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Ratio Transformation Lomax Distribution with Applications

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Abstract

It has been noted in the literature on probability theory that the classical probability distributions do not adequately fit real-world data and do not exhibit non-monotonic hazard rate behavior. To overcome this limitation, researchers are focusing on the improvement of these distributions. In this manuscript, we have introduced a new probability model called Ratio Transformation Lomax Distribution (RTLD) as a new generalization of Lomax distribution. A thorough mathematical analysis of the new distribution is provided in closed form such as density function, distribution function, the r-th moment, survival function, hazard function, moment generating function, generalized entropy and also the order statistics. The new model's parameters are calculated using the method of maximum likelihood estimation. The proposed distribution's performance and adaptability is backed by three sets of real lifetime data as well as simulated data.

Keywords: Ratio Transformation Lomax distribution, hazard rate function, moments, maximum likelihood estimation

1. Introduction

In several literary contexts, the Lomax distribution has been employed. It has been frequently utilised for reliability modelling and life testing. But it does not provide an acceptable fit for several applications, particularly when the risk rates include bimodal or bathtub-shaped hazards. To overcome these limitations, researchers have created a variety of extensions and changes to the Lomax distribution to model various sorts of data. In the statistical literature, a variety of probability models are available to simulate various real-life random processes. Every year more distinct models with high degrees of flexibility are developed because no single distribution can fully represent all phenomena. As a result, researchers are focusing on creating new families of distribution and releasing a new variety of families of distribution in order to more thoroughly analyse real-world data in a variety of applications. Among these, some of the extensions of the Lomax distribution found in the literature are exponentiated Weibull-Lomax distribution proposed by [7], power Lomax distribution introduced by [8], a new extension of Lomax distribution formulated by [5], Marshall-0lkin alpha power Lomax distribution presented by [4]. The generalization of probability models has been very popular in recent years. There are variety of approaches for generalizing probability distributions, such as Alpha Power Transformation (APT) proposed by [12], exponentiation, mixture and Weighted Technique ,Power Transformation, and several others. Recently, [11] proposed a new method for generating distributions known as Ratio Transformation(RT) method. In this manuscript, our motive is the generalization of Lomax distribution to develop the new probability model called as Ratio Transformation Lomax Distribution (RTLD) by using Ratio Transformation (RT) method. The primary justification for making this generalization is that RTLD's hazard rate displays a variety of complex shapes, such as constant, increasing-decreasing, decreasing-increasing, etc., which overcomes Lomax distribution's drawbacks. Additionally, when considering a real-world data sets, the new distribution performs better than the baseline distribution and certain well-known competitive models. The remaining portions of the manuscript are structured as follows: In section 2, the Ratio Transformation (RT) method is discussed. In section 3, the RTLD's pdf and cdf are defined, and its sub-cases are covered. In section 4, the reliability analysis of the RTLD is presented. In sections 5, 6, 7, and 8 the statistical properties ,generating functions, order statistics, and information measure of the RTLD are respectively discussed. A very effective method is used to carry out the parameter estimation in section 9. Sections 10, 11 and 12, respectively, provide information on the simulation study, applicability of RTLD and its conclusion.

2. Ratio Transformation (RT) Method

The Ratio Transformation (RT) family of probability distributions, as proposed by [11] is highlighted in this section. Suppose the continuous random variable X has cdf $F(x)$. Therefore, the RT of $F(x)$ denoted by $F_{RT}(x)$ for $x \in \mathbb{R}$ and is defined by

$$F_{RT}(X) = \frac{F(x)}{1+\eta-\eta^{F(x)}} \; ; \qquad \eta > 0 \tag{1}$$

The pdf of the Ratio Transformation(RT) distribution is defined as follows

$$f_{RT}(X) = f(x)\frac{\left(1+\eta-\eta^{F(x)}\left(1-F(x)log\eta\right)\right)}{\left(1+\eta-\eta^{F(x)}\right)^2} \; ; \qquad \eta > 0 \tag{2}$$

3. Ratio Transformation Lomax Distribution (RTLD)

Suppose the random variable X has the Lomax distribution with shape parameter β and scale parameter θ respectively, then its probability density function(pdf) and Cumulative distribution function (cdf) are respectively given by

$$f(x;\beta,\theta) = \frac{\beta}{\theta}\left(1+\frac{x}{\theta}\right)^{-(\beta+1)} \; ; \; x>0, \beta>0, \theta>0 \tag{3}$$

$$F(x;\beta,\theta) = 1-\left(1+\frac{x}{\theta}\right)^{-\beta} \; ; \; x>0, \beta>0, \theta>0 \tag{4}$$

The RTLD is constructed from the Lomax distribution by using the (3) and (4)into (2) and(1) respectively. Therefore, the cdf of the RTLD is obtained as;

$$F_{RTLD}(x;\beta,\theta,\eta) = \frac{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}{1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}} \; ; \qquad ; \; x>0, \eta>0, \beta>0, \theta>0 \tag{5}$$

and the corresponding pdf is

$$f_{RTLD}(x,\beta,\theta,\eta) = \frac{\frac{\beta}{\theta}\left(1+\frac{x}{\theta}\right)^{-(\beta+1)}\left(1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}\left(1-(1-\left(1+\frac{x}{\theta}\right)^{-\beta})log\eta\right)\right)}{\left(1+\eta-\eta^{1-(1+\frac{x}{\theta})^{-\beta}}\right)^2} \; ; \; x>0, \eta>0, \beta>0, \theta>0 \tag{6}$$

Table 1: *Sub-Cases of RTLD*

η	θ	β	**Reduced Model**
-	1	-	Two parameter RTLD
1	-	-	Two parameter Lomax distribution
1	1	-	Beta Prime distribution
1	1	1	$F(2,2)$
1	$\frac{1}{\theta_q(q-1)}$	$\frac{(2-q)}{(q-1)}$	q-exponential distribution

Figure 1 and 2 have been displayed to provide a visual representation of the potential shapes of pdf and cdf of RTLD. Figure 3 represents the hazard rate plots of the RTLD for different parameter values.
Remark: For $\eta = 1$ in 6, RTLD becomes the two parametric Lomax distrbution. The important sub-cases of RTLD are presented in Table 1

4. Reliability analysis of the RTLD

This section primarily focuses on calculating the reliability (survival function), hazard rate (failure rate), reverse hazard function, cumulative hazard function, and mills ratio expressions for RTLD respectively.

4.1. Survival function

The survival function/reliability function is the complement of the cumulative distribution function and it is defined as the probability that a system will survive beyond a specified time. For the RTLD, the survival function denoted as $R_{RTLD}(x)$ is given by

$$R_{RTLD}(x) = 1 - F_{RTLD}(x;\beta,\theta,\eta) = \frac{\eta\left(1-\eta^{-\left(1+\frac{x}{\theta}\right)^{-\beta}}\right)+\left(1+\frac{x}{\theta}\right)^{-\beta}}{1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}} \tag{7}$$

4.2. Hazard Rate

Hazard rate also known as hazard function , force of mortality or failure rate. The expression for the hazard rate of RTLD is expressed as

$$h(x;\eta,\beta,\theta) = \frac{f_{RTLD}(x,\beta,\theta,\eta)}{R_{RTLD}(x,\beta,\theta,\eta)}$$

$$h(x;\eta,\beta,\lambda) = \frac{\frac{\beta}{\theta}\left(1+\frac{x}{\theta}\right)^{-(\beta+1)}\left(1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}\left(1-(1-\left(1+\frac{x}{\theta}\right)^{-\beta})log\eta\right)\right)\left(1+\eta-\eta^{1-(1+\frac{x}{\theta})^{-\beta}}\right)^{-1}}{\eta\left(1-\eta^{-\left(1+\frac{x}{\theta}\right)^{-\beta}}\right)+\left(1+\frac{x}{\theta}\right)^{-\beta}} \tag{8}$$

4.3. Reverse Hazard function

The reverse hazard function for the RTLD is expressed as

$$h_r(x;\eta,\beta,\theta) = \frac{f_{RTLD}(x,\beta,\theta,\eta)}{F_{RTLD}(x;\beta,\theta,\eta)}$$

Using equation (6) and(5) , the reverse hazard function for the RTLD is obtained as

$$h_r(x;\eta,\beta,\theta) = \frac{\frac{\beta}{\theta}\left(1+\frac{x}{\theta}\right)^{-(\beta+1)}\left(1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}\left(1-(1-\left(1+\frac{x}{\theta}\right)^{-\beta})log\eta\right)\right)\left(1+\eta-\eta^{1-(1+\frac{x}{\theta})^{-\beta}}\right)^{-1}}{1-\left(1+\frac{x}{\theta}\right)^{-\beta}} \tag{9}$$

4.4. Cumulative Hazard function

The Cumulative hazard function for the RTLD is obtained as

$$\Lambda_{RTLD}(x;\eta,\beta,\theta) = -\log R_{RTLD}(x)$$

$$\Lambda_{RTLD}(x;\eta,\beta,\theta) = \log\left\{\frac{1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}}{\eta\left(1-\eta^{-\left(1+\frac{x}{\theta}\right)^{-\beta}}\right)+\left(1+\frac{x}{\theta}\right)^{-\beta}}\right\} \tag{10}$$

4.5. Mills Ratio

The Mills ratio for the RTLD is obtained as

$$M.R = \frac{F_{RTLD}(x;\beta,\theta,\eta)}{R_{RTLD}(x)} = \left\{\frac{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}{\eta\left(1-\eta^{-\left(1+\frac{x}{\theta}\right)^{-\beta}}\right)+\left(1+\frac{x}{\theta}\right)^{-\beta}}\right\} \tag{11}$$

5. STATISTICAL PROPERTIES OF RTLD

This part focuses on discussing the related measures that are connected to the formulated model, including the raw moments, central moments, pearson's coefficients, coefficient of variation, and index of dispersion.

5.1. Raw Moments

The r^{th}moment of the RTLD about origin $\mu_r^{'}$ is given by

$$\mu_r^{'} = E(x^r) = \int_0^\infty x^r f_{RTLD}(x,\beta,\theta,\eta)dx$$

$$\mu_r^{'} = \int_0^\infty x^r \frac{\frac{\beta}{\theta}\left(1+\frac{x}{\theta}\right)^{-(\beta+1)}\left(1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}\left(1-(1-\left(1+\frac{x}{\theta}\right)^{-\beta})log\eta\right)\right)}{\left(1+\eta-\eta^{1-(1+\frac{x}{\theta})^{-\beta}}\right)^2}dx \tag{12}$$

Here, r^{th} moment of the RTLD is obtained by using the following series representations.

$$\eta^{-x} = \sum_{k=0}^{\infty}\frac{(-log\eta)^k x^k}{k!} \tag{13}$$

$$(1-x)^{-2} = \sum_{k=0}^{\infty}(k+1)x^k\ ; \qquad |x|<1, \tag{14}$$

$$(1-x)^{-1} = \sum_{k=0}^{\infty}x^k\ ; \qquad |x|<1, \tag{15}$$

By substituting $y = \left(1 + \frac{x}{\theta}\right)^{-\beta}$ in (12) and solving the integral further,we obtain r^{th}moment of the RTLD about origin $\mu_r^{'}$ as

$$\mu_r^{'} = \sum_{k,m=0}^{\infty} \frac{\beta\theta^r \eta^k (-log\eta)^m}{(1+\eta)^{k+2} m!} \left\{ k^m (1+\eta) A + (1+k)^{m+1} log\eta \left[A - C \right] \right\} \tag{16}$$

where

$A = B(r+1, \beta(m+1) - r)$ and $C = B(r+1, \beta(m+2) - r)$ represents the beta functions of second type. Using equation (16) and substituting $r = 1, 2, 3, 4$, the first four moments about origin of the RTLD are obtained as

$$\mu_1^{'} = \sum_{k,m=0}^{\infty} \frac{\beta\theta \eta^k (-log\eta)^m}{(1+\eta)^{k+2} m!} \left\{ k^m (1+\eta) A_1^{'} + (1+k)^{m+1} log\eta \left[A_1^{'} - C_1^{'} \right] \right\} \tag{17}$$

where

$$A_1^{'} = B(2, \beta(m+1) - 1)$$

$$C_1^{'} = B(2, \beta(m+2) - 1)$$

The equation (17) represents the mean of the RTLD.

$$\mu_2^{'} = \sum_{k,m=0}^{\infty} \frac{\beta\theta^2 \eta^k (-log\eta)^m}{(1+\eta)^{k+2} m!} \left\{ k^m (1+\eta) A_2^{'} + (1+k)^{m+1} log\eta \left[A_2^{'} - C_2^{'} \right] \right\} \tag{18}$$

where

$$A_2^{'} = B(3, \beta(m+1) - 2)$$

$$C_2^{'} = B(3, \beta(m+2) - 2)$$

$$\mu_3^{'} = \sum_{k,m=0}^{\infty} \frac{\beta\theta^3 \eta^k (-log\eta)^m}{(1+\eta)^{k+2} m!} \left\{ k^m (1+\eta) A_3^{'} + (1+k)^{m+1} log\eta \left[A_3^{'} - C_3^{'} \right] \right\} \tag{19}$$

where

$$A_3^{'} = B(4, \beta(m+1) - 3)$$

$$C_3^{'} = B(4, \beta(m+2) - 3)$$

$$\mu_4^{'} = \sum_{k,m=0}^{\infty} \frac{\beta\theta^4 \eta^k (-log\eta)^m}{(1+\eta)^{k+2} m!} \left\{ k^m (1+\eta) A_4^{'} + (1+k)^{m+1} log\eta \left[A_4^{'} - C_4^{'} \right] \right\} \tag{20}$$

where

$$A_4^{'} = B(5, \beta(m+1) - 4)$$

$$C_4^{'} = B(5, \beta(m+2) - 4)$$

5.2. Moments about Mean (Central Moments)

The moments about the mean also known as central moments of RTLD are obtained as

$$\mu_2 = \mu_2^{'} - (\mu_1^{'})^2$$

$$\mu_2 = \sum_{k,m=0}^{\infty} \frac{\beta\theta^2\eta^k(-log\eta)^m}{(1+\eta)^{k+2}m!} \left\{ k^m(1+\eta)A_2^{'} + (1+k)^{m+1}log\eta \left[A_2^{'} - C_2^{'} \right] \right\}$$
$$-\left\{ \sum_{k,m=0}^{\infty} \frac{\beta\theta\eta^k(-log\eta)^m}{(1+\eta)^{k+2}m!} \left\{ k^m(1+\eta)A_1^{'} + (1+k)^{m+1}log\eta \left[A_1^{'} - C_1^{'} \right] \right\} \right\}^2 \quad (21)$$

The equation (21) represents the variance of RTLD.

$$\mu_3 = \sum_{k,m=0}^{\infty} \frac{\beta\theta^3\eta^k(-log\eta)^m}{(1+\eta)^{k+2}m!} \left\{ k^m(1+\eta)A_3^{'} + (1+k)^{m+1}log\eta \left[A_3^{'} - C_3^{'} \right] \right\}$$
$$-3\left(\sum_{k,m=0}^{\infty} \frac{\beta\theta^2\eta^k(-log\eta)^m}{(1+\eta)^{k+2}m!} \left\{ k^m(1+\eta)A_2^{'} + (1+k)^{m+1}log\eta \left[A_2^{'} - C_2^{'} \right] \right\} \right)$$
$$\left(\sum_{k,m=0}^{\infty} \frac{\beta\theta\eta^k(-log\eta)^m}{(1+\eta)^{k+2}m!} \left\{ k^m(1+\eta)A_1^{'} + (1+k)^{m+1}log\eta \left[A_1^{'} - C_1^{'} \right] \right\} \right)$$
$$+2\left\{ \sum_{k,m=0}^{\infty} \frac{\beta\theta\eta^k(-log\eta)^m}{(1+\eta)^{k+2}m!} \left\{ k^m(1+\eta)A_1^{'} + (1+k)^{m+1}log\eta \left[A_1^{'} - C_1^{'} \right] \right\} \right\}^3 \quad (22)$$

$$\mu_4 = \sum_{k,m=0}^{\infty} \frac{\beta\theta^4\eta^k(-log\eta)^m}{(1+\eta)^{k+2}m!} \left\{ k^m(1+\eta)A_4^{'} + (1+k)^{m+1}log\eta \left[A_4^{'} - C_4^{'} \right] \right\}$$
$$-4\left(\sum_{k,m=0}^{\infty} \frac{\beta\theta^3\eta^k(-log\eta)^m}{(1+\eta)^{k+2}m!} \left\{ k^m(1+\eta)A_3^{'} + (1+k)^{m+1}log\eta \left[A_3^{'} - C_3^{'} \right] \right\} \right)$$
$$\left(\sum_{k,m=0}^{\infty} \frac{\beta\theta\eta^k(-log\eta)^m}{(1+\eta)^{k+2}m!} \left\{ k^m(1+\eta)A_1^{'} + (1+k)^{m+1}log\eta \left[A_1^{'} - C_1^{'} \right] \right\} \right)$$
$$+6\left(\sum_{k,m=0}^{\infty} \frac{\beta\theta^2\eta^k(-log\eta)^m}{(1+\eta)^{k+2}m!} \left\{ k^m(1+\eta)A_2^{'} + (1+k)^{m+1}log\eta \left[A_2^{'} - C_2^{'} \right] \right\} \right)$$
$$\left(\sum_{k,m=0}^{\infty} \frac{\beta\theta\eta^k(-log\eta)^m}{(1+\eta)^{k+2}m!} \left\{ k^m(1+\eta)A_1^{'} + (1+k)^{m+1}log\eta \left[A_1^{'} - C_1^{'} \right] \right\} \right)$$
$$-3\left\{ \sum_{k,m=0}^{\infty} \frac{\beta\theta\eta^k(-log\eta)^m}{(1+\eta)^{k+2}m!} \left\{ k^m(1+\eta)A_1^{'} + (1+k)^{m+1}log\eta \left[A_1^{'} - C_1^{'} \right] \right\} \right\}^4 \quad (23)$$

As a result, these equations may be used to calculate the skewness measure, kurtosis, coefficient of variation and index of dispersion for the RTLD.

5.3. Pearson's Coefficients

The following four coefficients can be obtained for the RTLD based upon the first four moments about the mean using the above section as:

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}$$

$$\gamma_1 = \sqrt{\beta_1}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2}$$

$$\gamma_2 = \beta_2 - 3$$

5.4. Coefficient of Variation

$$CV = \frac{\sqrt{\mu_2}}{\mu_1^{'}}$$

On using the equations (17) and(21), the coefficient of variation can be obtained for RTLD.

5.5. Index of Dispersion

The index of dispersion is defined as :

$$D = \frac{\mu_2}{\mu_1^{'}}$$

On using the equations (17) and(21), the index of dispersion can be obtained for RTLD.

6. Generating Functions RTLD

6.1. Moment Generating Function

Moment generating function (MGF) is used to represent all the moments of a distribution. The MGF for RTLD distribution is given in the following theorem.

Theorem 1. Let X follows the RTLD distribution, then the moment generating function, $M_X(t)$ is

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \sum_{k,m=0}^{\infty} \frac{\beta\theta^r \eta^k (-log\eta)^m}{(1+\eta)^{k+2} m!} \left\{ k^m (1+\eta) A + (1+k)^{m+1} log\eta \, [A-C] \right\} \tag{24}$$

Proof: The moment generating function of RTLD distribution is defined as

$$M_x(t) = \int_0^{\infty} e^{tx} f(x) dx$$

Using the series representation of e^{tx}, we have

$$\sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} x^r f(x; \eta, \beta, \theta) dx$$

Using equation (16) we obtain the moment generating function for RTLD as

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \sum_{k,m=0}^{\infty} \frac{\beta\theta^r \eta^k (-log\eta)^m}{(1+\eta)^{k+2} m!} \left\{ k^m (1+\eta) A + (1+k)^{m+1} log\eta \, [A-C] \right\} \tag{25}$$

6.2. Characteristic Function

The characteristic function for RTLD distribution is given in the following theorem.

Theorem 2. Let X follows the RTLD distribution, then the characteristic function, $\phi_X(t)$is

$$\phi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \sum_{k,m=0}^{\infty} \frac{\beta\theta^r \eta^k (-log\eta)^m}{(1+\eta)^{k+2} m!} \left\{ k^m (1+\eta) A + (1+k)^{m+1} log\eta \, [A-C] \right\} \tag{26}$$

Proof: The characteristic function for the RTLD can be obtained using the relation $\phi_X(t) = M_x(it)$

$$\phi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \sum_{k,m=0}^{\infty} \frac{\beta\theta^r \eta^k (-log\eta)^m}{(1+\eta)^{k+2} m!} \left\{ k^m (1+\eta) A + (1+k)^{m+1} log\eta \, [A-C] \right\} \tag{27}$$

6.3. Cumulant Function

The cumulant function for the RTLD can be obtained using the relation $k_x(t) = \log M_x(t)$

$$k_v(t) = \log \sum_{r=0}^{\infty} \frac{t^r}{r!} \sum_{k,m=0}^{\infty} \frac{\beta\theta^r \eta^k (-log\eta)^m}{(1+\eta)^{k+2} m!} \left\{ k^m (1+\eta) A + (1+k)^{m+1} log\eta \left[A - C \right] \right\} \tag{28}$$

7. Order Statistics of RTLD

The order statistics connected to the RTLD is devoted in this section. Let $X_{(t;n)}$ be the t^{th} order statistics with the random sample $x_{(1)}, x_{(2)}, x_{(3)}, ...x_{(m)}$ derived from the RTLD having the probability density function (pdf) $f(x;\eta,\beta,\theta)$ and cumulative distribution function (cdf) $F(x;\eta,\beta,\theta)$. Therefore, the probability density function (pdf) and cumulative distribution function (cdf) of $x_{(t;n)}$ say $f_{(t;n)}(x)$ and $F_{(t;n)}(x)$ respectively is defined as

$$f_{(t;n)}(x) = \frac{n!}{(t-1)!(n-t)!} \left[F(x;\eta,\beta,\theta) \right]^{t-1} \left[1 - F(x;\eta,\beta,\theta) \right]^{n-t} f(x;\eta,\beta,\theta) \tag{29}$$

$$F_{(t;n)}(x) = \sum_{j=t}^{n} \binom{n}{j} \left[F(x;\eta,\beta,\theta) \right]^{j} \left[1 - F(x;\eta,\beta,\theta) \right]^{n-j} \tag{30}$$

Using equation(5) and equation(6) in equation(29) and equation(30), the pdf and cdf of t^{th} ordered statistics for the RTLD is derived and is expressed as

$$f_{(t;n)}(x) = \frac{n!}{(t-1)!(n-t)!} \left[\frac{1 - \left(1+\frac{x}{\theta}\right)^{-\beta}}{1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}} \right]^{t-1} \left[\frac{\left(1+\frac{x}{\theta}\right)^{-\beta} + \eta - \eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}}{1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}} \right]^{n-t}$$
$$\left\{ \frac{\frac{\beta}{\theta}\left(1+\frac{x}{\theta}\right)^{-(\beta+1)} \left(1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}} \left(1 - (1 - \left(1+\frac{x}{\theta}\right)^{-\beta}) log\eta\right)\right)}{\left(1+\eta-\eta^{1-(1+\frac{x}{\theta})^{-\beta}}\right)^2} \right\} \tag{31}$$

$$F_{(t;n)}(x) = \sum_{j=t}^{n} \binom{n}{j} \left[\frac{1 - \left(1+\frac{x}{\theta}\right)^{-\beta}}{1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}} \right]^{j} \left[\frac{\left(1+\frac{x}{\theta}\right)^{-\beta} + \eta - \eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}}{1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}} \right]^{n-j} \tag{32}$$

In order to obtain the expression for pdf of smallest(minimum) order statistics $x_{(1)}$and the largest (maximum) order statistics $x_{(m)}$ of RTLD , we assume $t = 1$ and n respectively and is expressed in the form as

$$f_{(1;n)}(x) = n \left[\frac{\left(1+\frac{x}{\theta}\right)^{-\beta} + \eta - \eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}}{1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}} \right]^{n-1} \left\{ \frac{\frac{\beta}{\theta}\left(1+\frac{x}{\theta}\right)^{-(\beta+1)} \left(1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}} \left(1 - (1 - \left(1+\frac{x}{\theta}\right)^{-\beta}) log\eta\right)\right)}{\left(1+\eta-\eta^{1-(1+\frac{x}{\theta})^{-\beta}}\right)^2} \right\} \tag{33}$$

$$f_{(n;n)}(v) = n \left[\frac{1 - \left(1+\frac{x}{\theta}\right)^{-\beta}}{1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}} \right]^{n-1} \left\{ \frac{\frac{\beta}{\theta}\left(1+\frac{x}{\theta}\right)^{-(\beta+1)} \left(1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}} \left(1 - (1 - \left(1+\frac{x}{\theta}\right)^{-\beta}) log\eta\right)\right)}{\left(1+\eta-\eta^{1-(1+\frac{x}{\theta})^{-\beta}}\right)^2} \right\} \tag{34}$$

7.1. Median order statistics

Theorem 3. The Pdf of median order statistics for the RTLD is given as

$$f_{(n+1;n)}(x) = \frac{(2n+1)!}{(n)!(n)!} \left[\frac{1 - \left(1+\frac{x}{\theta}\right)^{-\beta}}{1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}} \right]^{n} \left[\frac{\left(1+\frac{x}{\theta}\right)^{-\beta} + \eta - \eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}}{1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}} \right]^{n}$$
$$\left\{ \frac{\frac{\beta}{\theta}\left(1+\frac{x}{\theta}\right)^{-(\beta+1)} \left(1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}} \left(1 - (1 - \left(1+\frac{x}{\theta}\right)^{-\beta}) log\eta\right)\right)}{\left(1+\eta-\eta^{1-(1+\frac{x}{\theta})^{-\beta}}\right)^2} \right\} \tag{35}$$

Proof The pdf of median order statistics, $x_{(n+1)}$ is defined as

$$f_{(n+1;n)}(x) = \frac{(2n+1)!}{n!n!}\left[F(x;\eta,\beta,\theta)\right]^n\left[1-F(v;\eta,\beta,\theta)\right]^n f(v;\eta,\beta,\theta)$$

$$f_{(n+1;n)}(x) = \frac{(2n+1)!}{(n)!(n)!}\left[\frac{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}{1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}}\right]^n\left[\frac{\left(1+\frac{x}{\theta}\right)^{-\beta}+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}}{1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}}\right]^n \left\{\frac{\frac{\beta}{\theta}\left(1+\frac{x}{\theta}\right)^{-(\beta+1)}\left(1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}\left(1-(1-\left(1+\frac{x}{\theta}\right)^{-\beta})log\eta\right)\right)}{\left(1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}\right)^2}\right\} \tag{36}$$

8. Information measure of RTLD

Entropy is a quantitative measures of the amount of uncertainty in a random variable. In this section we derive the expression for generalized entropy of RTLD.

Theorem 4. The generalized entropy for the RTLD is expressed as

$$I(\alpha) = \frac{1}{\alpha(\alpha-1)}\left\{\frac{\sum_{k,m=0}^{\infty}\frac{\beta\theta^{\alpha}\eta^k(-log\eta)^m}{(1+\eta)^{k+2}m!}\left\{k^m(1+\eta)A+(1+k)^{m+1}log\eta\left[A-C\right]\right\}}{\left\{\sum_{k,m=0}^{\infty}\frac{\beta\theta\eta^k(-log\eta)^m}{(1+\eta)^{k+2}m!}\left\{k^m(1+\eta)A_1'+(1+k)^{m+1}log\eta\left[A_1'-C_1'\right]\right\}\right\}^{\alpha}}-1\right\} \tag{37}$$

Proof: The generalized entropy is defined as

$$I(\alpha) = \frac{x_{\alpha}\mu^{-\alpha}-1}{\alpha(\alpha-1)}$$

where

$$x_{\alpha} = \int_{-\infty}^{\infty} x^{\alpha} f(x)dx$$

and μ represents mean. For RTLD, we have

$$x_{\alpha} = \sum_{k,m=0}^{\infty}\frac{\beta\theta^{\alpha}\eta^k(-log\eta)^m}{(1+\eta)^{k+2}m!}\left\{k^m(1+\eta)A+(1+k)^{m+1}log\eta\left[A-C\right]\right\} \tag{38}$$

where

$A = B(\alpha+1,\beta(m+1)-\alpha)$ and $C = B(\alpha+1,\beta(m+2)-\alpha)$ represents the beta functions of second type.

$$\mu^{-\alpha} = \left\{\sum_{k,m=0}^{\infty}\frac{\beta\theta\eta^k(-log\eta)^m}{(1+\eta)^{k+2}m!}\left\{k^m(1+\eta)A_1'+(1+k)^{m+1}log\eta\left[A_1'-C_1'\right]\right\}\right\}^{-\alpha} \tag{39}$$

where

$$A_1' = B(2,\beta(m+1)-1)$$

$$C_1' = B(2,\beta(m+2)-1)$$

Therefore, the expression for the generalized entropy of RTLD is obtained as

$$I(\alpha) = \frac{1}{\alpha(\alpha-1)}\left\{\frac{\sum_{k,m=0}^{\infty}\frac{\beta\theta^{\alpha}\eta^k(-log\eta)^m}{(1+\eta)^{k+2}m!}\left\{k^m(1+\eta)A+(1+k)^{m+1}log\eta\left[A-C\right]\right\}}{\left\{\sum_{k,m=0}^{\infty}\frac{\beta\theta\eta^k(-log\eta)^m}{(1+\eta)^{k+2}m!}\left\{k^m(1+\eta)A_1'+(1+k)^{m+1}log\eta\left[A_1'-C_1'\right]\right\}\right\}^{\alpha}}-1\right\} \tag{40}$$

9. Estimation of Parameters

This section is devoted to maximum likelihood estimation procedure for estimating unknown parameters η, β, θ of RTLD.

9.1. Maximum Likelihood Estimation(MLE)

Suppose $x_1, x_2, x_3, ...x_n$ be the random sample derived from the RTLD having the probability density function (pdf) $f(x; \eta, \beta, \theta)$. Therefore, for n observations , the logarithm of the likelihood function of RTLD is obtained as

$$l = nlog\beta + n\beta log\theta - (\beta+1)\sum_{i=1}^{n} log(x_i+\theta) - 2\sum_{i=1}^{n} log\left(1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}\right)$$
$$+\sum_{i=1}^{n} log\left[1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}\left(1-log\eta\ (1-\left(1+\frac{x}{\theta}\right)^{-\beta})\right)\right] \quad (41)$$

The MLEs of η, θ and β are obtained by partially differentiating (41) with respect to the corresponding parameters and equating to zero, we have

$$\frac{\partial l}{\partial \eta} = \sum_{i=1}^{n} \frac{1+(1-\left(1+\frac{x}{\theta}\right)^{-\beta})^2\eta^{-\left(1+\frac{x}{\theta}\right)^{-\beta}} log\eta}{1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}\left(1-log\eta(1-\left(1+\frac{x}{\theta}\right)^{-\beta})\right)} - 2\sum_{i=1}^{n}\frac{1-(1-\left(1+\frac{x}{\theta}\right)^{-\beta})\eta^{-\left(1+\frac{x}{\theta}\right)^{-\beta}}}{1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}} \quad (42)$$

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} + nlog\theta - \sum_{i=1}^{n} log(x_i+\theta) + 2\sum_{i=1}^{n}\frac{\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}} log\eta\ \left(1+\frac{x}{\theta}\right)^{-\beta} log\left(1+\frac{x}{\theta}\right)}{1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}}$$
$$-\sum_{i=1}^{n}\frac{1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}(1+\frac{x}{\theta})^{-\beta}log\eta(1+\frac{x}{\theta}) + \left(1-log\eta\ (1-\left(1+\frac{x}{\theta}\right)^{-\beta})\right)\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}} log\eta(1+\frac{x}{\theta})^{-\beta}log(1+\frac{x}{\theta})}{1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}\left(1-log\eta(1-\left(1+\frac{x}{\theta}\right)^{-\beta})\right)} \quad (43)$$

$$\frac{\partial l}{\partial \theta} = \frac{n\beta}{\theta} - (\beta+1)\sum_{i=1}^{n}\frac{1}{(x_i+\theta)} - 2\sum_{i=1}^{n}\frac{\beta\theta^{-2}x\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}} log\eta\ \left(1+\frac{x}{\theta}\right)^{-(\beta+1)}}{1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}}$$
$$-\sum_{i=1}^{n}\frac{\left[\beta\theta^{-2}xlog\eta\ \left(1+\frac{x}{\theta}\right)^{-(\beta+1)}\right]\left\{(1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta})} + \left(1-log\eta\ (1-\left(1+\frac{x}{\theta}\right)^{-\beta})\right)\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}\right\}}{1+\eta-\eta^{1-\left(1+\frac{x}{\theta}\right)^{-\beta}}\left(1-log\eta(1-\left(1+\frac{x}{\theta}\right)^{-\beta})\right)} \quad (44)$$

The above three non-linear equations (42),(43) and (44) are not in closed form.Therefore,we shall solve these equations with the help of R software.

10. SIMULATION ILLUSTRATION

In this section, the effectiveness of the (MLEs) of RTLD is explored. To demonstrate the behavior of MLEs in terms of random generating sample sizes of n= 50, 100, 200, 300, 400 a simulation research was conducted using R Software. The procedure is repeated 500 times. Different sets of parameter combinations are selected as (1,0.5,1)and (0.5,1,1) with reference to the usual order (θ, η, β). The average MLE values, bias, and related empirical mean squared errors (MSEs) were determined for each scenario. From Table 2 and Table 3 the simulation findings are shown. The estimates are stable and near to the genuine parameter values, as presented in Tables 2 and 3. In all circumstances, the MSE drops as the sample size increases.

Table 2: *Results of the simulation study for the RTLD model at parameter combination set as* $(\theta = 1, \eta = 0.5, \beta = 1)$.

Sample	MLE			BIAS			MSE		
n	$\hat{\theta}$	$\hat{\eta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\eta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\eta}$	$\hat{\beta}$
50	1.88737	0.81308	1.48624	0.88737	0.31308	0.48624	5.31104	4.05814	1.24
100	1.36960	0.77404	1.23155	0.369604	0.27404	0.23155	1.26655	3.01730	0.32662
200	1.17796	0.61684	1.12613	0.177968	0.11684	0.126133	0.456912	0.24202	0.085288
300	1.09238	0.59553	1.07560	0.09238	0.0955	0.07560	0.381143	0.08623	0.06454
400	1.02375	0.59153	1.02596	0.02375	0.09153	0.02596	0.23470	0.06871	0.033937

Table 3: *Results of the simulation study for the RTLD model at parameter combination set as* $(\theta = 0.5, \eta = 1, \beta = 1)$.

Sample	MLE			BIAS			MSE		
n	$\hat{\theta}$	$\hat{\eta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\eta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\eta}$	$\hat{\beta}$
50	0.82485	1.9607	1.36407	0.3248	0.9607	0.36407	1.07362	6.2609	0.59748
100	0.78055	1.64962	1.20104	0.28055	0.6496	0.2010	1.02573	4.14133	0.25056
200	0.60470	1.16408	1.07572	0.1047	0.1640	0.07572	0.16791	1.0627	0.04460
300	0.59327	1.14369	1.06681	0.0932	0.14369	0.06681	0.122244	0.50144	0.033350
400	0.56298	1.13550	1.0422	0.0629	0.135501	0.04229	0.0884	0.5013	0.01751

11. Application

This section concentrates on application of the proposed model to real life data sets. The significance and superiority of RTLD are highlighted in this part by the use of three real-life data sets. The MLEs of the model parameters are computed along with the corresponding Standard Error (SE) and goodness-of-fit statistics for these models are compared with other competing models. We compare the fits of the RTLD distribution with some competitive models which are listed in Table 4. To choose the best model among the compared models, performance comparing tools such as Akaike Information Criteria (AIC),Bayesian Information Criteria (BIC) and Akaike Information Criteria Corrected (AICc) are exploited. These Criterions choose the superior distribution as the one which is having the smallest value of AIC,BIC,and AICc. Furthermore, the Kolmogorov -Smirnov (KS)-distance and associated p- value is obtained to assess the goodness of fit. The superior probability model is considered the one which is having the least value of KS and maximum value of p- value.
The performance comparing tools are mentioned below:

- Akaike Information Criterion(AIC) is calculated as

$$\text{AIC} = -2\hat{l} + 2m$$

- Bayesian Information Criterion (BIC) is defined as

$$\text{BIC} = -2\hat{l} + m\ln(n)$$

- Akaike Information Criterion Corrected(AICC) is defined as.

$$\text{AICC} = \text{AIC} + \frac{2m(m+1)}{n-m-1}$$

where

$\hat{l}$ is the log-likelihood function of the model given the data.
m represents the number of parameters involved in the given model.
n is the sample size.

Table 5 and Table 6 displays the MLE's with corresponding Standard Error (SE) and the comparison of performance of RTLD with compared distributions for data set 1 ,which represents the COVID-19 vaccination rate from different countries. Table 7 and Table 8 presents the MLE's along with corresponding Standard

Table 4: *Competitive models of the RTLD model.*

Competitive models of the RTLD model	
Distribution(s)	**Author(s)**
(1) Sine Power Lomax (SPL)	[14]
(2) Length Biased Weighted Lomax Distribution(LBWLD)	[2]
(3) Topp-Leone Lomax (TLLo)	[15]
(4) Power Lomax (PL)	[16]
(5) Exponentiated Lomax (EL)	[1]
(6) Weibull Lomax(WL)	[17]
(7) Lomax (L)	[10]

Table 5: *MLE's of RTLD and compared distributions with corresponding standard error (given in parenthesis) for Covid -19 vaccination rate data set .*

Model	$\hat{\eta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$
RTLD	0.3446	2.7564	25.9981		
	(0.1854)	(2.3906)	(31.3471)	-	-
PL		0.9390		1.6917	6.7054
	-	(0.2478)	-	(1.6295)	(6.9138)
SPL		0.9402		0.7755	0.2037
	-	(0.2714)	-	(0.7075)	(0.2041)
EL		0.1952		1.3525	1.0322
	-	(0.2336)	-	(0.6882)	(0.3999)
TLLo		0.6762		0.1952	1.0322
	-	(0.3441)	-	(0.2336)	(0.3999)
L		5.5782	1.3924		
	-	(3.3975)	(0.5346)	-	-

Table 6: *Comparison of RTLD and compared distributions for Covid -19 vaccination rate data set*

Model	-2$\hat{l}$	AIC	AICC	BIC	K-S	p-value
RTLD	**284.6826**	**290.6826**	**291.2540**	**296.1685**	**0.0845**	**0.8697**
PL	285.6881	291.6881	292.2595	297.1740	0.1013	0.6938
SPL	285.8228	291.8229	292.3943	297.3088	0.1030	0.6751
EL	285.8365	291.8365	292.4079	297.3224	0.10699	0.6294
TLLo	285.8365	291.8365	292.4079	297.3224	0.10699	0.6294
L	288.7432	292.7432	293.0222	296.4004	0.108	0.6124

Error (SE) and comparison of performance of RTLD with compared distributions for data set 2. In addition to these, the result findings of the RTLD for the data set 3 , that represents the organic carbon content percentage in the soil of the district Ganderbal and are discussed in Table 9 and Table 10. The results shown in Table 6, Table 8 and 10 reveals that RTLD is having a minimum value of AIC ,BIC and AICC, and thus

Table 7: *MLE's of RTLD and compared distributions with corresponding standard error (given in parenthesis) for the dataset of the life of fatigue fracture of Kevlar 373/epoxy data .*

Model	$\hat{\eta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$
RTLD	20.9773	3.31514	0.635077		
	(17.388)	(0.892)	(0.550)	-	-
SPL		1.543		1.768	0.127
	-	(0.246)	-	(1.385)	(0.106)
PL		1.591		3.629	9.746
	-	(0.243)	-	(3.024)	(8.519)
TLLo		15.937		0.023	1.772
	-	(17.019)	-	(0.026)	(0.303)
EL		0.026		27.846	1.794
	-	(0.027)	-	(26.984)	(0.309)
WL	9288.5276	41544.7577	1.32698	19.90441	
	(17578.45506)	(548.674)	(0.11376)	(30.64844)	-
L		112,212.8	219,815.9		
	-	(11,863.8471)	(231.3384)	-	-

Table 8: *Comparison of RTLD and compared distributions for the data set of the life of fatigue fracture of Kevlar 373/epoxy data*

Model	-2$\hat{l}$	AIC	AICC	BIC	K-S	p-value
RTLD	**239.2362**	**245.2362**	**245.5695**	**252.2284**	**0.0669**	**0.863**
SPL	242.6924	248.6924	249.0257	255.6846	0.0829	0.6416
PL	243.0583	249.0583	249.3916	256.0505	0.0844	0.6204
TLLo	244.5824	250.5824	250.9157	257.5746	0.0907	0.53
EL	244.6087	250.6087	250.9421	257.6009	0.0906	0.52
WL	245.0592	253.0592	253.6226	262.3822	0.11003	0.2943
L	254.2288	258.2289	258.3931	262.8902	0.16631	0.02635

outperforms the base model of Lomax and a few well-known competitive models as shown in Table 4 for the provided data set 1, data set 2 and data set 3 (Given in Appendix A). The claim is further supported by Figures 4 and 5. Also the P-P plots of the RTLD model for all the given 3 data sets are shown in Figure 6 ,supports the results presented in Table 6, Table 8 and 10.
**Note:* The findings of the TLLO and EL models for data set 1 and data set 2 are nearly equal,because of their similar nature but slight numerical variations are seen without rounding.

12. Conclusion

In this manuscript, the main contribution is to propose a flexible generalization of Lomax distribution that can acts as a potential substitute for the base model in various situations. In this regard , we use the Ratio Transformation (RT) method and introduced a new model called as RTLD. Some of its key characteristics are discussed, and parameters are determined using a fairly potent estimation technique. The application of

Table 9: *MLE's of RTLD and compared distributions with corresponding standard error (given in parenthesis) for the dataset of the Organic carbon content percentage in the soil of district Ganderbal .*

Model	$\hat{\eta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$
RTLD	18.0874	112511.8	24858.69		
	(5.0829)	(6957.718)	(241.6045)	-	-
SPL		2.2218		48.3872	0.01316
	-	(0.2901)	-	(60.3950)	(0.0164)
LBWLD	82006.2083				34926.7946
	(8393.8383)	-	-	-	(141.5032)
L		58010.4142	49436.1672		
	-	(10617.0983)	(227.7487)	-	-

Table 10: *Comparison of RTLD and compared distributions for the data set of the Organic carbon content percentage in the soil of district Ganderbal*

Model	**-2$\hat{l}$**	AIC	AICC	BIC	K-S	p-value
RTLD	**36.7457**	**42.7457**	**43.3457**	**48.0982**	**0.0787**	**0.948**
SPL	42.9822	48.9822	49.5822	54.3347	0.137	0.3808
LBWLD	56.3150	60.3150	60.60774	63.8834	0.22998	0.01904
L	73.91046	77.91046	78.20314	81.47884	0.30589	0.01

RTLD from a practical perspective is demonstrated through the incorporation of a three real life data sets. The goodness of fit measure is used to assess the effectiveness of the proposed model to other existing known models. The acquired findings are quite encouraging and demonstrate that the RTLD model outperforms the competing models for the provided data sets.

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Appendix A

Data set 1: The first data represents the COVID-19 vaccination rate from 46 different countries in southern Africa . The data has been previously analyzed by [3]. The data is as follows: 0.042, 0.205, 0.285, 0.319, 0.464, 0.550, 0.889, 0.895, 0.939, 0.986, 1.000, 1.088, 1.212, 1.244, 1.450, 1.593, 1.844, 2.039, 2.157, 2.167, 2.334, 2.440, 2.657, 3.685, 3.879, 4.493, 4.800, 4.944, 5.155, 5.674, 7.602, 10.004, 12.238, 12.520, 12.553, 13.063, 15.105, 15.229, 15.629, 15.848, 18.641, 18.940, 29.885, 58.162, 61.838, 72.286.
Data set 2 The data set represents the life of fatigue fracture of Kevlar 373/epoxy that are subject to constant pressure at the 90 % stress level until all had failed. For previous studies on the data sets, see, [9] and [6] The data are: 0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960.
Data set 3 The data set represents the organic carbon(%) content in the soil of district Ganderbal. For

previous studies on the data set, see, [13].The data set are: 0.99, 0.81, 0.57, 1.11, 0.97, 0.78, 0.85, 0.85, 0.91, 0.79, 0.66, 0.99, 0.94, 1.17, 1.06, 0.99, 0.84, 1.47, 1.14, 1.41, 0.2, 0.6, 0.03,0.12, 1.11, 0.25, 1.14, 0.63, 0.45, 0.76, 1.2, 1.08, 1.26, 1.08, 0.27, 0.15, 0.75, 0.33, 0.75, 0.63, 1.47, 1.21, 1.24, 1.48.

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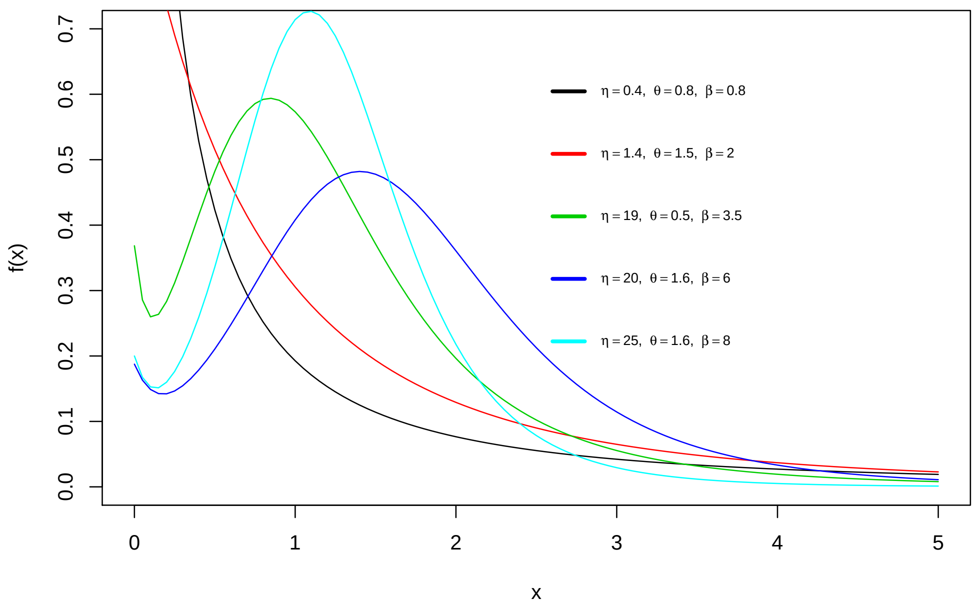
13. figures

Here in this section the figures related to the probability density functions, distribution functions, hazard rate and different plots for the three given data sets



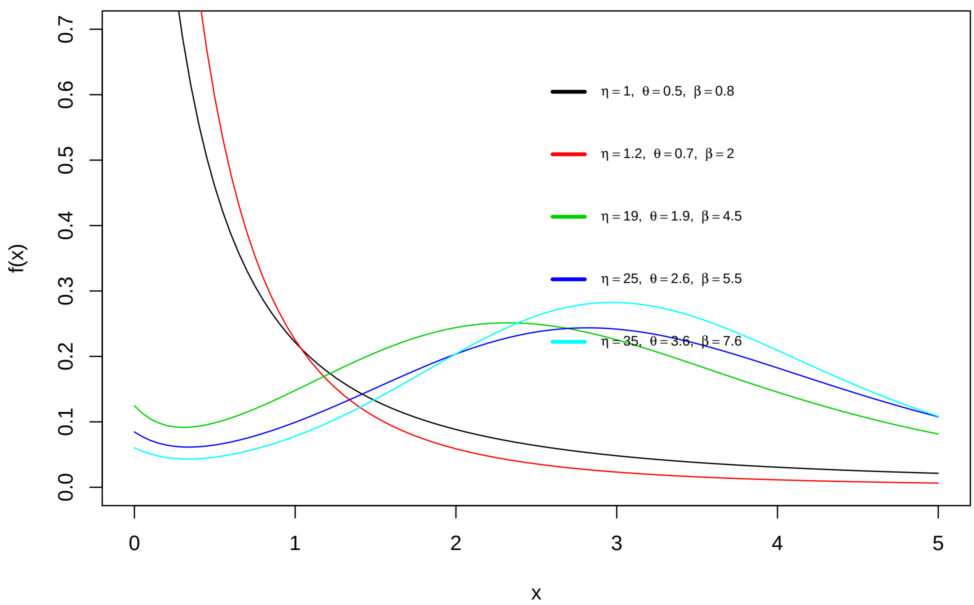


Figure 1: *Probability density plots of the RTLD for various values of* $\eta, \beta, \theta > 0$

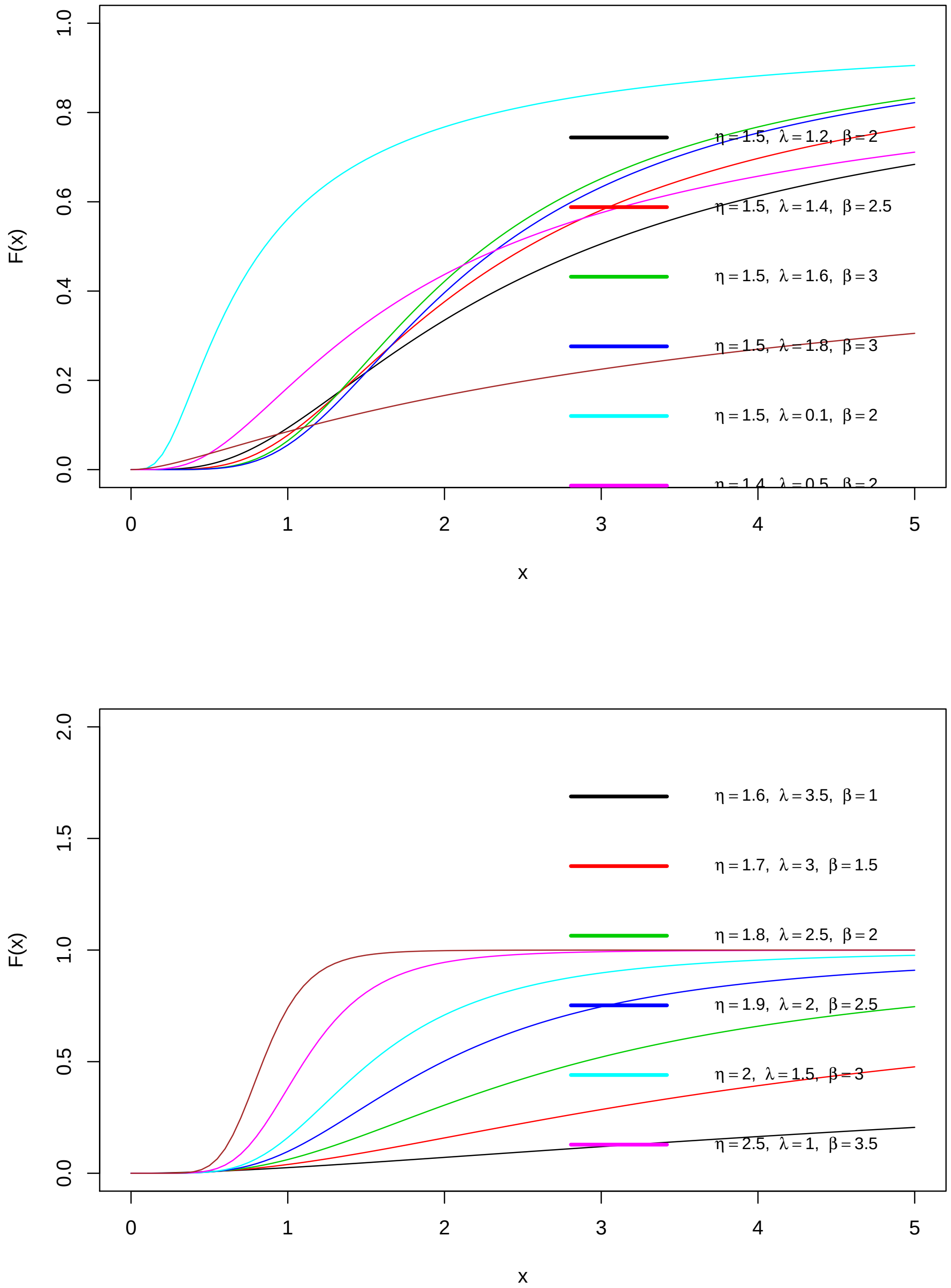


Figure 2: *Distribution function plots of the RTLD for various values of* $\eta, \beta, \theta > 0$

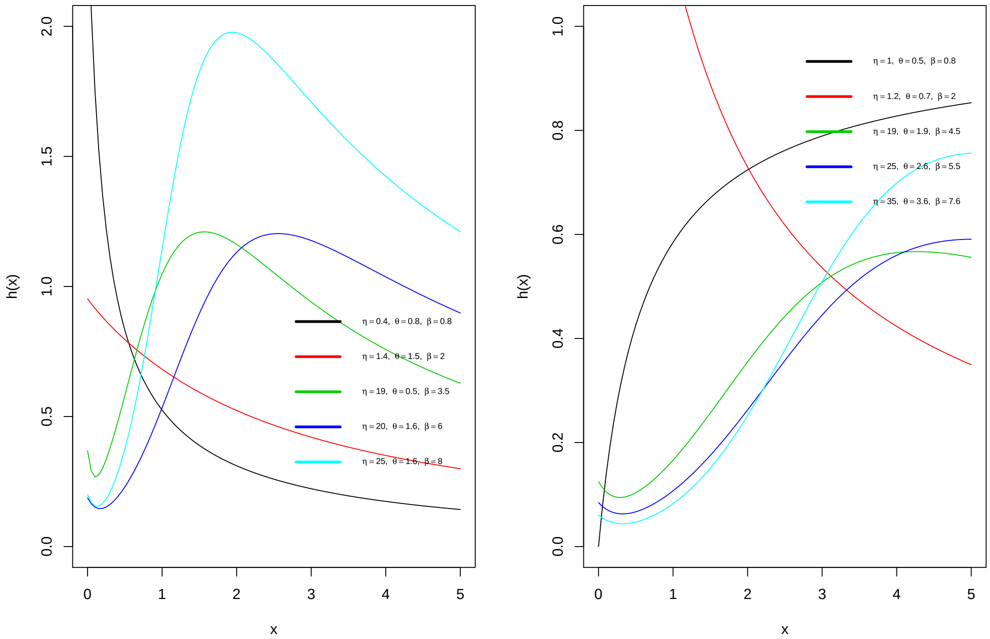


Figure 3: *Hazard rate plots of the RTLD for various values of* $\eta, \beta, \theta > 0$

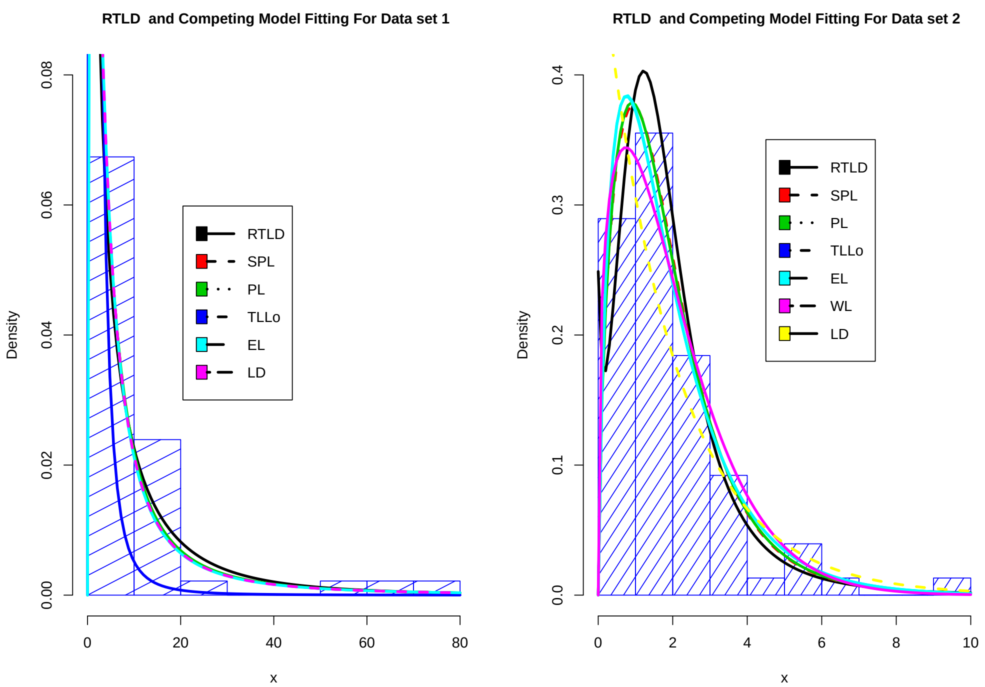


Figure 4: *Plot of the Fitted densities.*

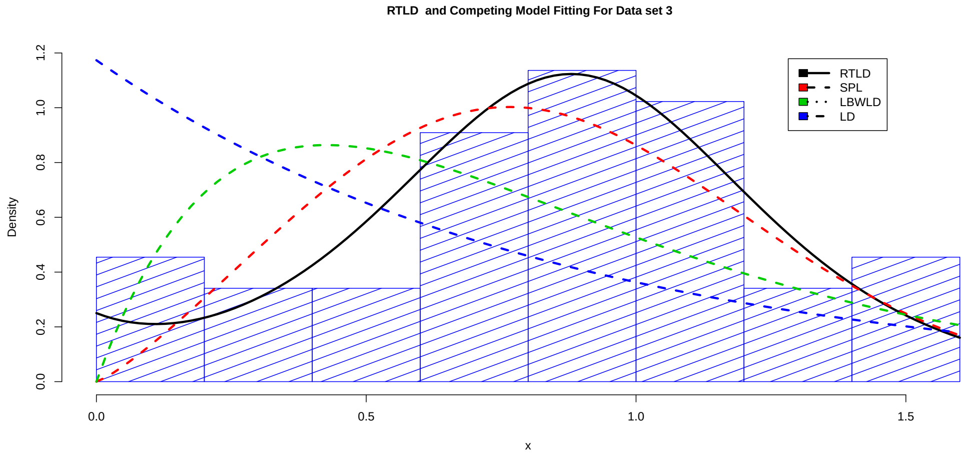


Figure 5: *Plot of the Fitted densities.*

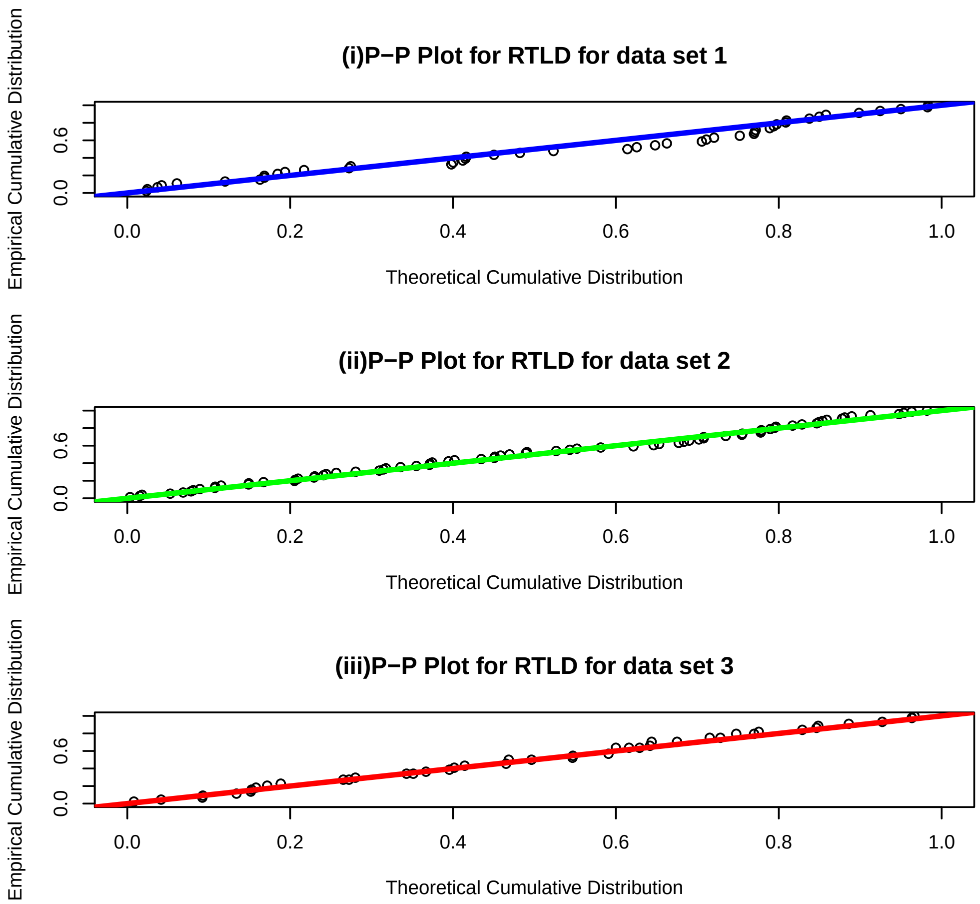


Figure 6: *P-P plot of the RTLD model for data set 1,data set 2 and data set 3 respectively.*