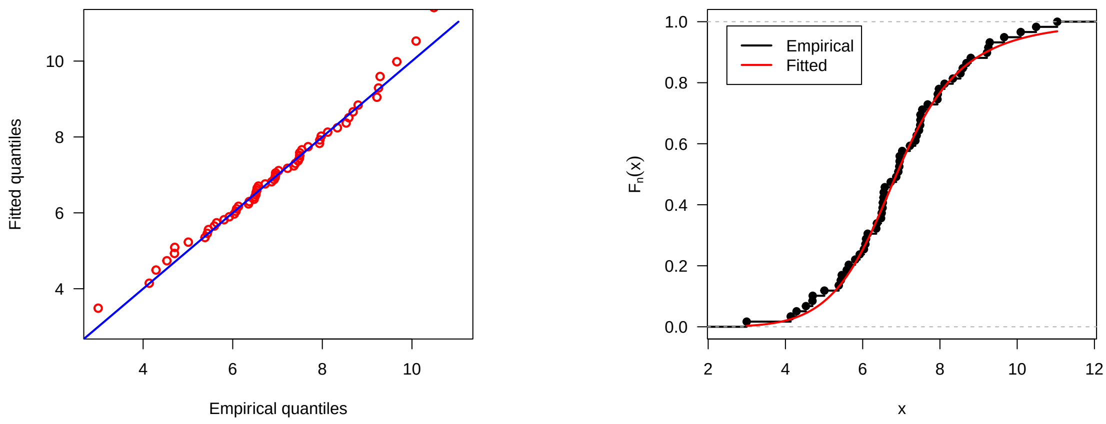

Figure 2: *Q-Q and KS plot (data set 1)*

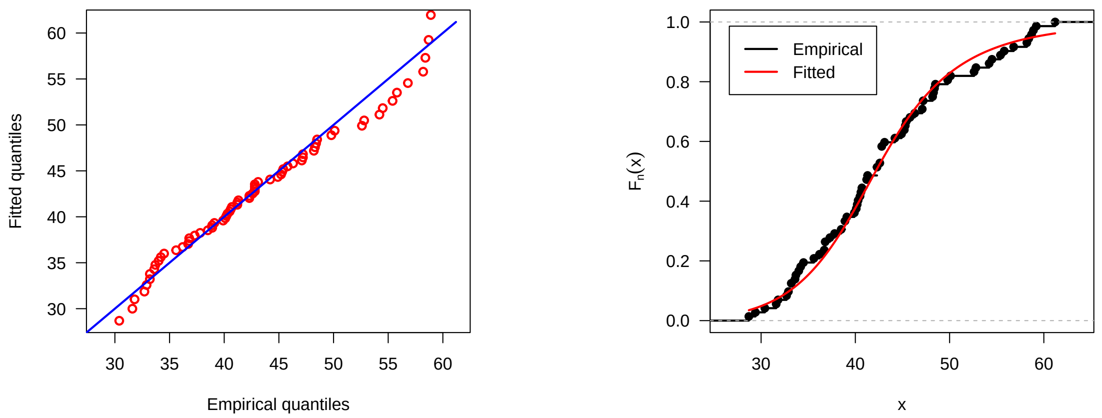

Figure 3: *Q-Q and KS plot (data set 2)*

statistics. These measures are frequently used to contrast non-nested models and show how well a certain CDF matches the empirical distribution of a particular data and computed as

$$KS = \max_{1 \leqslant j \leqslant n} \left(c_j - \frac{j-1}{n}, \frac{j}{n} - c_j \right)$$

$$W = -n - \frac{1}{n} \sum_{j=1}^{n} (2j-1) \left[\log c_j + \log \left(1 - c_{n+1-j} \right) \right]$$

$$A^2 = \frac{1}{12n} + \sum_{j=1}^{n} \left[\frac{(2j-1)}{2n} - c_j \right]^2$$

where $c_j = CDF(x_j)$; the x_j's are the ordered observations. The results are reported in Tables 7 and 8 and found that IPC model gets small test statistics values (except the value of AD for WPC and Burr XII-PC) and the highest p-value, hence the proposed model fits the data under study very well. This result is also verified by figures 4 and 5.

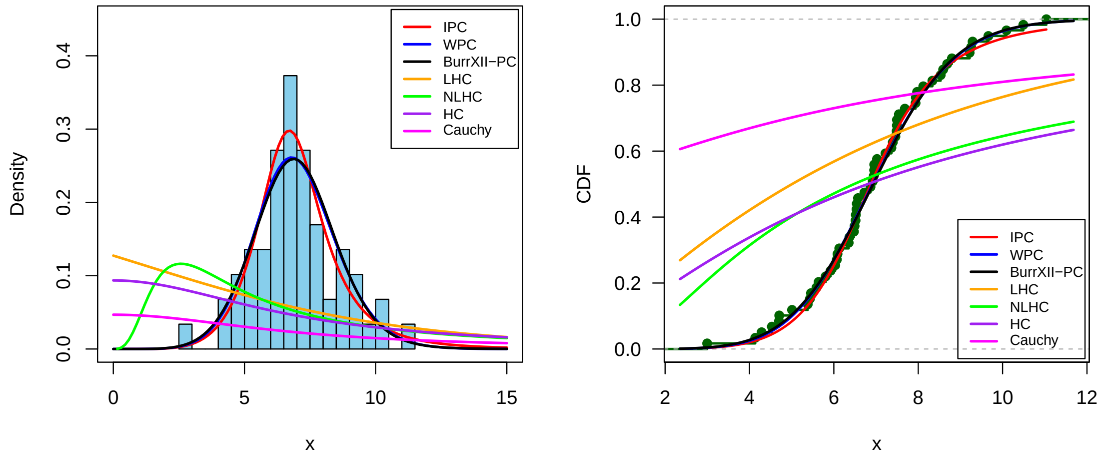

Figure 4: *PDF and CDF fit attained by competing models (data set 1)*

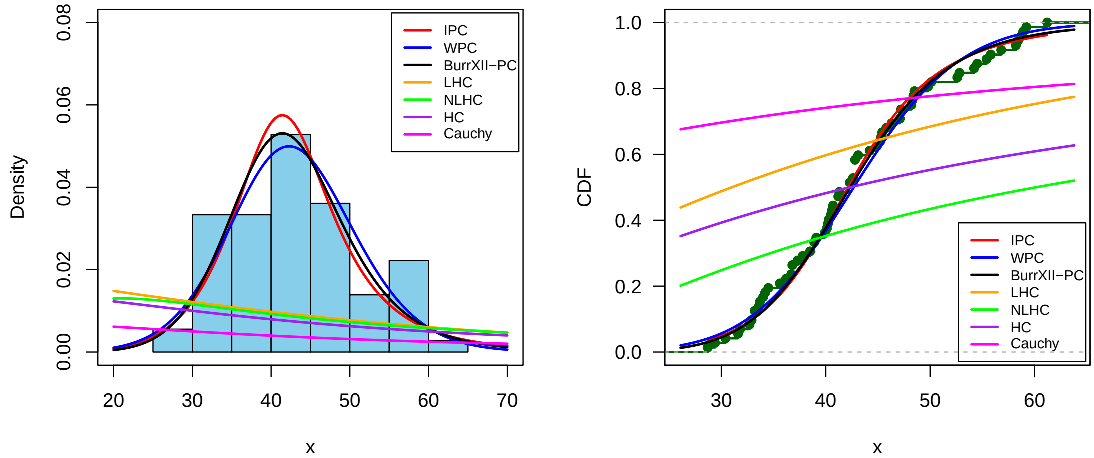

Figure 5: *PDF and CDF fit attained by competing models (data set 2)*

8. CONCLUSION

In this paper, we present the inverse power Cauchy (IPC) distribution, a novel class of continuous distributions. The plots of the PDF, and HRF, explicit expressions for CDF, quantile function, survival function, reverse HRF, skewness, kurtosis, moments, order statistics, and entropy have all been covered. The parameters of the suggested model were estimated using the maximum likelihood estimation (MLE) approach. For the purpose of illustrating the IPC distribution, two real data sets are taken into account. We come to the conclusion that the IPC distribution outperforms by comparing it to several other lifetime models the WPC, BurrXI-PC, LHC, NLHC, HC, and Cauchy distributions that are taken into account. We anticipate that the fields of applied statistics, probability theory, and survival analysis may choose to use this distribution.

Conflict of interest: The authors declare that there is no conflict of interest.

Table 7: *The goodness-of-fit test statistic with p-value (data set 1)*

Model	***KS(p-value)***	***W(p-value)***	A^2 ***(p-value)***
IPC	0.0480(0.9982)	0.1780(0.9953)	0.0199(0.9973)
WPC	0.05602(0.9876)	0.1541(0.9983)	0.0257(0.9887)
BurrXII-PC	0.0589(0.979)	0.1659(0.9971)	0.0282(0.9825)
LHC	0.4152(1.004e-09)	15.561(1.017e-05)	3.2415(7.108e-09)
NLHC	0.4348(1.156e-10)	15.321(1.017e-05)	3.214(8.475e-09)
HC	0.3517(4.956e-07)	14.226(1.017e-05)	2.8148(9.604e-08)
Cauchy	0.6569(7.772e-16)	33.963(1.017e-05)	7.4666(¡ 2.2e-16)

Table 8: *The goodness-of-fit test statistic with p-value (data set 2)*

Model	***KS(p-value)***	***W(p-value)***	A^2 ***(p-value)***
IPC	0.0623(0.9429)	0.4434(0.8044)	0.0453(0.9055)
WPC	0.0776(0.7792)	0.5364(0.7094)	0.0595(0.8189)
BurrIX-PC	0.0599(0.9581)	0.3916(0.8566)	0.0406(0.9315)
LHC	0.4719(2.387e-14)	20.943(8.333e-06)	4.4141(2.2e-16)
NLHC	0.4940(1.11e-15)	22.016(8.333e-06)	4.6737(¡ 2.2e-16)
HC	0.3852(1.054e-09)	18.986(8.333e-06)	3.8139(2.697e-10)
Cauchy	0.6897(¡ 2.2e-16)	42.534(8.333e-06)	9.3442(¡ 2.2e-16)

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