# E-Bayesian estimations for Chen distribution under Type II censoring with medical application

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**Abstract**

*The study focuses on the E-Bayesian estimation of a Type-II censored sample from the Chen distribution. Three distinct prior distributions for the hyper-parameters and three different loss functions are considered here for deriving the E-Bayes estimators of the scale parameter and hazard rate of above said distribution under Type-II censoring. Also derived analytical expressions for the E-MSE of the proposed estimators. Additionally, several features of the E-Bayesian estimators and E-MSEs are derived. This paper compares E-Bayesian estimation with traditional estimation methods like MLE and Bayesian. The applicability of the proposed estimators is demonstrated using a real data application. Furthermore, the credible intervals of the scale parameter estimators are also provided. The numerical analysis demonstrates that the proposed method is simpler and more feasible than traditional techniques.*

**Keywords:** Chen distribution; Type II censoring; Bayesian estimation; E-Bayesian estimation; E-MSE.

## 1 Introduction

Experiments in reliability and life-testing are done to learn more about the time of a significant event of interest. Examples of situations where the time of occurrence is significant are when a component fails, a disease abrogates, or a biological unit dies. For some reason, most investigations might not have complete information on the lifetimes or failure times of the experimental units. For example, in a medical trial, patients may withdraw from treatment, or the funding is only available for a specific period. In industrial trials, it is planned to remove accidentally damaged units before they fail to reduce testing time and costs. Censored data are those obtained from such experiments. The censoring schemes that appear the most frequently in the literature are Type-I and Type-II censoring. The experiment's endpoint is fixed, while the amount of failures reported is random in Type-I censoring. In contrast, the experiment's endpoint is random, and the number of failures is fixed in Type-II censoring. Type-II censoring is more cost-effective than Type-I censoring when comparing two censoring schemes. Inference under Type-II censoring for different parametric family distributions has been thoroughly studied in the literature. For more details, one can refer to [23], [7], [4], [9], [10].

A number of distributions with hazard rate functions that are constant, increasing, or decreasing in nature are discussed in the reliability literature. These distributions include generalized exponential, gamma, Weibull, and lognormal. These are the most often used models, and we use them to investigate various phenomena that occur in real life. However, these models do not work well with data sets showing bathtub-shaped hazard rates. In order to analyze real data with bathtub-shaped failure rates, several authors introduced probability models like modified Weibull by [18] and extended Weibull by [19], but these models are still unsuitable for producing accurate bathtub shape failure rates. Chen [8] showed a two-parameter lifespan distribution with a bathtub-shaped or increasing failure rate function. The hazard rate for this distribution initially declines, keeps the same, and increases. Chen distribution is a useful model for examining the lifespan of mechanical and electronic devices and humans. In addition to well-known models like lognormal and gamma, it can also be used to model positively skewed data. Due to the two parameters, closed-form confidence intervals for the shape parameter and joint confidence regions, this distribution is adaptable.

The MLEs of the Chen distribution parameter using samples that have been progressively Type-II censored are calculated by [24]. The MLE and Bayes estimators for the parameters of the Chen distribution using complete and censored samples are derived by [22], [2], [17], [16]. Kayal et al. [15] developed point and interval estimates of the multicomponent stress-strength reliability model of an s-out-of-j system using both classical and Bayesian techniques under the presumption that both the stress and the strength variables follow a Chen distribution. The literature on estimations of Chen distributions mentioned above mainly focuses on MLE or Bayesian techniques.

In addition to the Bayesian approach, the E-Bayesian estimation method, was developed in the literature. Originally, Han [13] addressed the definition of E-Bayesian estimation. Since the prior distribution of the hyperparameters is taken into account, the E-Bayesian approach is more reliable than Bayesian. The term "E-Bayesian estimation" refers to the expectation of the parameter's Bayesian estimate for all hyperparameters. In recent days so many works related to E-Bayesian inference of parameters and reliability functions of different distributions using complete and censored samples are discussed in the literature. For more details, one can refer to [1], [12],[3], [21], [20], [5]. The works mentioned above were a source of inspiration for further research on E-Bayes estimators for the scale parameter and hazard rate of Chen distributions under Type-II censoring schemes. The present work aims to develop E-Bayes estimators for the scale parameter and hazard rate of the Chen distribution using a Type-II censoring scheme and to calculate E-MSEs for the proposed estimators.

The organization of the remaining part of the work is as follows. In part 2, we go through the MLE of the scale parameter and hazard rate of the Chen distribution under the Type-II censoring scheme. Section 3 discusses the estimators' MSE as well as the Bayesian estimation of the scale parameter and hazard rate. Section 4 developed how to obtain E-Bayesian estimators of the scale parameter, hazard rate, and their associated E-MSEs. Section 5 of the article discusses the features that all of these estimators possess. In Section 6, it is discussed how well the estimators work with real data set. The final findings of the proposed study are provided in Section 7.

## 2 Maximum Likelihood Estimation

In this section, using a Type-II censoring technique, we derive the MLE of the scale parameter and hazard rate of the Chen distribution. The pdf, cdf and hazard function of the Chen distribution are respectively given by

$$f(x;\theta,\lambda) = \theta\lambda x^{\lambda-1}e^{x^{\lambda}+\theta(1-e^{x^{\lambda}})}, \quad x>0, \quad \lambda>0, \quad \theta>0, \tag{1}$$

$$F(x;\theta,\lambda) = 1-e^{\theta(1-e^{x^{\lambda}})}, \quad x>0, \quad \lambda>0, \quad \theta>0 \tag{2}$$

and

$$h(t) = \theta\lambda t^{\lambda-1}e^{t^{\lambda}}, \quad t>0. \tag{3}$$

With pdf and cdf defined in (1) and (2), respectively, assume that n distinct units selected from a population are put to the test and that the associated lifetimes are distributed identically. Let X = $(X_{(1)}, X_{(2)}, ..., X_{(r)})$ be the Type-II censored sample taken from (1) with r failure times. The likelihood function for Type-II censored sample is given by

$$\begin{aligned} L(\lambda,\theta|\underline{x}) &= \frac{n!}{(n-r)!}\pi_{i=1}^{r}f(x_{(i)})[1-F(x_{(r)})]^{n-r} \\ &= \frac{n!}{(n-r)!}\theta^{r}\nu(\lambda,\underline{x})e^{-\theta T}. \end{aligned} \tag{4}$$

where $\nu(\lambda, \underline{x}) = \lambda^r \pi_{i=1}^r x_i^{\lambda-1} e^{\sum_{i=1}^r x_i^\lambda}$, $\underline{x} = (x_{(1)}, x_{(2)}, ..., x_{(r)})$ and $T = \sum_{i=1}^r e^{x_i^\lambda} + (n-r)e^{x_r^\lambda} - n$. The log-likelihood function is provided by Chen distribution when $\lambda$ is known is given by

$$\ln(L) = \ln \theta^r + \ln e^{-\theta T}.$$

The normal equation is obtained by differentiating the log-likelihood with respect to the scale parameter $\theta$ and equating them to zero.

$$\frac{\partial \ln(L)}{\partial \theta} = 0 \implies \frac{r}{\theta} - T = 0.$$

By solving the above equation we can obtain MLE of the parameter $\theta$ as

$$\hat{\theta}_{ML} = \frac{r}{T}. \tag{5}$$

# 3 Bayesian Estimation

The Bayes estimators of the parameter $\theta$ are obtained in this section based on the squared error loss function (SELF), entropy loss function (ELF), and precautionary loss function (PLF). For developing the Bayesian estimation, we assume the gamma distribution as conjugate prior with probability density function

$$\pi(\theta|a,b) = \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta}, \quad \theta > 0, \quad a, b > 0, \tag{6}$$

where $a$ and $b$ are the hyper parameters. The posterior density of $\theta$ can be expressed as the following using the prior density (6) and likelihood function (4) as

$$q(\theta|\underline{x}) = \frac{(b+T)^{r+a}}{\Gamma(r+a)} \theta^{r+a-1} e^{-\theta(b+T)}, \quad \theta > 0. \tag{7}$$

We arrived at the Bayes estimators of $\theta$ and the hazard rate of (1) under three distinct loss functions in the subsequent theorem.

**Theorem 1.** For the Type-II censored sample $\underline{X} = (X_{(1)}, X_{(2)}, ..., X_{(r)})$ from (1) under SELF, ELF, and PLF together with the likelihood function (4) and prior distribution (6), we obtain the Bayes estimators of $\theta$ and hazard rate, respectively, provided as

i) Under SELF

$$\hat{\theta}_{B1} = \frac{r+a}{b+T}, \tag{8}$$

$$\hat{h}(t)_{B1} = \frac{r+a}{b+T} \lambda t^{\lambda-1} e^{t^\lambda}. \tag{9}$$

ii) Under ELF

$$\hat{\theta}_{B2} = \frac{r+a-1}{b+T}, \tag{10}$$

$$\hat{h}(t)_{B2} = \frac{r+a-1}{b+T} \lambda t^{\lambda-1} e^{t^\lambda}. \tag{11}$$

iii) Under PLF

$$\hat{\theta}_{B3} = \sqrt{\frac{(r+a+1)(r+a)}{(b+T)^2}}, \tag{12}$$

$$\hat{h}(t)_{B3} = \sqrt{\frac{(r+a+1)(r+a)}{(b+T)^2}} \lambda t^{\lambda-1} e^{t^\lambda}. \tag{13}$$

**Proof.**

i) The mean of (7) serves as the Bayes estimator of $\theta$ under SELF and is written as

$$\hat{\theta}_{B1} = E(\theta|\underline{x}) = \frac{r+a}{b+T}.$$

Likewise, the Bayes estimator of hazard rate is

$$\hat{h}(t)_{B1} = E\left(\theta\lambda t^{\lambda-1}e^{t^{\lambda}}|\underline{x}\right) = \frac{r+a}{b+T}\lambda t^{\lambda-1}e^{t^{\lambda}}.$$

ii) The following is the expression of a Bayes estimator of $\theta$ using ELF:

$$\hat{\theta}_{B2} = \left[E\left(\frac{1}{\theta}|\underline{x}\right)\right]^{-1} = \frac{r+a-1}{b+T}.$$

Likewise, the Bayes estimator of hazard rate is

$$\hat{h}(t)_{B2} = \left[E\left(\left(\theta\lambda t^{\lambda-1}e^{t^{\lambda}}\right)^{-1}|\underline{x}\right)\right]^{-1} = \frac{r+a-1}{b+T}\lambda t^{\lambda-1}e^{t^{\lambda}}.$$

iii) The following is the expression of a Bayes estimator of $\theta$ using PLF:

$$\hat{\theta}_{B3} = \sqrt{E(\theta^2|\underline{x})} = \sqrt{\frac{(r+a)(r+a+1)}{(b+T)^2}}.$$

Likewise, the Bayes estimator of hazard rate is

$$\hat{h}(t)_{B3} = \sqrt{E\left(\left(\theta\lambda t^{\lambda-1}e^{t^{\lambda}}\right)^2|\underline{x}\right)} = \sqrt{\frac{(r+a+1)(r+a)}{(b+T)^2}}\lambda t^{\lambda-1}e^{t^{\lambda}}.$$

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We determined the MSE of the Bayes estimators of $\theta$ and the hazard rate of (1) for three distinct loss functions in the subsequent theorem.

**Theorem 2.** For the Type-II censored sample $\underline{X} = (X_{(1)}, X_{(2)}, ..., X_{(r)})$ from (1), the MSE of the Bayes estimators of $\theta$ and the hazard rate under SELF, ELF, and PLF, respectively as

i) Under SELF

$$MSE(\hat{\theta}_{B1}) = \frac{r+a}{(b+T)^2}, \tag{14}$$

$$MSE(\hat{h}(t)_{B1}) = (\lambda t^{\lambda-1}e^{t^{\lambda}})^2\frac{r+a}{(b+T)^2}. \tag{15}$$

ii) Under ELF

$$MSE(\hat{\theta}_{B2}) = \frac{r+a+1}{(b+T)^2}, \tag{16}$$

$$MSE(\hat{h}(t)_{B2}) = (\lambda t^{\lambda-1}e^{t^{\lambda}})^2\frac{r+a-1}{(b+T)^2}. \tag{17}$$

iii) Under PLF

$$MSE(\hat{\theta}_{B3}) = \frac{2(r+a)}{(b+T)^2}[(r+a+1) - \sqrt{(r+a+1)(r+a)}], \tag{18}$$

$$MSE(\hat{h}(t)_{B3}) = (\lambda t^{\lambda-1}e^{t^{\lambda}})^2\frac{2(r+a)}{(b+T)^2}[(r+a+1) - \sqrt{(r+a+1)(r+a)}]. \tag{19}$$

**Proof.**

i) MSE of the Bayes estimator of $\theta$ under SELF is defined as

$$\begin{aligned} MSE(\hat{\theta}_{B1}(a,b)) &= E(\theta^2|x) - 2\hat{\theta}_{B1}(a,b)E(\theta|x) + [\hat{\theta}_{B1}(a,b)]^2 \\ &= \frac{r+a}{(b+T)^2}. \end{aligned}$$

Likewise, the MSE of the Bayes estimator of hazard rate under SELF is as follows:

$$\begin{aligned} MSE(\hat{h}(t)_{B1}) &= E[h(t)^2|x] - 2\hat{h}(t)_{B1}E[h(t)|x] + [\hat{h}(t)_{B1}]^2 \\ &= (\lambda t^{\lambda-1}e^{t^\lambda})^2 \frac{r+a}{(b+T)^2}. \end{aligned}$$

ii) MSE of Bayes estimator of $\theta$ using ELF is defined as

$$\begin{aligned} MSE(\hat{\theta}_{B2}(a,b)) &= E(\theta^2|x) - 2\hat{\theta}_{B2}(a,b)E(\theta|x) + [\hat{\theta}_{B2}(a,b)]^2 \\ &= \frac{r+a-1}{(b+T)^2}. \end{aligned}$$

Likewise, the MSE of the Bayes estimator of hazard rate under ELF is as follows:

$$\begin{aligned} MSE(\hat{h}(t)_{B2}) &= E[h(t)^2|x] - 2\hat{h}(t)_{B2}E[h(t)|x] + [\hat{h}(t)_{B2}]^2 \\ &= \frac{r+a-1}{b+T}\lambda t^{\lambda-1}e^{t^\lambda}. \end{aligned}$$

iii) MSE of the Bayes estimator of $\theta$ using PLF is defined as

$$\begin{aligned} \hat{\theta}_{B3} &= E(\theta^2|x) - 2\hat{\theta}_{B3}(a,b)E(\theta|x) + [\hat{\theta}_{B3}(a,b)]^2 \\ &= \frac{2(r+a)}{(b+T)^2}[(r+a+1) - \sqrt{(r+a+1)(r+a)}]. \end{aligned}$$

Likewise, the MSE of the Bayes estimator of hazard rate under PLF is as follows:

$$\begin{aligned} \hat{h}(t)_{B3} &= E[h(t)^2|x] - 2\hat{h}(t)_{B3}E[h(t)|x] + [\hat{h}(t)_{B3}]^2 \\ &= (\lambda t^{\lambda-1}e^{t^\lambda})^2 \frac{2(r+a)}{(b+T)^2}[(r+a+1) - \sqrt{(r+a+1)(r+a)}]. \end{aligned}$$

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# 4 E-Bayesian Estimation and its E-MSE

Han [13] is the author who first introduced E-Bayesian estimation in literature. Here we will obtain the E-Bayes estimator of the scale parameter and hazard rate of the Chen distribution under Type II censoring based on SELF, ELF and PLF and derive the properties exhibited by these estimators. Three different prior distributions of the hyper-parameters are considered to examine the impact of various prior distributions on the E-Bayesian estimate of $\theta$. According to [13], it is important to establish that the prior distribution of a and b, indicated by $\pi(\theta|a,b)$, is a decreasing function in $\theta$. Finding the first derivative of $\pi(\theta|a,b)$ with respect to $\theta$ and obtaining the result as

$$\frac{\partial \pi(\theta|a,b)}{\partial \theta} = \frac{b^a \theta^{a-2} e^{-b\theta}}{\Gamma a}[(a-1) - b\theta].$$

As a result, the function $\frac{\partial \pi(\theta|a,b)}{\partial \theta}$ <0 and $\pi(\theta|a,b)$ is a decreasing function of $\theta$ for 0< $a$ <1 and b>0. Given 0< $a$ <1, the gamma density function's tail will be thinner the larger b. The thinner-tailed prior distribution frequently affects the robustness of the Bayesian estimate, according to [6], which took the robustness of the Bayesian estimate into account. As a result, b must not exceed a specified upper bound c, where $c > 0$ is an unknown constant. As a result, the restriction of 0< $a$ <1 and 0< $b < c$ should be used when choosing the hyper-parameters a and b.

**Definition 4.1.** $\hat{\theta}_{EB} = \int\int_D \hat{\theta}_B(a,b)\pi(a,b)dadb = E[\hat{\theta}(a,b)]$ is referred to as the E-Bayesian estimate of $\theta$ when $\hat{\theta}_B(a,b)$ is continuous and is considered finite. D is the domain of a and b, $\hat{\theta}_B(a,b)$ is the Bayesian estimation of $\theta$ with hyper-parameters a and b, and $\pi(a,b)$ is the prior density of a and b over D.

The expectation of the Bayesian estimation of $\theta$ for the hyperparameters is what definition 4.1 defines as the E-Bayesian estimation of $\theta$. The definition of E-MSE of the E-Bayes estimators of $\theta$ presented by [11] is provided below.

**Definition 4.2.** $E - MSE(\hat{\theta}_{EB}) = \int\int_D MSE(\hat{\theta}_B(a,b))\pi(a,b)dadb = E[MSE(\hat{\theta}_B(a,b))]$ is referred to as the E-MSE of E-Bayes estimation of $\theta$ when $MSE(\hat{\theta}_B(a,b))$ is continuous and is considered finite. D is the domain of $a$ and b, $MSE(\hat{\theta}(a,b))$ is the MSE of the Bayesian estimation of $\theta$ with hyper-parameters $a$ and $b$, and $\pi(a,b)$ is the prior density of a and $b$ over D.

The E-Bayesian estimators of the parameter $\theta$ are obtained in this section using three different prior distributions for the hyper-parameters a and b. These prior distributions were chosen to demonstrate how the various prior distributions affected the E-Bayesian estimation of the parameter $\theta$. The prior distributions we used are given by

$$\pi_1(a,b) = \frac{2(c-b)}{c^2}, \quad 0 < a < 1 \quad 0 < b < c. \tag{20}$$

$$\pi_2(a,b) = \frac{1}{c}, \quad 0 < a < 1 \quad 0 < b < c. \tag{21}$$

$$\pi_3(a,b) = \frac{2b}{c^2}, \quad 0 < a < 1 \quad 0 < b < c. \tag{22}$$

These prior distributions are used to ensure that $\pi(\theta|a,b)$ is a decreasing function in $\theta$. The E-Bayes estimators of the parameter $\theta$ and the hazard rate function using $\pi_1(a,b)$ are derived in the subsequent theorem under various loss functions.

**Theorem 3.** We have the E-Bayes estimators of $\theta$ and hazard rate, which are provided, for the censored sample $\underline{X} = (X_{(1)}, X_{(2)}, ..., X_{(r)})$ from (1) using the prior distribution (20) under SELF, ELF, and PLF, respectively as

i) Under SELF

$$\hat{\theta}_{ES1} = \frac{2r+1}{c^2}\left\{(T+c)ln\left(\frac{T+c}{T}\right) - c\right\}, \tag{23}$$

$$\hat{h}(t)_{ES1} = \lambda t^{\lambda-1}e^{t^\lambda}\frac{2r+1}{c^2}\left\{(T+c)ln\left(\frac{T+c}{T}\right) - c\right\}. \tag{24}$$

ii) Under ELF

$$\hat{\theta}_{EE1} = \frac{2r-1}{c^2}\left\{(T+c)ln\left(\frac{T+c}{T}\right) - c\right\}, \tag{25}$$

$$\hat{h}(t)_{EE1} = \lambda t^{\lambda-1}e^{t^\lambda}\frac{2r-1}{c^2}\left\{(T+c)ln\left(\frac{T+c}{T}\right) - c\right\}. \tag{26}$$

iii) Under PLF

$$\hat{\theta}_{EP1} = \frac{2}{c^2}\left\{(T+c)ln\left(\frac{T+c}{T}\right) - c\right\}\int_0^1 \sqrt{(r+a)(r+a+1)}da, \tag{27}$$

$$\hat{h}(t)_{EP1} = \lambda t^{\lambda-1}e^{t^\lambda}\frac{2}{c^2}\left\{(T+c)ln\left(\frac{T+c}{T}\right) - c\right\}\int_0^1 \sqrt{(r+a)(r+a+1)}da. \tag{28}$$

**Proof.**

i) The following is the expression of the E-Bayes estimator of $\theta$ using SELF:

$$\hat{\theta}_{ES1} = \int_0^1 \int_0^c \hat{\theta}_{B1}(a,b)\pi_1(a,b)dadb$$

Using (8) and (20), the above equation simplifies to

$$\hat{\theta}_{ES1} = \frac{2r+1}{c^2}\left\{(T+c)ln\left(\frac{T+c}{T}\right) - c\right\}$$

Likewise, the E-Bayes estimator of hazard rate using SELF using (9) and (20) is as follows:

$$\begin{aligned}\hat{h}(t)_{ES1} &= \int\int_D \hat{h}_{B1}(a,b)\pi_1(a,b)dadb \\ &= \lambda t^{\lambda-1}e^{t^\lambda}\frac{2r+1}{c^2}\left\{(T+c)ln\left(\frac{T+c}{T}\right) - c\right\}.\end{aligned}$$

ii) The following is the expression of the E-Bayes estimator of $\theta$ using ELF:

$$\hat{\theta}_{EE1} = \int_0^1 \int_0^c \hat{\theta}_{B2}(a,b)\pi_1(a,b)dadb.$$

Using (10) and (20), the above equation simplifies to

$$\hat{\theta}_{EE1} = \frac{2r-1}{c^2}\left\{(T+c)ln\left(\frac{T+c}{T}\right) - c\right\}.$$

Likewise, the E-Bayes estimator of hazard rate using ELF using (11) and (20) is as follows:

$$\begin{aligned}\hat{h}(t)_{EE1} &= \int_0^1 \int_0^c \hat{h}_{B2}(a,b)\pi_1(a,b)dadb \\ &= \lambda t^{\lambda-1}e^{t^\lambda}\frac{2r-1}{c^2}\left\{(T+c)ln\left(\frac{T+c}{T}\right) - c\right\}.\end{aligned}$$

iii) The following is the expression of the E-Bayes estimator of $\theta$ using PLF:

$$\hat{\theta}_{EP1} = \int_0^1 \int_0^c \hat{\theta}_{B3}\pi_1(a,b)dadb.$$

Using (12) and (20), the above equation simplifies to

$$\hat{\theta}_{EP1} = \frac{2}{c^2}\left\{(T+c)ln\left(\frac{T+c}{T}\right) - c\right\}\int_0^1 \sqrt{(r+a)(r+a+1)}da.$$

Likewise, the E-Bayes estimator of hazard rate using PLF using (13) and (20) is as follows:

$$\hat{h}(t)_{EP1} = \lambda t^{\lambda-1}e^{t^\lambda}\frac{2}{c^2}\left\{(T+c)ln\left(\frac{T+c}{T}\right) - c\right\}\int_0^1 \sqrt{(r+a)(r+a+1)}da.$$

■

The E-Bayes estimators of the parameter $\theta$ and the hazard rate function using $\pi_2(a,b)$ are derived in the subsequent theorem under various loss functions.

**Theorem 4.** We have the E-Bayes estimators of $\theta$ and hazard rate, which are provided, for the censored sample $\underline{X} = (X_{(1)}, X_{(2)}, ..., X_{(r)})$ from (1) using the prior distribution (21) under SELF, ELF, and PLF, respectively as

i) Under SELF

$$\hat{\theta}_{ES2} = \frac{2r+1}{2c}ln\left(\frac{T+c}{T}\right), \tag{29}$$

$$\hat{h}(t)_{ES2} = \lambda t^{\lambda-1}e^{t^\lambda}\frac{2r+1}{2c}ln\left(\frac{T+c}{T}\right). \tag{30}$$

ii) Under ELF

$$\hat{\theta}_{EE2} = \frac{2r-1}{2c} ln\left(\frac{T+c}{T}\right), \tag{31}$$

$$\hat{h}(t)_{EE2} = \lambda t^{\lambda-1} e^{t^{\lambda}} \frac{2r-1}{2c} ln\left(\frac{T+c}{T}\right). \tag{32}$$

iii) Under PLF

$$\hat{\theta}_{EP2} = \frac{1}{c} ln\left(\frac{T+c}{T}\right) \int_0^1 \sqrt{(r+a)(r+a+1)}da, \tag{33}$$

$$\hat{h}(t)_{EP2} = \lambda t^{\lambda-1} e^{t^{\lambda}} \frac{1}{c} ln\left(\frac{T+c}{T}\right) \int_0^1 \sqrt{(r+a)(r+a+1)}da. \tag{34}$$

**Proof.** The proof is excluded since it is similar to that of Theorem 3. ■

The E-Bayes estimators of the parameter $\theta$ and the hazard rate function using $\pi_3(a,b)$ are derived in the subsequent theorem under various loss functions.

**Theorem 5.** We have the E-Bayes estimators of $\theta$ and hazard rate, which are provided, for the censored sample $\underline{X} = (X_{(1)}, X_{(2)}, ..., X_{(r)})$ from (1) using the prior distribution (22) under SELF, ELF, and PLF, respectively as

i) Under SELF

$$\hat{\theta}_{ES3} = \frac{2r+1}{c^2}\left\{c - Tln\left(\frac{T+c}{T}\right)\right\}, \tag{35}$$

$$\hat{h}(t)_{ES3} = \lambda t^{\lambda-1} e^{t^{\lambda}} \frac{2r+1}{c^2}\left\{c - Tln\left(\frac{T+c}{T}\right)\right\}. \tag{36}$$

ii) Under ELF

$$\hat{\theta}_{EE3} = \frac{2r-1}{c^2}\left\{c - Tln\left(\frac{T+c}{T}\right)\right\}, \tag{37}$$

$$\hat{h}(t)_{EE3} = \lambda t^{\lambda-1} e^{t^{\lambda}} \frac{2r-1}{c^2}\left\{c - Tln\left(\frac{T+c}{T}\right)\right\}. \tag{38}$$

iii) Under PLF

$$\hat{\theta}_{EP3} = \frac{2}{c^2}\left\{c - Tln\left(\frac{T+c}{T}\right)\right\} \int_0^1 \sqrt{(r+a)(r+a+1)}da, \tag{39}$$

$$\hat{h}(t)_{EP3} = \lambda t^{\lambda-1} e^{t^{\lambda}} \frac{2}{c^2}\left\{c - Tln\left(\frac{T+c}{T}\right)\right\} \int_0^1 \sqrt{(r+a)(r+a+1)}da. \tag{40}$$

**Proof.** The proof is excluded since it is similar to that of Theorem 3. ■

The E-MSE of the E-Bayes estimators of the parameter $\theta$ using different priors are derived in the subsequent theorem under various loss functions.

**Theorem 6.** The E-MSE of the E-Bayes estimators of $\theta$ using the priors $\pi_1(a,b)$, $\pi_2(a,b)$, and $\pi_3(a,b)$ under SELF, ELF, and PLF are presented, respectively, for the E-Bayes estimators of $\theta$ of Type-II censored sample $\underline{X} = (X_{(1)}, X_{(2)}, ..., X_{(r)})$ from (1) as

i) Under SELF

$$E - MSE(\hat{\theta}_{ES1}) = \frac{2r+1}{c^2}\left\{ln\left(\frac{T}{T+c}\right) + \frac{c}{T}\right\}, \tag{41}$$

$$E - MSE(\hat{\theta}_{ES2}) = \frac{2r+1}{2}\left\{\frac{1}{T(c+T)}\right\}, \tag{42}$$

$$E - MSE(\hat{\theta}_{ES3}) = \frac{2r+1}{c^2}\left\{ln\left(\frac{T+c}{T}\right) - \frac{c}{c+T}\right\}. \tag{43}$$

ii) Under ELF

$$E-MSE(\hat{\theta}_{EE1}) = \frac{2r+3}{c^2}\left\{ln\left(\frac{T}{T+c}\right)+\frac{c}{T}\right\}, \quad (44)$$

$$E-MSE(\hat{\theta}_{EE2}) = \frac{2r+3}{2}\left\{\frac{1}{T(c+T)}\right\}, \quad (45)$$

$$E-MSE(\hat{\theta}_{EE3}) = \frac{2r+3}{c^2}\left\{ln\left(\frac{T+c}{T}\right)-\frac{c}{c+T}\right\}. \quad (46)$$

iii) Under PLF

$$\begin{aligned} E-MSE(\hat{\theta}_{EP1}) &= \frac{4}{c^2}\left\{ln\left(\frac{T}{T+c}\right)+\frac{c}{T}\right\}\int_0^1 (r+a) \\ &\quad [(r+a+1)-\sqrt{(r+a)(r+a+1)}]da, \end{aligned} \quad (47)$$

$$E-MSE(\hat{\theta}_{EP2}) = \frac{2}{T(c+T)}\int_0^1 (r+a)[(r+a+1)-\sqrt{(r+a)(r+a+1)}]da, \quad (48)$$

$$\begin{aligned} E-MSE(\hat{\theta}_{EP3}) &= \frac{4}{c^2}\left[ln\left(\frac{T+c}{T}\right)-\frac{c}{c+T}\right]\int_0^1 (r+a) \\ &\quad [(r+a+1)-\sqrt{(r+a)(r+a+1)}]da. \end{aligned} \quad (49)$$

**Proof.**

i) Under SELF, the E-MSE of the estimator, $\hat{\theta}_{ES1}$ can be obtained from (14) and (20) by using the definition (4.2) and is given by

$$\begin{aligned} E-MSE(\hat{\theta}_{ES1}) &= \int\int_D MSE(\hat{\theta}_{B1}(a,b))\pi_1(a,b)dadb \\ &= \frac{2r+1}{c^2}\left\{ln\left(\frac{T}{T+c}\right)+\frac{c}{T}\right\}. \end{aligned}$$

Similarly, the E-MSE of $\hat{\theta}_{ES2}$ and $\hat{\theta}_{ES3}$ can be obtained from (14), (21) and (22) and by using the definition (4.2) and are given, respectively, by

$$\begin{aligned} E-MSE(\hat{\theta}_{ES2}) &= \int\int_D MSE(\hat{\theta}_{B1}(a,b))\pi_2(a,b)dadb \\ &= \frac{2r+1}{2}\left\{\frac{1}{T(c+T)}\right\}, \end{aligned}$$

and

$$\begin{aligned} E-MSE(\hat{\theta}_{ES3}) &= \int\int_D MSE(\hat{\theta}_{B1}(a,b))\pi_3(a,b)dadb \\ &= \frac{2r+1}{c^2}\left\{ln\left(\frac{T+c}{T}\right)-\frac{c}{c+T}\right\}. \end{aligned}$$

ii) Under ELF, the E-MSE of the estimator, $\hat{\theta}_{EE1}$ can be obtained from (15) and (20), and using the definition (4.2) and is given by

$$\begin{aligned} E-MSE(\hat{\theta}_{EE1}) &= \int\int_D MSE(\hat{\theta}_{B2}(a,b))\pi_1(a,b)dadb \\ &= \frac{2r+3}{c^2}\left\{ln\left(\frac{T}{T+c}\right)+\frac{c}{T}\right\}. \end{aligned}$$

Similarly, the E-MSE of $\hat{\theta}_{EE2}$ and $\hat{\theta}_{EE3}$ can be obtained from (15), (21) and (22) and by using the definition (4.2) and are given, respectively, by

$$E-MSE(\hat{\theta}_{EE2}) = \int\int_D MSE(\hat{\theta}_{B2}(a,b))\pi_2(a,b)dadb$$

$$= \frac{2r+3}{2}\left\{\frac{1}{T(c+T)}\right\},$$

and

$$\begin{aligned} E-MSE(\hat{\theta}_{EE3}) &= \int\int_D MSE(\hat{\theta}_{B2}(a,b))\pi_3(a,b)dadb \\ &= \frac{2r+3}{c^2}\left\{ln\left(\frac{T+c}{T}\right)-\frac{c}{c+T}\right\}. \end{aligned}$$

iii) Under PLF, the E-MSE of the estimator, $\hat{\theta}_{EP1}$ can be obtained from (16) and (20) by using the definition (4.2) and is given by

$$\begin{aligned} E-MSE(\hat{\theta}_{EP1}) &= \int\int_D MSE(\hat{\theta}_{B3}(a,b))\pi_1(a,b)dadb \\ &= \frac{4}{c^2}\left\{ln\left(\frac{T}{T+c}\right)+\frac{c}{T}\right\}\int_0^1(r+a) \\ &\quad [(r+a+1)-\sqrt{(r+a)(r+a+1)}]da. \end{aligned}$$

Similarly, the E-MSE of $\hat{\theta}_{EP2}$ and $\hat{\theta}_{EP3}$ can be obtained from (16), (20) and (21) and by using the definition (4.2) and are given, respectively, by

$$\begin{aligned} E-MSE(\hat{\theta}_{EP2}) &= \int\int_D MSE(\hat{\theta}_{B3}(a,b))\pi_2(a,b)dadb \\ &= \frac{2}{T(c+T)}\int_0^1(r+a)[(r+a+1)-\sqrt{(r+a)(r+a+1)}]da, \end{aligned}$$

and

$$\begin{aligned} E-MSE(\hat{\theta}_{EP3}) &= \int\int_D MSE(\hat{\theta}_{B3}(a,b))\pi_2(a,b)dadb \\ &= \frac{4}{c^2}\left[ln\left(\frac{T+c}{T}\right)-\frac{c}{c+T}\right] \\ &\quad \int_0^1(r+a)[(r+a+1)-\sqrt{(r+a)(r+a+1)}]da. \end{aligned}$$

■

The E-MSE of the E-Bayes estimators of the hazard rate $h(t)$ using different priors are derived in the subsequent theorem under various loss functions.

**Theorem 7.** The E-MSE of the E-Bayes estimators of $h(t)$ using the priors $\pi_1(a,b)$, $\pi_2(a,b)$, and $\pi_3(a,b)$ under SELF, ELF, and PLF are presented, respectively, for the E-Bayes estimators of $h(t)$ of Type-II censored sample $\underline{X}=(X_{(1)},X_{(2)},...,X_{(r)})$ from (1) as

i) Under SELF

$$E-MSE(\hat{h}(t)_{ES1})=(\lambda t^{\lambda-1}e^{t^\lambda})^2\frac{2r+1}{c^2}\left\{ln\left(\frac{T}{T+c}\right)+\frac{c}{T}\right\}, \quad (50)$$

$$E-MSE(\hat{h}(t)_{ES2})=(\lambda t^{\lambda-1}e^{t^\lambda})^2\frac{2r+1}{2}\left\{\frac{1}{T(c+T)}\right\}, \quad (51)$$

$$E-MSE(\hat{h}(t)_{ES3})=(\lambda t^{\lambda-1}e^{t^\lambda})^2\frac{2r+1}{c^2}\left\{ln\left(\frac{T+c}{T}\right)-\frac{c}{c+T}\right\}. \quad (52)$$

ii) Under ELF

$$E-MSE(\hat{h}(t)_{EE1})=(\lambda t^{\lambda-1}e^{t^\lambda})^2\frac{2r+3}{c^2}\left\{ln\left(\frac{T}{T+c}\right)+\frac{c}{T}\right\}, \quad (53)$$

$$E-MSE(\hat{h}(t)_{EE2})=(\lambda t^{\lambda-1}e^{t^\lambda})^2\frac{2r+3}{2}\left\{\frac{1}{T(c+T)}\right\}, \quad (54)$$

$$E-MSE(\hat{h}(t)_{EE3})=(\lambda t^{\lambda-1}e^{t^\lambda})^2\frac{2r+3}{c^2}\left\{ln\left(\frac{T+c}{T}\right)-\frac{c}{c+T}\right\}. \quad (55)$$

iii) Under PLF

$$E - MSE(\hat{h}(t)_{EP1}) = (\lambda t^{\lambda-1}e^{t^\lambda})^2 \frac{4}{c^2}\left\{ln\left(\frac{T}{T+c}\right)+\frac{c}{T}\right\}\int_0^1 (r+a) [(r+a+1)-\sqrt{(r+a)(r+a+1)}]da, \quad (56)$$

$$E - MSE(\hat{h}(t)_{EP2}) = (\lambda t^{\lambda-1}e^{t^\lambda})^2 \frac{2}{T(c+T)}\int_0^1 (r+a) [(r+a+1)-\sqrt{(r+a)(r+a+1)}]da, \quad (57)$$

$$E - MSE(\hat{h}(t)_{EP3}) = (\lambda t^{\lambda-1}e^{t^\lambda})^2 \frac{4}{c^2}\left[ln\left(\frac{T+c}{T}\right)-\frac{c}{c+T}\right]\int_0^1 (r+a) [(r+a+1)-\sqrt{(r+a)(r+a+1)}]da. \quad (58)$$

**Proof.** The proof is excluded since it is similar to that of Theorem 6. ■

# 5 Properties of E-Bayesian estimation

We now go over some important features of E-Bayesian estimators and their E-MSE. The relationship between E-Bayes estimators of $\theta$ under various loss functions is given in the subsequent theorem.

**Theorem 8.** Using the priors $\pi_1(a,b), \pi_2(a,b)$ and $\pi_3(a,b)$ under various loss functions, the relationship between E-Bayes estimators of $\theta$ when $0< c <$T is given as

a) under SELF

i) $\hat{\theta}_{ES3} < \hat{\theta}_{ES1} < \hat{\theta}_{ES2}$,

ii) $\lim_{T\to\infty}\hat{\theta}_{ES1} = \lim_{T\to\infty}\hat{\theta}_{ES2} = \lim_{T\to\infty}\hat{\theta}_{ES3}$,

b) under ELF

i) $\hat{\theta}_{EE3} < \hat{\theta}_{EE1} < \hat{\theta}_{EE2}$,

ii) $\lim_{T\to\infty}\hat{\theta}_{EE1} = \lim_{T\to\infty}\hat{\theta}_{EE2} = \lim_{T\to\infty}\hat{\theta}_{EE3}$,

c) under PLF

i) $\hat{\theta}_{EP3} < \hat{\theta}_{EP1} < \hat{\theta}_{EP2}$,

ii) $\lim_{T\to\infty}\hat{\theta}_{EP1} = \lim_{T\to\infty}\hat{\theta}_{EP2} = \lim_{T\to\infty}\hat{\theta}_{EP3}$.

**Proof.**

a) Under SELF

i) From (23) and (29), we have

$$\hat{\theta}_{ES1} - \hat{\theta}_{ES2} = \frac{2r+1}{c}\left[\left(\frac{1}{2}+\frac{c}{T}\right)\ln\left(\frac{T+c}{T}\right)-1\right]. \quad (59)$$

For $-1 < x < +1$, we have, $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - ... = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}$.
Let $x = \frac{c}{T}$, when $0 < c < T, 0 < \frac{c}{T} < 1$, we get

$$\begin{aligned}
&\left[\left(\frac{1}{2}+\frac{c}{T}\right)\ln\left(\frac{T+c}{T}\right)\right]-1\\
&\quad=\left(\frac{1}{2}+\frac{c}{T}\right)\left[\frac{c}{T}-\frac{1}{2}\left(\frac{c}{T}\right)^2+\frac{1}{3}\left(\frac{c}{T}\right)^3-\frac{1}{4}\left(\frac{c}{T}\right)^4+...\right]-1\\
&\quad=\left[\frac{1}{2}\left(\frac{c}{T}\right)-\frac{1}{4}\left(\frac{c}{T}\right)^2+\frac{1}{6}\left(\frac{c}{T}\right)^3-\frac{1}{8}\left(\frac{c}{T}\right)^4+...\right]+\left[\left(\frac{c}{T}\right)^2-\frac{1}{2}\left(\frac{c}{T}\right)^3\right.\\
&\qquad\left.+\frac{1}{3}\left(\frac{c}{T}\right)^4-\frac{1}{4}\left(\frac{c}{T}\right)^5+...\right]-1\\
&\quad=\frac{1}{2}\left(\frac{c}{T}\right)+\frac{3}{4}\left(\frac{c}{T}\right)^2\left[1-\frac{4}{9}\left(\frac{c}{T}\right)\right]+\frac{5}{24}\left(\frac{c}{T}\right)^4\left[1-\frac{18}{25}\left(\frac{c}{T}\right)\right]+...-1\\
&\quad<0.
\end{aligned} \tag{60}$$

So, we can say that,

$$\hat{\theta}_{ES1} < \hat{\theta}_{ES2}. \tag{61}$$

Now from (23) and (35), we have

$$\begin{aligned}
&\hat{\theta}_{ES1}-\hat{\theta}_{ES3}\\
&\quad=\frac{2r+1}{c^2}\left[(T+c)\ln\left(\frac{T+c}{T}\right)-c\right]-\frac{2r+1}{c^2}\left[c-T\ln\left(\frac{T+c}{T}\right)\right]\\
&\quad=\frac{2r+1}{c^2}\left[(2T+c)\ln\left(\frac{T+c}{T}\right)-2c\right]\\
&\quad=\frac{2r+1}{c}\left[\frac{(2T+c)}{c}\ln\left(\frac{T+c}{T}\right)-2\right].
\end{aligned} \tag{62}$$

$$\begin{aligned}
&\left[\left(\frac{2T}{c}+1\right)\ln\left(\frac{T+c}{T}\right)\right]-2\\
&\quad=\left[2-\left(\frac{c}{T}\right)+\left(\frac{2}{3}\right)\left(\frac{c}{T}\right)^2-\left(\frac{1}{2}\right)\left(\frac{c}{T}\right)^3+...\right]+\left[\left(\frac{c}{T}\right)-\frac{1}{2}\left(\frac{c}{T}\right)^2\right.\\
&\qquad\left.+\frac{1}{3}\left(\frac{c}{T}\right)^3-\frac{1}{4}\left(\frac{c}{T}\right)^4+...\right]-2\\
&\quad=\left(\frac{1}{6}\right)\left(\frac{c}{T}\right)^2\left[1-\left(\frac{c}{T}\right)\right]+\frac{3}{20}\left(\frac{c}{T}\right)^4\left[1-\frac{8}{9}\left(\frac{c}{T}\right)\right]+...\\
&\quad>0.
\end{aligned} \tag{63}$$

So we can say that,

$$\hat{\theta}_{ES1} > \hat{\theta}_{ES3}. \tag{64}$$

From (61) and (64),

$$\hat{\theta}_{ES3} < \hat{\theta}_{ES1} < \hat{\theta}_{ES2}. \tag{65}$$

ii) From (59) and (60) and by applying limit $T \to \infty$

$$\begin{aligned}
&\lim_{T\to\infty}(\hat{\theta}_{ES1}-\hat{\theta}_{ES2})\\
&\quad=\left(\frac{2r+1}{c}\right)\lim_{T\to\infty}\{\frac{1}{2}\left(\frac{c}{T}\right)+\frac{3}{4}\left(\frac{c}{T}\right)^2\left[1-\frac{4}{9}\left(\frac{c}{T}\right)\right]+\frac{5}{24}\left(\frac{c}{T}\right)^4\left[1-\frac{18}{25}\left(\frac{c}{T}\right)\right]\\
&\qquad+...-1\}\\
&\quad=0.
\end{aligned} \tag{66}$$

From (62) and (63) and by applying limit $T \to \infty$

$$\lim_{T\to\infty} (\hat{\theta}_{ES1} - \hat{\theta}_{ES3}) = \left(\frac{2r+1}{c}\right) \lim_{T\to\infty} \left\{ \left(\frac{1}{6}\right) \left(\frac{c}{T}\right)^2 \left[1 - \left(\frac{c}{T}\right)\right] + \frac{3}{20} \left(\frac{c}{T}\right)^4 \left[1 - \frac{8}{9} \left(\frac{c}{T}\right)\right] + \ldots \right\} = 0. \tag{67}$$

Hence from (66) and (67),

$$\lim_{T\to\infty} \hat{\theta}_{ES1} = \lim_{T\to\infty} \hat{\theta}_{ES2} = \lim_{T\to\infty} \hat{\theta}_{ES3}. \tag{68}$$

The remaining part of the proof is removed since it is comparable to that presented above. ■
The relationship between E-Bayes estimators of $h(t)$ under various loss functions is given in the subsequent theorem.

**Theorem 9.** Using the priors $\pi_1(a,b), \pi_2(a,b)$ and $\pi_3(a,b)$ under various loss functions, the relationship between E-Bayes estimators of $h(t)$ when $0< c <$T is given as

a) under SELF
   i) $\hat{h}(t)_{ES3} < \hat{h}(t)_{ES1} < \hat{h}(t)_{ES2}$,
   ii) $\lim_{T\to\infty} \hat{h}(t)_{ES1} = \lim_{T\to\infty} \hat{h}(t)_{ES2} = \lim_{T\to\infty} \hat{h}(t)_{ES3}$,

b) under ELF
   i) $\hat{h}(t)_{EE3} < \hat{h}(t)_{EE1} < \hat{h}(t)_{EE2}$,
   ii) $\lim_{T\to\infty} \hat{h}(t)_{EE1} = \lim_{T\to\infty} \hat{h}(t)_{EE2} = \lim_{T\to\infty} \hat{h}(t)_{EE3}$,

c) under PLF
   i) $\hat{h}(t)_{EP3} < \hat{h}(t)_{EP1} < \hat{h}(t)_{EP2}$,
   ii) $\lim_{T\to\infty} \hat{h}(t)_{EP1} = \lim_{T\to\infty} \hat{h}(t)_{EP2} = \lim_{T\to\infty} \hat{h}(t)_{EP3}$.

The proof is excluded since it is similar to that of Theorem 8.

The relationship between E-MSE of the E-Bayes estimators of $\theta$ under various loss functions is given in the subsequent theorem.

**Theorem 10.** Using the priors $\pi_1(a,b), \pi_2(a,b)$ and $\pi_3(a,b)$ under various loss functions, the relationship between E-MSE of the E-Bayes estimators of $\theta$ when $0< c <$T is given as

a) under SELF
   i) $E - MSE(\hat{\theta}_{ES3}) < E - MSE(\hat{\theta}_{ES1}) < E - MSE(\hat{\theta}_{ES2})$,
   ii) $\lim_{T\to\infty} E - MSE(\hat{\theta}_{ES1}) = \lim_{T\to\infty} E - MSE(\hat{\theta}_{ES2}) = \lim_{T\to\infty} E - MSE(\hat{\theta}_{ES3})$.

b) under ELF
   i) $E - MSE(\hat{\theta}_{EE3}) < E - MSE(\hat{\theta}_{EE1}) < E - MSE(\hat{\theta}_{EE2})$,
   ii) $\lim_{T\to\infty} E - MSE(\hat{\theta}_{EE1}) = \lim_{T\to\infty} E - MSE(\hat{\theta}_{EE2}) = \lim_{T\to\infty} E - MSE(\hat{\theta}_{EE3})$.

c) under PLF
   i) $E - MSE(\hat{\theta}_{EP3}) < E - MSE(\hat{\theta}_{EP1}) < E - MSE(\hat{\theta}_{EP2})$,
   ii) $\lim_{T\to\infty} E - MSE(\hat{\theta}_{EP1}) = \lim_{T\to\infty} E - MSE(\hat{\theta}_{EP2}) = \lim_{T\to\infty} E - MSE(\hat{\theta}_{EP3})$.

**Proof.**

a)

i) From (41) and (43), we have

$$\begin{aligned} E-MSE(\hat{\theta}_{ES3}) &- E-MSE(\hat{\theta}_{ES1}) \\ &= \frac{1}{c}\left[\ln\left(\frac{T+c}{T}\right)-\left(\frac{c}{c+T}\right)-\ln\left(\frac{T}{T+c}\right)-\frac{c}{T}\right] \\ &= \frac{1}{c}\left[2\ln\left(\frac{T+c}{T}\right)-\frac{c(2T+c)}{T(c+T)}\right] \\ &= \frac{2}{c}\ln\left(\frac{T+c}{T}\right)-\frac{(2T+c)}{T(c+T)} \\ &< 0. \end{aligned} \tag{69}$$

So, we can say that,

$$E-MSE(\hat{\theta}_{ES3}) < E-MSE(\hat{\theta}_{ES1}). \tag{70}$$

Now, from (41) and (42), we have

$$\begin{aligned} E-MSE(\hat{\theta}_{ES2}) &- E-MSE(\hat{\theta}_{ES1}) \\ &= \frac{c}{2}\frac{1}{T(c+T)}-\frac{1}{c}\left\{\ln\left(\frac{T+c}{T}\right)-\frac{c}{c+T}\right\} \\ &= \frac{c}{2T(c+T)}+\frac{1}{c+T}-\frac{1}{c}\ln\left(\frac{T+c}{T}\right) \\ &= \frac{c+2T}{(c+T)2T}-\frac{1}{c}\left[\frac{c}{T}-\frac{1}{2}\left(\frac{c}{T}\right)^2+\frac{1}{3}\left(\frac{c}{T}\right)^3-...\right] \\ &= \frac{c+2T}{(c+T)2T}-\frac{1}{T}+\frac{1}{2T}\left(\frac{c}{T}\right)\left[1-\frac{2}{3}\left(\frac{c}{T}\right)\right] \\ &\quad +\frac{1}{4T}\left(\frac{c}{T}\right)^3\left[1-\frac{4}{5}\left(\frac{c}{T}\right)\right]+... \\ &> 0. \end{aligned} \tag{71}$$

So, we can say that,

$$E-MSE(\hat{\theta}_{ES2}) > E-MSE(\hat{\theta}_{ES1}). \tag{72}$$

From (70) and (72),

$$E-MSE(\hat{\theta}_{ES3}) < E-MSE(\hat{\theta}_{ES1}) < E-MSE(\hat{\theta}_{ES2}). \tag{73}$$

ii) From (69) and by applying limit $T \to \infty$

$$\begin{aligned} \lim_{T\to\infty}(E &- MSE(\hat{\theta}_{ES3}) - E - MSE(\hat{\theta}_{ES1})) \\ &= \lim_{T\to\infty}\left[\frac{2}{c}\ln\left(\frac{T+c}{T}\right)-\frac{(2T+c)}{T(c+T)}\right] \\ &= \lim_{T\to\infty}\left[\frac{2}{T}-\frac{c}{T^2}+\frac{2c^2}{3T^3}-\frac{c^3}{2T^4}+...\right] \\ &\quad -\lim_{T\to\infty}\frac{2+\frac{c}{T}}{T(\frac{c}{T}+1)} \\ &= 0. \end{aligned} \tag{74}$$

From (71) and by applying limit $T \to \infty$

$$\begin{aligned} \lim_{T\to\infty}(E &- MSE(\hat{\theta}_{ES2}) - E - MSE(\hat{\theta}_{ES1})) \\ &= \lim_{T\to\infty}\frac{\frac{c}{T}+2}{2T(\frac{c}{T}+1)}-\frac{1}{T}+\frac{c}{2T^2}\left[1-\frac{2}{3}\left(\frac{c}{T}\right)\right]+ \\ &\quad \frac{c^3}{4T^4}\left[1-\frac{4}{5}\left(\frac{c}{T}\right)\right]+... \\ &= 0. \end{aligned} \tag{75}$$

Hence from (73) and (74),

$$\lim_{T\to\infty} E-MSE(\hat{\theta}_{ES1}) = \lim_{T\to\infty} E-MSE(\hat{\theta}_{ES2}) = \lim_{T\to\infty} E-MSE(\hat{\theta}_{ES3}). \tag{76}$$

The remaining part of the proof is removed since it is comparable to that presented above. ■
The relationship between E-MSE of the E-Bayes estimators of $h(t)$ under various loss functions is given in the subsequent theorem.

**Theorem 11.** Using the priors $\pi_1(a,b), \pi_2(a,b)$ and $\pi_3(a,b)$ under various loss functions, the relationship between E-MSE of the E-Bayes estimators of $h(t)$ when 0< $c$ <T is given as

a) under SELF

i) $E - MSE(\hat{h}(t)_{ES3}) < E - MSE(\hat{h}(t)_{ES1}) < E - MSE(\hat{h}(t)_{ES2})$,

ii) $\lim_{T\to\infty} E - MSE(\hat{h}(t)_{ES1}) = \lim_{T\to\infty} E - MSE(\hat{h}(t)_{ES2}) = \lim_{T\to\infty} E - MSE(\hat{h}(t)_{ES3})$.

b) under ELF

i) $E - MSE(\hat{h}(t)_{EE3}) < E - MSE(\hat{h}(t)_{EE1}) < E - MSE(\hat{h}(t)_{EE2})$,

ii) $\lim_{T\to\infty} E - MSE(\hat{h}(t)_{EE1}) = \lim_{T\to\infty} E - MSE(\hat{h}(t)_{EE2}) = \lim_{T\to\infty} E - MSE(\hat{h}(t)_{EE3})$.

c) under PLF

i) $E - MSE(\hat{h}(t)_{EP3}) < E - MSE(\hat{h}(t)_{EP1}) < E - MSE(\hat{h}(t)_{EP2})$,

ii) $\lim_{T\to\infty} E - MSE(\hat{h}(t)_{EP1}) = \lim_{T\to\infty} E - MSE(\hat{h}(t)_{EP2}) = \lim_{T\to\infty} E - MSE(\hat{h}(t)_{EP3})$.

**Table 1:** *The AE (first row), MSE (second row) and ACI for MLE, Bayesian and E-Bayesian estimates of $\theta$ for real data.*

| | n=148 | | | | |
|---|---|---|---|---|---|
| | r=30 | r=60 | r=90 | r=120 | ACI |
| $\hat{\theta}_{MLE}$ | 0.0140806<br>$9.58331 * 10^{-4}$ | 0.0213817<br>$5.65431 * 10^{-4}$ | 0.0302629<br>$2.45881 * 10^{-4}$ | 0.0346765<br>$1.07767 * 10^{-4}$ | |
| $\hat{\theta}_{B1}$ | 0.014312<br>$1.0999 * 10^{-5}$ | 0.021557<br>$9.3838 * 10^{-6}$ | 0.0304276<br>$1.11832 * 10^{-5}$ | 0.0348179<br>$1.03929 * 10^{-5}$ | (0.0113113, 0.0473139) |
| $\hat{\theta}_{B2}$ | 0.0138428<br>$1.13596 * 10^{-5}$ | 0.0212007<br>$9.5389 * 10^{-6}$ | 0.0300914<br>$1.13068 * 10^{-5}$ | 0.0345289<br>$1.04791 * 10^{-5}$ | (0.0111863, 0.0467911) |
| $\hat{\theta}_{B3}$ | 0.0145447<br>$1.10877 * 10^{-5}$ | 0.0217345<br>$9.42226 * 10^{-6}$ | 0.0305953<br>$1.12139 * 10^{-5}$ | 0.0349621<br>$1.04144 * 10^{-5}$ | (0.0113736, 0.0475745) |
| $\hat{\theta}_{ES1}$ | 0.0143109<br>$1.09967 * 10^{-5}$ | 0.0215561<br>$9.38285 * 10^{-6}$ | 0.0304265<br>$1.11823 * 10^{-5}$ | 0.0348168<br>$1.03922 * 10^{-5}$ | (0.0113114, 0.0473116) |
| $\hat{\theta}_{ES2}$ | 0.0143087<br>$1.09922 * 10^{-5}$ | 0.0215542<br>$9.38095 * 10^{-6}$ | 0.0304243<br>$1.11805 * 10^{-5}$ | 0.0348148<br>$1.0391 * 10^{-5}$ | (0.0113117, 0.0473072) |
| $\hat{\theta}_{ES3}$ | 0.0143065<br>$1.09877 * 10^{-5}$ | 0.0215524<br>$9.37904 * 10^{-6}$ | 0.030422<br>$1.11788 * 10^{-5}$ | 0.0348127<br>$1.03897 * 10^{-5}$ | (0.011312, 0.0473028) |
| $\hat{\theta}_{EE1}$ | 0.0138417<br>$1.13573 * 10^{-5}$ | 0.0211998<br>$9.53794 * 10^{-6}$ | 0.0300903<br>$1.13059 * 10^{-5}$ | 0.0345279<br>$1.04785 * 10^{-5}$ | (0.0111864, 0.0467889) |
| $\hat{\theta}_{EE2}$ | 0.0138396<br>$1.13526 * 10^{-5}$ | 0.021198<br>$9.536 * 10^{-6}$ | 0.0300881<br>$1.13041 * 10^{-5}$ | 0.0345258<br>$1.04772 * 10^{-5}$ | (0.0111867, 0.0467845) |
| $\hat{\theta}_{EE3}$ | 0.0138374<br>$1.1348 * 10^{-5}$ | 0.0211961<br>$9.53407 * 10^{-6}$ | 0.0300859<br>$1.13023 * 10^{-5}$ | 0.0345238<br>$1.04759 * 10^{-5}$ | (0.011187, 0.0467801) |
| $\hat{\theta}_{EP1}$ | 0.0300255<br>$1.10854 * 10^{-5}$ | 0.0353436<br>$9.4213 * 10^{-6}$ | 0.0443292<br>$1.1213 * 10^{-5}$ | 0.0472391<br>$1.04137 * 10^{-5}$ | (0.0113737, 0.0475723) |
| $\hat{\theta}_{EP2}$ | 0.0145414<br>$1.10809 * 10^{-5}$ | 0.0217316<br>$9.41939 * 10^{-6}$ | 0.0305919<br>$1.12113 * 10^{-5}$ | 0.0349589<br>$1.04124 * 10^{-5}$ | (0.011374, 0.0475678) |
| $\hat{\theta}_{EP3}$ | 0.0145391<br>$1.10763 * 10^{-5}$ | 0.0217297<br>$9.41748 * 10^{-6}$ | 0.0305896<br>$1.12095 * 10^{-5}$ | 0.0349568<br>$1.04112 * 10^{-5}$ | (0.0113743, 0.0475634) |

 

# 6 Results

In this section, we examine how well the estimators developed in this article performed.

## 6.1 *Real Data Analysis*

We used the real data set given by [14] for real-life situations indicating the graft survival periods in months of 148 renal transplant patients to examine the performance of the estimators developed in this research. The data were fitted using the Chen distribution, and the p-value and test statistic values for the Kolmogorov-Smirnov test are 0.5844 and 0.0626, respectively. The MLEs for the unknown Chen distribution parameters are calculated to be $\hat{\theta}$ =0.0429 and $\hat{\lambda}$ =0.3863. We generate Type-II censored samples by choosing different values for r (30, 60, 90 and 120). We presume that the shape parameter is always known and equal to its MLE, i.e., $\hat{\lambda}$ = 0.3863. Using the bootstrapping concept, we computed the AE, MSE, E-MSE and 95% average credible interval (ACI) of the estimators and are given in Tables 1 and 2.

**Table 2:** *The AE (first row), MSE (second row) and ACI for MLE, Bayesian and E-Bayesian estimates of $h(t)$ for real data.*

| | n=148 | | | | |
|---|---|---|---|---|---|
| | r=30 | r=60 | r=90 | r=120 | ACI |
| $\hat{\lambda}_{MLE}$ | 0.0205438 | 0.0334508 | 0.0501238 | 0.0557942 | |
| | $1.82571 * 10^{-3}$ | $1.28701 * 10^{-3}$ | $7.69232 * 10^{-4}$ | $7.28922 * 10^{-4}$ | |
| $\hat{h}_{B1}$ | 0.0198053 | 0.0325953 | 0.0490964 | 0.0556378 | (0.00534318, 0.0855648) |
| | $2.10708 * 10^{-5}$ | $2.20851 * 10^{-5}$ | $3.14922 * 10^{-5}$ | $3.08391 * 10^{-5}$ | |
| $\hat{h}_{B2}$ | 0.019156 | 0.0320565 | 0.0485539 | 0.0551761 | (0.00528414, 0.0846193) |
| | $2.17617 * 10^{-5}$ | $2.24501 * 10^{-5}$ | $3.18402 * 10^{-5}$ | $3.1095 * 10^{-5}$ | |
| $\hat{h}_{B3}$ | 0.0201274 | 0.0328636 | 0.0493669 | 0.0558682 | (0.00537262, 0.0860362) |
| | $2.12408 * 10^{-5}$ | $2.21756 * 10^{-5}$ | $3.15787 * 10^{-5}$ | $3.09028 * 10^{-5}$ | |
| $\hat{h}_{ES1}$ | 0.0198039 | 0.0325939 | 0.0490946 | 0.0556361 | (0.00534354, 0.0855612) |
| | $2.10668 * 10^{-5}$ | $2.2083 * 10^{-5}$ | $3.14898 * 10^{-5}$ | $3.08372 * 10^{-5}$ | |
| $\hat{h}_{ES2}$ | 0.019801 | 0.0325912 | 0.0490911 | 0.0556328 | (0.00534425, 0.0855542) |
| | $2.10586 * 10^{-5}$ | $2.2079 * 10^{-5}$ | $3.14851 * 10^{-5}$ | $3.08335 * 10^{-5}$ | |
| $\hat{h}_{ES3}$ | 0.0197981 | 0.0325885 | 0.0490876 | 0.0556295 | (0.00534496, 0.0855471) |
| | $2.10505 * 10^{-5}$ | $2.20749 * 10^{-5}$ | $3.14804 * 10^{-5}$ | $3.08298 * 10^{-5}$ | |
| $\hat{h}_{EE1}$ | 0.0191546 | 0.0320552 | 0.0485521 | 0.0551744 | (0.00528449, 0.0846158) |
| | $2.17575 * 10^{-5}$ | $2.2448 * 10^{-5}$ | $3.18378 * 10^{-5}$ | $3.10931 * 10^{-5}$ | |
| $\hat{h}_{EE2}$ | 0.0191518 | 0.0320525 | 0.0485487 | 0.0551712 | (0.0052852, 0.0846088) |
| | $2.17491 * 10^{-5}$ | $2.24439 * 10^{-5}$ | $3.1833 * 10^{-5}$ | $3.10894 * 10^{-5}$ | |
| $\hat{h}_{EE3}$ | 0.0191489 | 0.0320499 | 0.0485452 | 0.0551679 | (0.0052859, 0.0846018) |
| | $2.17407 * 10^{-5}$ | $2.24398 * 10^{-5}$ | $3.18282 * 10^{-5}$ | $3.10856 * 10^{-5}$ | |
| $\hat{h}_{EP1}$ | 0.0201259 | 0.0328622 | 0.0493651 | 0.0558665 | (0.00537298, 0.0860327) |
| | $2.12367 * 10^{-5}$ | $2.21735 * 10^{-5}$ | $3.15763 * 10^{-5}$ | $3.09009 * 10^{-5}$ | |
| $\hat{h}_{EP2}$ | 0.020123 | 0.0328595 | 0.0493616 | 0.0558632 | (0.0053737, 0.0860256) |
| | $2.12285 * 10^{-5}$ | $2.21695 * 10^{-5}$ | $3.15716 * 10^{-5}$ | $3.08972 * 10^{-5}$ | |
| $\hat{h}_{EP3}$ | 0.02012 | 0.0328567 | 0.049358 | 0.0558599 | (0.00537441, 0.0860184) |
| | $2.12203 * 10^{-5}$ | $2.21654 * 10^{-5}$ | $3.15669 * 10^{-5}$ | $3.08935 * 10^{-5}$ | |

From Tables 1 and 2, it is to be noted that the approximated MSEs decrease as r increases. We can deduce from Tables that E-Bayesian estimators outperform MLE and Bayesian estimators in terms of MSE.

# 7 Discussion and Concluding Remark

The parameter and hazard rate of the Chen distribution based on Type II censoring are estimated using the MLE, Bayesian, and E-Bayesian approaches. The estimates are computed using real data, and various estimation techniques are compared. One of the study's key findings is the superiority of the proposed estimators versus existing estimators. The impact of various prior distributions and loss functions is also something we theoretically investigate. Important concluding remarks from our study are listed below:

- The lowest MSE of all the estimates is seen in the E-Bayesian estimations of $\theta$.
- The lowest E-MSE among all estimates is found in the E-Bayesian estimations of $\theta$ based on ELF with prior distribution $\pi_3(a, b)$.
- For a fixed value of $n$ and $r$ the E-MSE is less for E-Bayesian estimators as compared to Bayesian and MLE.
- Compared to Bayesian and MLE, the proposed estimators perform better in terms of minimum MSE.

Combining the findings mentioned above, we recommended the E-Bayesian technique, which outperforms previous estimates in terms of minimum MSE, to estimate the scale parameter and hazard rate functions of the Chen distribution based on the type-II censoring scheme using prior distribution $\pi_3(a, b)$.

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