ON THE INFERENCES AND APPLICATIONS OF WEIBULL HALF LAPLACE{EXPONENTIAL} DISTRIBUTION

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Abstract

The study of probability distribution has expanded the field of statistical modelling of real life data. It has also provided solution to the problems of skewed data which often violate the normality. This research work introduces a new T - Half-Lapalace{Exponential} family with a novel Half-Laplace distribution as baseline distribution with specific interest in three-parameter lifetime model called the Weibull-Half-Lapalace{Exponential} (W-HLa{E}) distribution. The W-HLa{E} model is capable of modeling various shapes of aging events. The W-HLa{E} distribution is derived by combining Half-Laplace and Weibull distribution using the quartile function of Exponential distribution. Some of its statistical properties such as the mean, mode, quantile function, median, variance, standard deviation, skewness, and kurtosis are derived. Other statistical properties such as survival function, hazard rate, moments, asymptotic limit, order statistics, and entropy which is the measure of uncertainty of a random variable are derived and studied. The parameter estimation method adopted in this study is the maximum likelihood method. The graphs of W-HLa{E} at different values of shape and scale parameters show that the distribution is unimodal hence the mode is given as $mode = \theta + (\frac{k-1}{k})^{\frac{1}{k}}$ *and it is positively skewed with a steep peak. A simulation study is carried on the new proposed distribution using maximum likelihood estimation. The simulation also supported the theoretical expression of the statistical properties of the proposed distribution such as the location parameter does not affect the variance, skewness, and kurtosis of the new distribution. The importance and the flexibility of the proposed distribution in modeling some real life data sets is demostrated inn the research. The results of the sudy shows that the proposed W-HLa{E} distribution perform better than other disribbutions in the literature.*

Keywords: Laplace distribution; Half-Laplace distribution; Weibull Half-Laplace distribution; Censored data; Lifetime data; Maximum likelihood estimation

1. INTRODUCTION

Numerous classical distributions have been extensively used over the past decades for modeling data in several areas such as engineering, actuarial, environmental and medical sciences, biological studies, demography, economics, finance, and insurance. However, there is a clear need for extended forms of these classical distributions. Due to that reason, researchers have developed and studied several methods for generating new families of distributions. The most outstanding characteristics of this distribution are that it is unimodal and symmetric. Laplace distribution is a mixture of normal laws (see Kotz *[10]*), as a possible explanation of the wide applicability of this distribution for modeling growth rates. Let $f(x, \theta, \beta)$ be the probability density function of Laplace distribution.

$$f(x,\theta,\beta) = \frac{1}{2\beta} e^{(-|x-\theta|/\beta)}, x \in (-\infty, \infty) \tag{1}$$

and CDF, F(x) is given as:

$$F(x;\theta,\beta) = \begin{cases} \frac{1}{2} e^{\left(-\frac{x-\theta}{\beta}\right)}, & if\ x < \theta \\ 1 - \frac{1}{2} e^{\left(-\frac{x-\theta}{\beta}\right)} & if\ x \geq \theta \end{cases} \tag{2}$$

The expected value of a Laplace distribution is given as $E(x) = \theta$. The expected value of a Laplace distribution is the same as the location parameter and a symmetric situation means that the mean is the same as mode and median. Laplace distribution can be compared with other symmetric distributions like Normal, Logistic, etc. except that Laplace has a higher spike and slightly thicker tails.

In recent years, Asymmetric Laplace distribution of [10] has received much attention in modeling currency exchange rates, interests, stock price changes which is a modification of Laplace distribution but not for survival data. Many researchers have developed compound distributions using different methods to fit survival data. In this research work, we shall reduce the classical Laplace distribution to a non-negative function. Most real-life quantities to be measured are non-negative values. With the assumption that the data is non-negative, therefore, it is necessary to reduce the Laplace distribution to one-sided (positively skewed distribution). Thus, the one-sided Laplace distribution, otherwise called the half-Laplace distribution is the positive side of the Laplace distribution.

Let X be a random variable on $R_+ = (0, \infty)$ given by the density function of Laplace distribution equation (1), where $x \geq \theta \geq 0 and\ \beta > 0$, then x is said to have a half Laplace distribution, denoted by $HL(\theta, \beta)$

$$f(x,\theta,\beta) = \frac{1}{\beta} e^{(-\frac{x-\theta}{\beta})}, x > \theta \geq 0; \beta > 0 \tag{3}$$

and the CDF of the half-Laplace distribution is

$$F(x,\theta,\beta) = 1 - \frac{1}{\beta} e^{(-\frac{x-\theta}{\beta})}, x > \theta \geq 0; \beta > 0 \tag{4}$$

Where θ is the location parameter and β is the shape parameter The half-Laplace distribution reduces to the exponential distribution when $\theta = 0$, we have the exponential distribution.

Recently, many researchers have developed and studied compound distributions using T-X which was introduced by Alzaatreh [5] and T-X{Y} by Aljarrah [3]. This was later modified by Alzaatreh [6] as T-gamma family, Alzaatreh [7] also constructed T-normal families. Almheidat [4] studied the T-Weibull family. Amalare [8] derived Lomax-Cauchy {Uniform}. Ogunsanya [13] developed and studied the extension of Cauchy distribution named Rayleigh Cauchy distribution, Ogunsanya [14] studied Weibull-Inverse Rayleigh distribution: Classical/ Bayesian approach and Job [11] applied Weibull Loglogistic{Exponential} distribution on some survival data.

In Section 2, we derive the new Weibull Half Laplace distribution, with statistical properties such as hazard function, survival function, Moments, skewness, kurtosis, order statistics, and Shannon entropy are determined. Section 3 shows the simulation study. Estimation of the parameters of W-HLa{E} distribution by maximum likelihood is performed in Section 4. In Section 5, the performance of the new W-HLa{E} distribution is demonstrated on two real data sets and the conclusion and summary of the work are expressed in Section 6.

2. DERIVATION OF WEIBULL-HALFLAPLACE {EXPONENTIAL} DISTRIBUTION

In this section, we investigate in details the properties, parameters estimation, and applications of a new distribution of the T-Half Laplace $\{Y\}$ family called Weibull- Half Laplace $\{Exponential\}$ (W-HLa{E}) distribution.

Let t be a random variable that follows a two-parameter Weibull distribution, then PDF is given as

$$f_T(t;\lambda,k)=\left\{\frac{k}{\lambda}\left(\frac{t}{\lambda}\right)^{k-1}e^{-\left(\frac{t}{\lambda}\right)^k}\right\};t\geqslant 0,\lambda,k\geqslant 0$$

And the CDF is

$$F_T(t;\lambda,k)=1-e^{-\left(\frac{t}{\lambda}\right)^k};t\geqslant 0,\lambda,k\geqslant 0 \tag{5}$$

Recall Equation (4), and given the quantile function of exponential distribution as *-blog[1-x]*

$$F_X(x)=F_T\{-blog[1-(F_R(x))]\}$$

then the CDF of proposed W-HLa{E} distribution is

$$F_X(x)=F_T\left\{-blog\left[1-\left(1-e^{\left(-\frac{x-\theta}{\beta}\right)}\right)\right]\right\} \tag{6}$$

Substituting the CDF of Weibull distribution in equation (6)

$$F_X(x)=1-e^{-\left(\frac{b}{\lambda\beta}\right)^k(x-\theta)^k} \tag{7}$$

Let $\frac{b}{\lambda\beta}=\gamma$ in (7), then we have

$$F_X(x)=1-e^{-(\gamma)^k(x-\theta)^k} \tag{8}$$

Hence the corresponding PDF using equation (8)

$$f_X(x)=k\left(\frac{b}{\lambda\beta}\right)^k(x-\theta)^{k-1}e^{\left(\frac{b}{\lambda\beta}\right)^k(x-\theta)^k} \tag{9}$$

Let $\frac{b}{\lambda\beta}=\gamma$ in (9) or differentiate equation (8) with respect to x, then we have

$$f_X(x)=k\gamma^k(x-\theta)^{k-1}e^{\gamma^k(x-\theta)^k} \tag{10}$$

where $k,\theta,\gamma\geq 0$ are parameters of W-HLa{E} distribution

3. STATISTICAL PROPERTIES OF W-HLA{E} DISTRIBUTION

The statistical properties of the W-HLa{E} distribution including quantile function, ordinary moments, and Shannon entropy are provided in this section

Proposition 1 (Quantile Function) If X is a random variable that has W-HLa{E} distribution $(x;k,\theta,\gamma)$ and let $Q_X(p)$, such that $0\leq p\leq 1$ denote the quantile function for the W-HLa{E} distribution. Then $Q_X(p)$ is given by

$$Q_X(p)=\theta+\frac{1}{\gamma}\{-log(1-p)\}^{\frac{1}{k}} \tag{11}$$

where $k,\theta,\gamma\geq 0$ are parameters of W-HLa{E} distribution

Proof:

From Equation (8) replace $F_X(x)$ with p and solve for x, we obtain (11), the quantile function of W-HLa{E} distribution.

Setting$p=0.25,0.50,and0.75$ in (30)he quartiles of the W-HLa{E} distribution can be obtained.

Lower Quartile

$$Q_X(0.25)=\theta+\frac{1}{\gamma}\{-log(1-0.25)\}^{\frac{1}{k}}$$

Median

$$Q_X(0.5)=\theta+\frac{1}{\gamma}\{-log(1-0.5)\}^{\frac{1}{k}}$$

Upper Quartile

$$Q_X(0.75) = \theta + \frac{1}{\gamma}\{-log(1-0.75)\}^{\frac{1}{k}}$$

Proposition 2 (Modal Function) *If X is a random variable that has W-HLa{E} distribution*$(x; k, \theta, \gamma)$ *and let* $X_{Mode}(x)$*, such that* $0 \le p \le 1$ *denote the mode function for the W-HLa{E} distribution. Then* $Q_X(p)$ *is given by*

$$X_{mode}(x) = \theta + \frac{1}{\gamma}\left[\frac{k-1}{k}\right]^{\frac{1}{k}} \tag{12}$$

where $k, \theta, \gamma > 0$ are parameters of W-HLa{E} distribution
Proof:
Differentiate (10) and equate to zero

$$\frac{d}{dx}f(x) = 0$$

$$\frac{d}{dx}\left[k\gamma^k(x-\theta)^{k-1}e^{-\gamma^k(x-\theta)^{k-1}}\right] = 0$$

$$\frac{k(k-1)\gamma^k(x-\theta)^{k-1}e^{-\gamma^k(x-\theta)^{k-1}}}{(x-\theta)} - \frac{k^2\gamma^{2k}(x-\theta)^{2k-1}e^{-\gamma^k(x-\theta)^{k-1}}}{(x-\theta)} = 0$$

$$\frac{k(k-1)\gamma^k(x-\theta)^{k-1}e^{-\gamma^k(x-\theta)^{k-1}}}{(x-\theta)} = \frac{k^2\gamma^{2k}(x-\theta)^{2k-1}e^{-\gamma^k(x-\theta)^{k-1}}}{(x-\theta)}$$

Solving for x, we have

$$X_{mode}(x) = \theta + \frac{1}{\gamma}\left[\frac{k-1}{k}\right]^{\frac{1}{k}}$$

3.1. Shape Properties of W-HLa{E} Distribution

Cumulative Distribution Function (CDF) of W-HLa{E} Distribution.
Equation (9) is now the CDF of the new probability distribution called W-HLa{E} Distribution.

3.2 Hazard Function

The hazard function of the W-HLa{E} distribution is derived from this definition

$$h_X(x) = f_X(x)/1 - F_X(x)$$

where $f_X(x)$ and $F_X(x)$ are the PDF and CDF of W-HLa{E} distribution given in (9) and (10) respectively. The hazard function h(x) can be written as

$$h_X(x) = \frac{k\gamma^k(x-\theta)^{k-1}e^{-\gamma^k(x-\theta)^k}}{1-\{1-e^{-\gamma^k(x-\theta)^k}\}} \tag{13}$$

Simplifying (13), we have

$$h_X(x) = k\gamma^k(x-\theta)^{k-1} \tag{14}$$

The log hazard of W-HLa{E}D, which is frequently used in modeling is given by

$$\lambda_X(t) = log(h_X(x))$$

$$\lambda_X(t) = log(k\gamma^k(x-\theta)^{k-1}) \tag{15}$$

Expanding equation (15)

$$\lambda_X(t) = log(k\gamma^k) + log(x-\theta)^{k-1}$$

$$\lambda_X(t) = log(k\gamma^k) + (k-1)log(x-\theta) \tag{16}$$

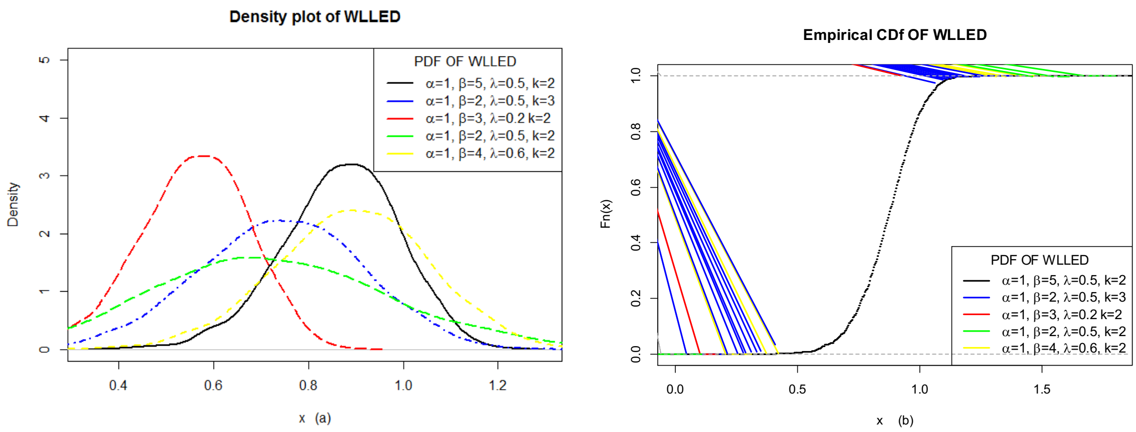

Figure 1: *(a) Density plot and (b) CDF plot of WLE distribution for sample size = 1000 and for various values of* k, θ *and* γ.

Figure 1(a) ahows that the PDF is positively skewed. Figure 1(b) shows that the CDF of W-HLa{E}D increases as x increases and remains constant as it approaches 1.

3.3 Survival Function

The survival function of the W-HLa{E} distribution is derived from this definition

$$S_X(x) = 1 - F_X(x)$$

where F(x) is the CDF of the W-HLa{E} distribution as defined in equation(9) The survival function S_x can be written as $S_X(x) = 1 - \{1 - e^{-\gamma^k (x-\theta)^k}\}$

$$S_X(x) = e^{-\gamma^k (x-\theta)^k} \tag{17}$$

For $x > 0, k, \theta and \gamma > 0$ *and* $t > 0$, the probability that a system having age x units of time will survive up to $x + t$ units of time is given by

$$S_X(x) = \frac{e^{-\gamma^k (x+t-\theta)^k}}{e^{-\gamma^k (x-\theta)^k}}$$

3.4 Cumulative Hazard Function

The cumulative hazard function, of the W-HLa{E} distribution, is given as

$$H_X(x) = -log_e\{e^{-\gamma^k (x-\theta)^k}\} \tag{18}$$

Simplifying (18) we have

$$H_X(x) = \gamma^k (x-\theta)^k \tag{19}$$

3.5 Asymptotic Behavior of W-HLa{E} Distribution

To investigate the asymptotic behavior of the proposed distribution model W-HLa{E}, we find the limit as $x \longrightarrow \theta$ and as $x \longrightarrow \infty$ of the W-HLa{E} distribution

$$lim_{x \to \theta} f(x) = lim_{x \to \theta} k\gamma^k (x-\theta)^{k-1} e^{-\gamma^k (x-\theta)^k} = 0$$

Proof: Since $lim_{x \to \theta}(x - \theta) = 0$, then
$lim_{x \to \theta} f(x) = 0$
Hence as x tend to a minimum value of the distribution, W-HLa{E} distribution becomes zero
Similarly,

$$lim_{x \to \infty} f(x) = lim_{x \to \theta} k\gamma^k (x-\theta)^{k-1} e^{-\gamma^k (x-\theta)^k} = 0$$

Proof: Since $lim_{x \to \infty} = \left(e^{-\gamma^k (x-\theta)^k}\right) = 0$, then
$lim_{x \to \infty} f(x) = 0$

As x tends to infinity the W-HLa{E} distribution becomes zero

3.6 Moments and Variance

In this subsection, we shall determine r^{th} moment of about the origin and n^{th} moment about the mean. Given

$$E(X) = \int_{\theta}^{\infty} x f_X(x) dx \tag{20}$$

Proposition 3 (First Moment about Origin) *If X is a random variable that has W-HLa{E} distribution$(x; k, \theta, \gamma)$ and let X_M denote the first moment about the origin of the W-HLa{E} distribution. Then $\grave{\mu}_1$ is given by*

$$X_M = \grave{\mu}_1 = \theta + \frac{1}{\gamma}\Gamma\left(1 + \frac{1}{k}\right)$$

where $k, \theta, \gamma \geq 0$ are parameters of W-HLa{E} distribution

Proof: Substituting (10) in (20)

$$E(X) = \int_{\theta}^{\infty} xk\gamma^k (x-\theta)^{k-1} e^{-\gamma^k (x-\theta)^k} dx \tag{21}$$

Let

$$z = \gamma^k (x-\theta)^k \tag{22}$$

then differentiate Equation (22) with respect to x

$$dx = \frac{dz}{k\gamma^k (x-\theta)^{k-1}} \tag{23}$$

Substitute (23) in (21), we have

$$E(X) = \int_{\theta}^{\infty} xk\gamma^k (x-\theta)^{k-1} e^{-\gamma^k (x-\theta)^k} \frac{dz}{k\gamma^k (x-\theta)^{k-1}} \tag{24}$$

$$E(X) = \int_{\theta}^{\infty} x e^{-\gamma^k (x-\theta)^k} dz \tag{25}$$

From (22), we make x the subject,

$$x = \theta + \frac{z^{1/k}}{\gamma} \tag{26}$$

$0 \leq z \leq \infty$ Substitute (22) and (26) in Eqn. (25), we have

$$E(X) = \int_{0}^{\infty} \left(\theta + \frac{z^{1/k}}{\gamma}\right) e^{-z} dz \tag{27}$$

$$E(X) = \int_{0}^{\infty} \theta e^{-z} dz + \int_{0}^{\infty} \frac{z^{1/k}}{\gamma} e^{-z} dz \tag{28}$$

By evaluating the limits, we have

$$E(X) = -\theta e^{-\gamma^k (x-\theta)^k} |_{\theta}^{\infty} + \frac{1}{\gamma}\Gamma\left(1 + \frac{1}{k}\right) \tag{29}$$

Hence equation (29) becomes

$$E(X) = \theta + \frac{1}{\gamma}\Gamma\left(1 + \frac{1}{k}\right) \tag{30}$$

Proposition 4 (Second Moment about Origin) *If X is a random variable that has W-HLa{E} distribution$(x; k, \theta, \gamma)$ and let $\grave{\mu}_2$ denote the second moment about the origin of the W-HLa{E} distribution. Then $\grave{\mu}_2$ is given by*

$$\grave{\mu}_2 = \theta + 2\frac{1}{\gamma}\Gamma\left(1 + \frac{1}{k}\right) + \frac{1}{\gamma^2}\Gamma\left(1 + \frac{2}{k}\right)$$

where $k, \theta, \gamma \geq 0$ are parameters of W-HLa{E} distribution

Proof: Given

$$E(X^2) = \grave{\mu}_2 = \int_{\theta}^{\infty} x^2 f_X(x) dx \tag{31}$$

Substitute equation (10) in equation (31)

$$\grave{\mu}_2 = \int_{\theta}^{\infty} x^2 k\gamma^k (x-\theta)^{k-1} e^{-\gamma^k (x-\theta)^k} dx \tag{32}$$

Using the above method adopted in proposition 4.4, we have,

$$\grave{\mu}_2 = \theta^2 + 2\frac{\theta}{\gamma}\Gamma\left(1 + \frac{1}{k}\right) + \frac{1}{\gamma^2}\Gamma\left(1 + \frac{2}{k}\right) \tag{33}$$

Corollary 1 *(rth moment) Let X be a random variable that follows W-HLa{E} distribution*$(x; k, \theta, \gamma)$ *and let* $\grave{\mu}_r$ *denote the rth moment about the origin of the W-HLa{E} distribution. Then* $\grave{\mu}_r$ *is given by*

$$\grave{\mu}_r = \theta^r + \sum_{i=1}^{r} \binom{r}{i} \frac{\theta^{r-1}}{\gamma^i} \Gamma\left(1 + \frac{i}{k}\right) \tag{34}$$

Where i=1,2,3,…,r

Proof: By Mathematical induction, it follows from equations (30) and (33) of propositions 3 and 4. respectively

Hence the first four moments of the proposed distribution are given

$$\grave{\mu}_1 = \theta + \frac{1}{\gamma}\Gamma\left(1 + \frac{1}{k}\right)$$
$$\grave{\mu}_2 = \theta^2 + 2\frac{\theta}{\gamma}\Gamma\left(1 + \frac{1}{k}\right) + \frac{1}{\gamma^2}\Gamma\left(1 + \frac{2}{k}\right)$$
$$\grave{\mu}_3 = \theta^3 + 3\frac{\theta^2}{\gamma}\Gamma\left(1 + \frac{1}{k}\right) + 3\frac{\theta}{\gamma^2}\Gamma\left(1 + \frac{2}{k}\right) + \frac{1}{\gamma^3}\Gamma\left(1 + \frac{3}{k}\right)$$
$$\grave{\mu}_4 = \theta^4 + 4\frac{\theta^3}{\gamma}\Gamma\left(1 + \frac{1}{k}\right) + 6\frac{\theta^2}{\gamma^2}\Gamma\left(1 + \frac{2}{k}\right) + 4\frac{\theta}{\gamma^3}\Gamma\left(1 + \frac{3}{k}\right) + \frac{1}{\gamma^4}\Gamma\left(1 + \frac{4}{k}\right)$$

Again using the relationship between moments about mean and moments about the origin, the moments about the mean of W-HLa{E} distribution are obtained as

$$Variance(\mu_2) = E(X^2) - [E(X)]^2$$
$$Variance(\mu_2) = \theta^2 + 2\frac{\theta}{\gamma}\Gamma\left(1 + \frac{1}{k}\right) + \frac{1}{\gamma^2}\Gamma\left(1 + \frac{2}{k}\right) - \left[\theta + \frac{1}{\gamma}\Gamma\left(1 + \frac{1}{k}\right)\right]^2$$
$$= \frac{1}{\gamma^2}\Gamma\left(1 + \frac{2}{k}\right) - \left[\frac{1}{\gamma}\Gamma\left(1 + \frac{1}{k}\right)\right]^2$$
$$= \frac{1}{\gamma^2}\left\{\Gamma\left(1 + \frac{2}{k}\right) - \left[\Gamma\left(1 + \frac{1}{k}\right)\right]^2\right\}$$

Therefore the standard deviation is given as

$$StandardDeviation(X) = \sqrt{\frac{1}{\gamma^2}\left\{\Gamma\left(1 + \frac{2}{k}\right) - \left[\Gamma\left(1 + \frac{1}{k}\right)\right]^2\right\}}$$

$$= \frac{1}{\gamma}\sqrt{\left\{\Gamma\left(1 + \frac{2}{k}\right) - \left[\Gamma\left(1 + \frac{1}{k}\right)\right]^2\right\}}$$

$$Co-efficient \text{ of } Variation(C.V) = \frac{StandardDeviation}{E(X)}$$

$$C.V = \frac{\frac{1}{\gamma^2}\sqrt{\left\{\Gamma\left(1 + \frac{2}{k}\right) - \left[\Gamma\left(1 + \frac{1}{k}\right)\right]^2\right\}}}{\theta + \frac{1}{\gamma}\Gamma\left(1 + \frac{1}{k}\right)}$$

Again using the relationship between moments about mean and moments about origin where $\mu_3 = \grave{\mu}_3 - 3\grave{\mu}_1\grave{\mu}_2 + 2\grave{\mu}_1^3$ and $\mu_4 = \grave{\mu}_4 - 4\grave{\mu}_1\grave{\mu}_3 + 6\grave{\mu}_1^2\grave{\mu}_2 + 3\grave{\mu}_1^4$ are third and fourth moments about the mean respectively. The moments about the mean of W-HLa{E} distribution are obtained as

$$\mu_3 = \theta^3 + 3\frac{\theta^2}{\gamma}\Gamma\left(1 + \frac{1}{k}\right) + 3\frac{\theta}{\gamma^2}\Gamma\left(1 + \frac{2}{k}\right) + \frac{1}{\gamma^3}\Gamma\left(1 + \frac{3}{k}\right) -$$
$$3\left[\theta + \frac{1}{\gamma}\Gamma\left(1 + \frac{1}{k}\right)\right]\left[\theta^2 + 2\frac{\theta}{\gamma}\Gamma\left(1 + \frac{1}{k}\right) + \frac{1}{\gamma^2}\Gamma\left(1 + \frac{2}{k}\right)\right] + 2\left[\theta + \frac{1}{\gamma}\Gamma\left(1 + \frac{1}{k}\right)\right]^3$$
$$\mu_4 = \theta^4 + 4\frac{\theta^3}{\gamma}\Gamma\left(1 + \frac{1}{k}\right) + 6\frac{\theta^2}{\gamma^2}\Gamma\left(1 + \frac{2}{k}\right) + 4\frac{\theta}{\gamma^3}\Gamma\left(1 + \frac{3}{k}\right) + \frac{1}{\gamma^4}\Gamma\left(1 + \frac{4}{k}\right)$$
$$-4\left[\theta + \frac{1}{\gamma}\Gamma\left(1 + \frac{1}{k}\right)\right]\left[\theta^3 + 3\frac{\theta^2}{\gamma}\Gamma\left(1 + \frac{1}{k}\right) + 3\frac{\theta}{\gamma^2}\Gamma\left(1 + \frac{2}{k}\right) + \frac{1}{\gamma^3}\Gamma\left(1 + \frac{3}{k}\right)\right]$$
$$+6\left[\theta + \frac{1}{\gamma}\Gamma\left(1 + \frac{1}{k}\right)\right]^2\left[\theta^2 + 2\frac{\theta}{\gamma}\Gamma\left(1 + \frac{1}{k}\right) + \frac{1}{\gamma^2}\Gamma\left(1 + \frac{2}{k}\right)\right] + 3\left[\theta + \frac{1}{\gamma}\Gamma\left(1 + \frac{1}{k}\right)\right]^4$$

Hence, the skewness and kurtosis are determined as follows;

$$Skewness = \frac{E(x-\mu)^3}{\sigma^2} = \frac{\mu_3}{\sigma^2}$$

$$Skewness = \frac{1}{\frac{1}{\gamma^2}\left\{\Gamma\left(1+\frac{2}{k}\right)-\left[\Gamma\left(1+\frac{1}{k}\right)\right]^2\right\}}\theta^3 + 3\frac{\theta^2}{\gamma}\Gamma\left(1+\frac{1}{k}\right) + 3\frac{\theta}{\gamma^2}\Gamma\left(1+\frac{2}{k}\right) + \frac{1}{\gamma^3}\Gamma\left(1+\frac{3}{k}\right) -$$

$$3\left[\theta + \frac{1}{\gamma}\Gamma\left(1+\frac{1}{k}\right)\right]\left[\theta^2 + 2\frac{\theta}{\gamma}\Gamma\left(1+\frac{1}{k}\right) + \frac{1}{\gamma^2}\Gamma\left(1+\frac{2}{k}\right)\right] + 2\left[\theta + \frac{1}{\gamma}\Gamma\left(1+\frac{1}{k}\right)\right]^3$$

Further simplification of above we have

$$= \frac{\dfrac{2\Gamma\left(1+\frac{1}{k}\right)^2 - 3\Gamma\left(1+\frac{1}{k}\right)\Gamma\left(1+\frac{2}{k}\right) + \Gamma\left(1+\frac{3}{k}\right)}{\gamma^2}}{\dfrac{1}{\gamma^2}\Gamma\left(1+\frac{2}{k}\right) - \left[\frac{1}{\gamma}\Gamma\left(1+\frac{1}{k}\right)\right]^2}$$

$$= \frac{2\Gamma\left(1+\frac{1}{k}\right)^2 - 3\Gamma\left(1+\frac{1}{k}\right)\Gamma\left(1+\frac{2}{k}\right) + \Gamma\left(1+\frac{3}{k}\right)}{\Gamma\left(1+\frac{2}{k}\right) - \left[\frac{1}{\gamma}\Gamma\left(1+\frac{1}{k}\right)\right]^2}$$

$$\mathrm{Kurtosis} = \frac{\mu_4}{\sigma^4}$$

$$Kurtosis = \left\{\frac{1}{\frac{1}{\gamma^4}\left\{\Gamma\left(1+\frac{2}{k}\right)-\left[\Gamma\left(1+\frac{1}{k}\right)\right]^2\right\}^2}\right\}\left\{\theta^4 + 4\frac{\theta^3}{\gamma}\Gamma\left(1+\frac{1}{k}\right) + 6\frac{\theta^2}{\gamma^2}\Gamma\left(1+\frac{2}{k}\right) + \right.$$

$$4\frac{\theta}{\gamma^3}\Gamma\left(1+\frac{3}{k}\right) + \frac{1}{\gamma^4}\Gamma\left(1+\frac{4}{k}\right) - 4\left[\theta + \frac{1}{\gamma}\Gamma\left(1+\frac{1}{k}\right)\right]\left[\theta^3 + 3\frac{\theta^2}{\gamma}\Gamma\left(1+\frac{1}{k}\right) + 3\frac{\theta}{\gamma^2}\Gamma\left(1+\frac{2}{k}\right) + \frac{1}{\gamma^3}\Gamma\left(1+\frac{3}{k}\right)\right] +$$

$$\left. 6\left[\theta + \frac{1}{\gamma}\Gamma\left(1+\frac{1}{k}\right)\right]^2\left[\theta^2 + 2\frac{\theta}{\gamma}\Gamma\left(1+\frac{1}{k}\right) + \frac{1}{\gamma^2}\Gamma\left(1+\frac{2}{k}\right)\right] + 3\left[\theta + \frac{1}{\gamma}\Gamma\left(1+\frac{1}{k}\right)\right]^4\right\}$$

3.7 Order Statistics

Order statistics is an important concept in probability theory. Let a random sample $X_1, X_2, \ldots X_n$, from the distribution function F(x) and corresponding pdf f(x), therefore the pdf of ith order statistic is given as

Proposition 5 *If X is a random variable that has W-HLa{E} distribution*$(x; k, \theta, \gamma)$ *and let* $f(x_i)$ *denote the pdf of ith order statistic which is given as*

$$f(X_i) = \frac{n!}{(i-1)!(n-i)!} k\theta^p\gamma^k \sum_{p,q=0}^{\infty} \binom{k-1}{p}\binom{i-1}{q}(-1)^{p+q}\left(e^{-\gamma^k(x-\theta)^k}\right)^{1+q+n-i} \quad (35)$$

where $k, \theta, \gamma \geq 0$ are parameters of W-HLa{E} distribution

Proof: Given

$$f(X_i) = \frac{n!}{(i-1)!(n-i)!} f(x)F(x)^{i-1}[1-F(x)]^{n-1} \quad (36)$$

hence the pdf of ith order statistic of W-HLa{E} distribution is determined by substituting equation (7) and (10) in equation (35) we have

$$f(X_i) = \frac{n!}{(i-1)!(n-i)!} k\gamma^k(x-\theta)^{k-1}e^{-\gamma^k(x-\theta)^k}\left[1-e^{-\gamma^k(x-\theta)^k}\right]^{i-1}\left[e^{-\gamma^k(x-\theta)^k}\right]^{n-1} \quad (37)$$

$$f(X_i) = \frac{n!}{(i-1)!(n-i)!} k\gamma^k \sum_{p=0}^{\infty}\binom{k-1}{p}(-1)^p\theta^p e^{-\gamma^k(x-\theta)^k}\sum_{q=0}^{\infty}\binom{i-1}{q}(-1)^q\left(e^{-\gamma^k(x-\theta)^k}\right)^q \quad (38)$$

$$f(X_i) = \frac{n!}{(i-1)!(n-i)!} k\theta^p\gamma^k \sum_{p,q=0}^{\infty}\binom{k-1}{p}\binom{i-1}{q}(-1)^{p+q}\left(e^{-\gamma^k(x-\theta)^k}\right)^{1+q+n-i}$$

Therefore the first and nth order statistics for W-HLa{E} distribution can be determined as thus

Corollary 2 *(nth Order Statistic) Let X be a random variable that follows W-HLa{E} distribution*$(x; k, \theta, \gamma)$ *and let* $f(x_1)$ *denote the first (1st) order statistic of the W-HLa{E} distribution. Then* $f(x_1)$ *is given by*

$$f(X_1) = \frac{n}{(q)!(-q)!} k\theta^p\gamma^k \sum_{p,q=0}^{\infty}\binom{k-1}{p}(-1)^{p+q}\left(e^{-\gamma^k(x-\theta)^k}\right)^{n+q}) \quad (39)$$

Where i=1,2,3,…,n

Proof: From equation (35), replace i with 1 we have equation (39)

Corollary 3 *(nth Order Statistic) Let X be a random variable that follows W-HLa{E} distribution$(x; k, \theta, \gamma)$ and let $f(x_n)$ denote the last (nth) order statistic of the W-HLa{E} distribution. Then $f(x_n)$ is given by*

$$f(X_i) = \frac{n}{(n-1-q)!(q)!} k\theta^p \gamma^k \sum_{p,q=0}^{\infty} \binom{k-1}{p} (-1)^{p+q} \left(e^{-\gamma^k (x-\theta)^k} \right)^{1+q}) \tag{40}$$

Where i=1,2,3,…,n

Proof: From equation (35), replace i with n we have equation (40)

3.8 Entropy

In information theory, entropy is an important concept and can be defined as a measure of the randomness or uncertainty associated with a random variable. However, the Shannon entropy for a random variable X with pdf $f_X(x)$ is defined as $E\{-log(f_X(x))\}$

Proposition 6 *If X is a random variable that has W-HLa{E} distribution$(x; k, \theta, \gamma)$ then Shannon's entropy is given as*

$$E\{-log(f_X(x))\} = \gamma^k - (k-1)\left(\frac{0.57722}{k}\right) - log(\gamma) - log(k)$$

where $k, \theta, \gamma \geq 0$ are parameters of W-HLa{E} distribution and $\Phi = -\int_0^{\infty} logze^{-z}dz \approx 0.57722$ is the Euler gamma constant

Proof: Substitute equation (10) in the definition of entropy we have

$$E\{-log(f_X(x))\} = E\left\{-log\left(k\gamma^k (x-\theta)^{k-1} e^{\gamma^k (x-\theta)^k}\right)\right\} \tag{41}$$

$$E\{-log(f_X(x))\} = E\{-[log(k) + klog(\gamma) + (k-1)log(x-\theta) - \gamma^k (x-\theta)^k]\}$$

$$= E\{\gamma^k (x-\theta)^k - [log(k) + klog(\gamma) + (k-1)log(x-\theta)]\} \tag{42}$$

Finding the expectation of $(x-\theta)$ with the proposed distribution W-HLa{E}, we have,

$$E((x-\theta)^k) = \int_{\theta}^{\infty} (x-\theta)^k k\gamma^k (x-\theta)^{k-1} e^{-\gamma^k (x-\theta)^k} dx \tag{43}$$

Recall Equation (22) and (23)and substitute them in equation (43)

$$E((x-\theta)^k) = \int_{\theta}^{\infty} (x-\theta)^k k\gamma^k (x-\theta)^{k-1} e^{-\gamma^k (x-\theta)^k} \times \frac{dz}{k\gamma^k (x-\theta)^{k-1}} \tag{44}$$

$$E((x-\theta)^k) = \int_0^{\infty} (x-\theta)^k e^{-\gamma^k (x-\theta)^k} dz \tag{45}$$

Substituting Equation (26) in Equation (45) and integrating the expression

$$E((x-\theta)^k) = \int_0^{\infty} ze^z dz = [-ze^z - e^z]_0^{\infty} = 1 \tag{46}$$

Hence

$$E(\gamma^k (x-\theta)^k) = \gamma^k E((x-\theta)^k) = \gamma^k \tag{47}$$

Also

$$E((x-\theta)^k) = \int_{\theta}^{\infty} log(x-\theta)^k f_X(x) dx \tag{48}$$

Therefore

$$E(log(x-\theta)^k) = \int_{\theta}^{\infty} log(x-\theta)^k k\gamma^k (x-\theta)^{k-1} e^{-\gamma^k (x-\theta)^k} dx \tag{49}$$

Substitute Equation (22) and (23)and substitute them in equation (49)

$$E(log(x-\theta)^k) = \int_{\theta}^{\infty} log(x-\theta)^k k\gamma^k (x-\theta)^{k-1} e^{-\gamma^k (x-\theta)^k} \times \frac{dz}{k\gamma^k (x-\theta)^{k-1}} \tag{50}$$

Then equation (50) becomes

$$= \int_{\theta}^{\infty} log(x-\theta)^k e^{-z} dz \tag{51}$$

$$= \int_{\theta}^{\infty} log\left(\frac{z^{\frac{1}{k}}}{\gamma}\right) e^{-z} dz \tag{52}$$

$$= \int_0^{\infty} log\left(z^{\frac{1}{k}}\right) e^{-z} dz + \int_0^{\infty} log\left(\frac{1}{\gamma}\right) e^{-z} dz$$

$$= \int_0^{\infty} log\left(z^{\frac{1}{k}}\right) e^{-z} dz - \int_0^{\infty} log(\gamma) e^{-z} dz \tag{53}$$

$$= \int_0^{\infty} log\left(z^{\frac{1}{k}}\right) e^{-z} dz - log(\gamma) \int_0^{\infty} e^{-z} dz \tag{54}$$

$$= \frac{1}{k}\int_0^{\infty} log(z) e^{-z} dz - log(\gamma) \int_0^{\infty} e^{-z} dz \tag{55}$$

Let $\Phi = -\int_0^{\infty} log(z)e^{-z}dz$ then equation (55) becomes

$$E(log(x-\theta)^k) = -\frac{\Phi}{k} - log(\gamma) \tag{56}$$

Substitute Equations (47) and (56) in Equation (42), we have

$$E\{-log(f_X(x))\} = \gamma^k - (k-1)\left(-\frac{\Phi}{k} - log(\gamma)\right) - \left(log(k) + klog(\gamma)\right) \tag{57}$$

$$E\{-log(f_X(x))\} = \gamma^k + (k-1)\left(\frac{\Phi}{k} + log(\gamma)\right) - log(k) - klog(\gamma)$$

$$E\{-log(f_X(x))\} = \gamma^k + (k-1)\left(\frac{\Phi}{k}\right) - log(\gamma) - log(k)$$

where $k, \theta, \gamma \geq 0$ are parameters of W-HLa{E} distribution and $\Phi = -\int_0^\infty logze^{-z}dz \approx 0.57722$ is the Euler gamma constant from Abramowitz [1]

$$E\{-log(f_X(x))\} = \gamma^k + (k-1)\left(\frac{0.57722}{k}\right) - log(\gamma) - log(k) \tag{58}$$

3.9 Mean Residual life

The Mean Residual Life (MRL) at a given time *t* measures the expected remaining life of an individual of age t. it otherwise called the life expectancy.

Proposition 7: *Let T be a random variable that follows W-HLa{E} distribution*(t, k, γ, θ) *and let* MRL *represents mean residual life at a given time t W-HLa{E} distribution. Then* MRL *is given by*

$$\text{MRL} = \frac{1}{e^{-(\gamma)^k(t-\theta)^k}}\left\{\frac{1}{\gamma}\Gamma\left(1+\frac{1}{k}\right) + \theta e^{-t} - \frac{1}{\gamma}\Gamma\left(1+\frac{1}{k}, t\right)\right\} - \text{t}$$

where $k, \gamma, \theta \geq 0$ are parameters of *t W-HLa{E}* distribution

Proof: Given

$$\text{MRL} = \frac{1}{1-S(t)}\left\{E(x) - \int_0^t \text{t}f_X(t)dt\right\} - \text{t} \tag{59}$$

And $S_X(x) = 1 - \text{F(x)}$

From equation (8) and (34)

$$F_X(t) = 1 - e^{-(\gamma)^k(t-\theta)^k}, \quad E(X) = \theta + \frac{1}{\gamma}\Gamma\left(1+\frac{1}{k}\right)$$

$$\text{MRL} = \frac{1}{e^{-(\gamma)^k(x-\theta)^k}}\left\{\theta + \frac{1}{\gamma}\Gamma\left(1+\frac{1}{k}\right) - \int_0^t \text{t}f_X(t)dt\right\} - \text{t} \tag{60}$$

$$\int_0^t \text{t}f_X(t)dt = \int_0^t \left(\theta + \frac{z^{1/k}}{\gamma}\right)e^{-t}dt$$

$$= \int_0^t \theta e^{-t}dt + \int_0^t \frac{t^{1/k}}{\gamma}e^{-t}dt$$

$$= -\theta e^{-t}{}_0^t + \frac{1}{\gamma}\gamma\left(1+\frac{1}{k}, t\right)$$

$$\int_0^t \text{t}f_X(t)dt = \theta - \theta e^{-t} + \frac{1}{\gamma}\Gamma\left(1+\frac{1}{k}, t\right) \tag{61}$$

Substituting equations (61) in equation (60)

$$\text{MRL} = \frac{1}{e^{-(\gamma)^k(x-\theta)^k}}\left\{\theta + \frac{1}{\gamma}\Gamma\left(1+\frac{1}{k}\right) - \left(\theta - \theta e^{-t} + \frac{1}{\gamma}\Gamma\left(1+\frac{1}{k}, t\right)\right)\right\} - \text{t}$$

Where $\Gamma\left(1+\frac{1}{k}, t\right)$ is an incomplete gamma function of variable t.

$$\text{MRL} = \frac{1}{e^{-(\gamma)^k(t-\theta)^k}}\left\{\frac{1}{\gamma}\Gamma\left(1+\frac{1}{k}\right) + \theta e^{-t} - \frac{1}{\gamma}\Gamma\left(1+\frac{1}{k}, t\right)\right\} - \text{t}$$

4. SIMULATION STUDY

The simulation was done using equation (11) which is the quantile function of W-HLa{E}. Let *p be* a uniform random variable on (0,1), then $X = \theta + \frac{1}{\gamma}\{-log(1-p)\}^{\frac{1}{k}}$, the descriptive summaries were obtained through Statistical software R3.4.4.version.

Table 1: *Descriptive summaries of simulation of W-HLa{E} distribution with various parameters*

Model	Parameters	Mean	Median	Max.	Variance	skewness	kurtosis	Cv
y_1	k=0.5; γ=0.5; θ=5	6.801	5.453	21.337	14.239	3.115	12.309	0.554
y_2	k=0.5; =0.5;θ=10	11.8	10.45	26.34	14.239	3.115	12.309	0.320
y_3	k=1; γ=3; θ=5	10.22	10.16	10.95	0.055	1.748	5.827	0.023
y_4	k=1.5; γ=1; θ=5	5.67	5.609	7.014	0.251	1.040	3.751	0.088
y_5	k=2; γ=2; θ=10	10.35	10.34	10.85	0.042	0.612	3.008	0.036
y_6	k=2; γ=2; θ=5	5.352	5.345	5.845	0.042	0.612	3.008	0.036
y_7	k=3; γ=1; θ=5	5.76	5.781	6.419	0.097	0.127	2.625	0.054
y_8	k=3; γ=1; θ=10	10.76	10.78	11.42	0.097	0.127	2.625	0.029

From table 1 it is observed that as parameter k increases the variance, skewness, and kurtosis function decreases for different values of the scale and location parameters of W-HLa{E} distribution hence, the skewness and kurtosis are decreasing functions of k.
When k=2 and γ=2, the mean is approximately equal to the median hence, the distribution tends to be symmetric. Table 1 also shows that the coefficient of variation is deeply affected by γ.
When the γ increases positively the value of the coefficient of variation decreases

5. ESTIMATION OF PARAMETERS FOR THE W-HLA{E} DISTRIBUTION

The Maximum Likelihood estimates of W-HLa{E} distribution will be obtained in this section
Definition: Let $x_1, x_2, \ldots, x_n$ denote a random sample drawn from W-HLa{E} distribution with parameters k, γ, θ. The likelihood function l (x, k, γ, θ) of W-HLa{E} distribution is defined to be the joint density of the random variables $x_1, x_2, \ldots, x_n$

$$l(x,k,\gamma,\theta) = \prod k\gamma^k (x-\theta)^{k-1} e^{\gamma^k (x-\theta)^k} \tag{62}$$

$$= k^n \gamma^{nk} \prod (x-\theta)^{k-1} e^{-\gamma^k (x-\theta)^k} \tag{63}$$

finding the loglikelihood function of equation (63), we have

$$logl(x,k,\gamma,\theta) = \log k^n + \log\gamma^{nk} + \sum_i^n \log(x-\theta)^{k-1} + \sum_i^n -\gamma^k((x-\theta)^k) \tag{64}$$

where L=logl(x, k, γ, θ)

$$L = n\log k + nk\log\gamma + (k-1)\sum_i^n \log(x-\theta) - \gamma^k \sum_i^n ((x-\theta)^k) \tag{66}$$

$$L = n\log k + nk\log\gamma + (k-1)\sum_i^n \log(x-\theta) - \gamma^k \sum_i^n ((x-\theta)^k) \tag{66}$$

Differentiating the log-likelihood function in (66) with respect to the parametersK, γ, θ we have

$$\frac{dL}{d\gamma} = \frac{nk}{\gamma} - k\gamma^{k-1} \sum_i^n (x_i-\theta)^k \tag{67}$$

$$\frac{dL}{dk} = \frac{n}{k} + n\log\gamma \sum_i^n (x_i-\theta)^k - \gamma^k \sum_i^n (x_i-\theta)^k \log(x_i-\theta) \tag{68}$$

$$\frac{dL}{d\theta} = k\gamma^k \sum_i^n (x_i-\theta)^{k-1} - (k-1)\sum_i^n \frac{1}{(x_i-\theta)} \tag{69}$$

The maximum likelihood estimates (MLE), $\hat{k}, \hat{\gamma}, \hat{\theta}$ for the parameters k, γ, θ respectively, are obtained by setting (67) - (69) to zero and solving them simultaneously.

6 APPLICATIONS

In this section, we shall be investing the importance of the new distribution W-HLa{E} distribution and apply it to three real-life data. In these applications, the maximum likelihood estimation method is used in these two applications to estimate the parameters of fitted distributions. The maximized log-likelihood, the Kolmogorov-Smirnov test (K-S) along with the corresponding p-value, the Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC) are reported to compare the W-HLa{E} distribution with the other distributions.

DATA SET I: Remission times of bladder cancer patients

Remission Time is the time taken for the signs and symptoms of a particular disease, in this case, cancer, to decrease or disappear after treatment. Though cancer may be considered in remission, cancer cells may remain in the body.

Table 2: *Remission times (in months) of bladder cancer patients' data*

0.080 0.200 0.400 0.500 0.510 0.810 0.900 1.050 1.190 1.260 1.350 1.400 1.460 1.760 2.020
2.020 2.070 2.090 2.230 2.260 2.460 2.540 2.620 2.640 2.690 2.690 2.750 2.830 2.870 3.020
3.250 3.310 3.360 3.360 3.480 3.520 3.570 3.640 3.700 3.820 3.880 4.180 4.230 4.260 4.330
4.340 4.400 4.500 4.510 4.870 4.980 5.060 5.090 5.170 5.320 5.320 5.340 5.410 5.410 5.490
5.620 5.710 5.850 6.250 6.540 6.760 6.930 6.940 6.970 7.090 7.260 7.280 7.320 7.390 7.590
7.620 7.630 7.660 7.870 7.930 8.260 8.370 8.530 8.650 8.660 9.020 9.220 9.470 9.740 10.06
10.34 10.66 10.75 11.25 11.64 11.79 11.98 12.02 12.03 12.07 12.63 13.11 13.29 13.80 14.24
14.76 14.77 14.83 15.96 16.62 17.12 17.14 17.36 18.10 19.13 20.28 21.73 22.69 23.63 25.74
25.82 26.31 32.15 34.26 36.66 43.01 46.12 79.05

Table 2 shows data of remission times(in month) of 128 bladder cancer patients selected at random as reported by Lee, et al [12], which was studied by Zea [16] to compare the fits of a different family of beta-Pareto(BP) and beta exponentiated Pareto (BEP) distributions. Almheidat [4] also applied to four parameters Cauchy-Weibull $\{logistic\}$ $(C-W\{L\})$ distribution in fitting this same data and just of recent Aldeni [2] applied uniform-exponential$\{generalisedlambda\}$ distribution $(U-E\{GL\})$ distribution to fit the same data

Table 3: *Descriptive Statistics of remission times of bladder cancer patients distributions*

Min.	Max.	Mean	1st Qu.	Median	3rd Qu.	Skewness	Kurtosis	SD
0.080	79.050	9.366	3.348	6.395	11.838	3.287	18.483	10.508

Table 4: *Performance of the distributions remission times of bladder cancer patients distributions Parameter estimates: Log-likelihood, AIC, and p-value (Standard errors in parentheses)*

Distributions	$WHLa\{E\}$	$U-E\{GL\}$	$C-W\{L\}$	BEP	BP
	k = 0.4847 (0.0267)	θ=0.2757 (0.0665)	α=2.3040 (1.0937)	a = 0.348 (0.0970)	a = 4.805 (0.0550)
	$\gamma = 0.4850$ (0.0651)	$\lambda_3 = 2.504$ (0.9285)	$\beta = 2.0205$ (0.4585)	b =159831 (183.7501)	b = 100.502 (0.2510)
	$\mu = 0.0785$ (-)	$\lambda_4 = 0.2894$ (0.0858)	k=3.0673 (0.7319)	k= 0.051 (0.0190)	k = 0.011 (0.0010)
			$\lambda = 12.663$ (2.6326)	$\beta = 0.080$ (2.0930)	$\beta = 0.080$

Table 5: *Performance of the distributions remission times of bladder cancer patients distributions Parameter estimates: Log-likelihood, AIC, and p-value*

Distributions	$WHLa\{E\}$	$U-E\{GL\}$	$C-W\{L\}$	BEP	BP
-Loglikelihood	271.3276	409.45	416.0965	432.41	480.446
AIC	548.6552	824.9	840.2	874.819	968.893
K-S	0.0078125	0.02876	0.06672	0.142	0.217
P-value	0.9922	0.9999	0.6189	0.0121	1.11E-05

Table 3 shows the summary of the dataset I and table 4 displaces the parameters estimates of W-HLa$\{E\}$ and four other distributios. Table 5 shows the values of parameter estimates, log-likelihood, AIC, K-S, and its p-value at 95%. Based on the above test statistics, W-HLa$\{E\}$ has the least AIC with 548.6552 and K-S Statistic (0.0078125) hence W-HLa{E} performed best among the five distribution models applied to remission time of bladder cancer. This implies that the new distribution can fit skewed data with long-tail better than any distribution. After an appropriate distribution has been identified and parameters estimated, we can estimate the probability of having a given duration of remission and other probabilities. For example, the probability of having a remission time longer than 10 months can be predicted as $P(X > x) = e^{-\gamma^k(x-\theta)^k}$ When $k = 0.4847, \gamma = 0.4850, \theta = 0.0785$

$$P(X > x) = e^{-0.4850^{0.4847}(10-0.0785)^{0.4847}}$$
$$P(X > 10) = 0.117$$

Data Set II: 72 pigs infected by virulent tubercle Bacilli (Bjerkedal, T, 1960)

The data in Table 6 are survival times (in days) of seventy-two pigs infected by virulent tubercle bacilli [9] The data in Table 6 are survival times (in days) of seventy-two pigs infected by virulent tubercle bacilli reported by Tahir [15] The Tables 4.4 shows the performance of W-HLa{E} and three other models (Logistic Frechet (LFr), Marshall-Olkin Frechet (MOFr), exponentiated-Frechet (EFr) and Frechet (Fr).

Table 6: *Infected Pigs data (in day)*

10 33 44 56 59 72 74 77 92 93 96 100 100 102 105 107 107 108 108 108 109 112
113 115 116 120 121 122 122 124 130 134 136 139 144 146 153 159 160 163 163 168
171 172 176 183 195 196 197 202 213 215 216 222 230 231 240 245 251 253 254 254
278 293 327 342 347 361 402 432 458 555

Table 7: *Descriptive Statistics of Infected Pigs data*

Min.	Max.	Mean	1st Qu.	Median	3rd Qu.	Skewness	Kurtosis	SD
43.00	598.00	141.85	82.75	102.50	149.25	2.5153	9.332	109.209

In this application, we obtain the descriptive statistics, maximum likelihood estimates of the parameters of the fitted distributions, and the values of the following statistics: AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion), and HQIC (Hanna Quinn Information Criterion).

In addition, we compute some goodness of fit statistics to verify which distribution provides the best fit to the data sets. We apply Kolmogorov-Smirnov (K-S) statistics. These statistics are described in detail in Tables 8 and 9. In general, the smaller the value of these statistics, the better the fit of the data by the distribution.

Table 8: *Parameter Estimates and Standard errors in parentheses for W-HLa{E} Distribution*

Distributions	$WHLa\{E\}$	LFr	$MOFr$	EFr
	$\theta = 42.99956$ (16.8814)	λ=32.5054 (0.0665)	a=212.7251 (115.7064)	a= 155.680 (5.7063)
Parameter Estimates	$\gamma = 0.4850$ (0.0651)	$\lambda_3 = 2.504$ (0.9285)	$\beta = 2.0205$ (0.4585)	$b = 159831$ (183.7501)
	$\mu = 0.0785$ (-)	$\lambda_4 = 0.2894$ (0.0858)	k=3.0673 (0.7319)	k= 0.051 (0.0190)
			$\lambda = 12.663$ (2.6326)	$\beta = 0.080$ (2.0930)

$skewness = 1.841076$ and $kurtosis = 7.49277$

Table 9: *Log-likelihood, AIC, BIC, HQIC, KS Statistic and p-value of 72 pigs infected by virulent tubercle Bacilli of different distributions*

Distributions	$WHLa\{E\}$	LFr	$MOFr$	EFr
-Loglikelihood	410.3371	426.2306	426.6764	449.7452
AIC	826.6741	858.4612	859.3527	903.4903
BIC	833.5041	865.2912	866.1827	908.0437
$HQIC$	829.3932	861.1803	866.1827	905.3030
K-S	0.027778	0.0695	0.1213	0.0923
P-value	0.9460	0.8773	0.2403	0.5710

From table 9, the new proposed W-HLa{E} model corresponds to the lowest values of the loglikelihood, AIC, BIC, HQIC, and K-S statistics among the fitted LFr, MOFr, and EFr models and therefore the W-HLa{E} model can be chosen as the best for the data set above. The distribution with the lowest Akaike Information Criteria (AIC) or BIC and the lowest Log-likelihood value is declared as ˜" best fit" distribution. In this case, W-HLa{E} distribution has the lowest Log-likelihood of -410.3371 with the lowest corresponding lowest AIC value of 826.6741. Hence, W-HLa{E}D is regarded as a best-fit model for this particular data used.

7. CONCLUSION

The research work introduced a new probability distribution called Weibull Half Laplace exponential distribution. Expressions for the probability density function, cumulative distribution function, survival function, and hazard function, and cumulative hazard function of the proposed distribution are derived. Some properties of the proposed distribution such as moments, order of statistics, and Shannon entropy have been studied. The simulation also supported the theoretical expression of the statistical properties of the proposed distribution such as the location parameter does not affect the variance, skewness, and kurtosis of the new distribution. From table 1, when k=2 and γ=2, the mean is approximately equal to the median hence, the distribution tends to be symmetric. The maximum likelihood method is adopted to estimate the parameters of the distribution. Coefficient of variation is deeply affected by γ. When the γ increases positively, the value of the coefficient of variation decreases It is shown, by means of two real data sets.
Two life data were applied to the new distribution and others established distributions by researchers in the field of probability distribution and we found out that W-HLa{E} distribution has the lowest log-likelihood and the smallest AIC, BIC, and HQIC for the two data sets in tables 6 and 9. W-HLa{E} distribution is a better fit for the two data sets.

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