A New Pi-Exponentiated Method for Constructing Distributions with an Application to Weibull Distribution

M. A. Lone*

•

Department of Statistics, University of Kashmir, Srinagar, India
murtazastat@gmail.com

T. R. Jan

•

Department of Statistics, University of Kashmir, Srinagar, India
drtrjan@gmail.com

Abstract

A novel method for generating families of continuous distributions is presented by introducing a new parameter referred as Pi-Exponentiated Transformation (PET). Various properties of the PET method have been obtained. The method has been specialized on two-parameter Weibull distribution, and a new distribution called Pi-Exponentiated Weibull (PEW) is attained. A comprehensive mathematical treatment of the new proposal is provided. Closed-form expressions for the density function, distribution function, reliability function, hazard rate function have been provided. The PEW distribution is quite flexible, and it can be used to model data with decreasing, increasing or bathtub shaped hazard rates. Simulation study has been carried out to assess the behavior of the model parameters. Finally, the effectiveness of the suggested method is demonstrated by examining two real-life data sets.

Keywords: Pi-Exponentiated Transformation; Quantile Function; Reliability Function; Mean Waiting Time; Maximum Likelihood Estimation.

1. Introduction

Classical distributions are extensively employed in many applicable domains, including engineering, environmental studies, medical sciences, economics, actuarial, finance, insurance etc. to represent lifetime data. These distributions have been successfully implemented in all the fields listed above. However, in many domains, like reliability engineering and medical science, these conventional distributions do not offer the perfect fit when the data follow non-monotonic failure rates. As a result, generalized versions of these classical distributions are required to model reliability engineering and medical science data. Therefore, researchers became inspired to develop new modifications to theses existing distributions. These modified distributions offer more flexibility to the baseline model by introducing one or more extra parameters. In recent advances in distribution theory, researchers have shown a keen interest in proposing new methods for expanding the family of lifetime distributions. This has been accomplished through a variety of methods. Some well-known methods are:

- The exponentiated transformation initiated by Mudholkar and Srivastava [16], and is given by
$$F(x;\alpha) = (\psi(x))^{\alpha};\ \alpha > 0, x \in \mathbb{R}.$$
Where $\psi(x)$ is the cumulative distribution function (cdf) of baseline model.

- The beta-generated technique was proposed by Eugene et al. [7] that makes use of the beta distribution as the generator with parameters a and b to establish the beta generated distributions.

$$F(x) = \int_{0}^{\psi(x)} r(s)ds.$$

Where $r(s)$ is the probability density function (pdf) of a beta random variable (rv) and $\psi(x)$ is the cdf of any rv X.

- The quadratic rank transmutation map approach proposed by Shaw and Buckley [19] and is given as

$$F(x;\xi) = (1+\xi)\psi(x) - \xi\psi(x)^2, \ |\xi| \leq 1, \ x \in \mathbb{R}.$$

Where $\psi(x)$ is the cdf of an existing distribution.

- Minimum Guarantee distribution proposed by Kumar et al. [9] and is given by

$$F(x) = e^{1-\frac{1}{\psi(x)}}, \ x \in \mathbb{R}.$$

Where $\psi(x)$ is the cdf of an existing distribution.

- Log-transformation proposed by Maurya et al. [15] and is given by

$$F(x) = 1 - \frac{log(2-\psi(x))}{log2}, \ x \in \mathbb{R}.$$

Where $\psi(x)$ is the cdf of an existing distribution.

- A new transmuted cumulative distribution function based on the Verhulst logistic function proposed by Kyurkchiev [10] and is given by

$$F(x) = \frac{2\psi(x)}{1+\psi(x)}.$$

Where $\psi(x)$ is the cdf of an existing distribution.

- Marshall and Olkin [14] proposed a general method for generating a new family of life distributions defined in terms of survival function as:

$$\bar{F}(x;\alpha) = \frac{\alpha\bar{\psi}(x)}{1-\bar{\alpha}\bar{\psi}(x)} = \frac{\alpha\bar{\psi}(x)}{\psi(x)+\alpha\bar{\psi}(x)} \quad ;\alpha > 0, x \in \mathbb{R}.$$

Where $\bar{\alpha} = 1-\alpha$ and $\bar{\psi}(x) = 1-\psi(x)$ is the survival function of the random variable X.

- Anwar et al. [8] presented a new method based on trigonometric function called Sine-Exponentiated-Transformation (SET). The cdf of SET family of distributions for $x \in \mathbb{R}$ is defined as

$$F_{SET}(x,\alpha) = \psi(x)\sin\left(\frac{\pi}{2}\psi^{\alpha}(x)\right) \ ;\alpha \geq 0.$$

Where $\psi(x)$ is the cdf of a continuous rv X.

- Lone et al. [11] proposed a new method for generating a family of continuous distributions called ratio trasformation (RT) method. The cdf of RT method for $x \in \mathbb{R}$ is defined as

$$F_{RT}(x;\alpha) = \frac{\psi(x)}{1+\alpha-\alpha^{\psi(x)}} \;;\alpha > 0.$$

  Where $\psi(x)$ is the cdf of a continuous rv X.

- Recently, Lone et al. [12] introduced an innovative method for generating a family of continuous distributions called the MTI method. They employed MTI method on Weibull distribution and derived a new three-parameter MTI Weibull (MTIW) distribution. The cdf of MTI method for $x \in \mathbb{R}$ is defined as

$$F_{MTI}(x;\alpha) = \frac{\alpha\psi(x)}{\alpha - log\alpha\ \bar{\psi}(x)} \;;\alpha > 0.$$

  Where $\bar{\psi}(x) = 1 - \psi(x)$ is the survival function of the random variable X.

In this manuscript a novel method for introducing greater flexibility to a family of distribution functions by bringing in new parameter to the given family has been introduced. This novel method has been refereed as PET. The proposed PET transformation is very simple and efficient method for introducing a new parameter to generalize the existing distributions. Some general properties of this class of distribution functions have been discussed. Then PET method has been specialized to a two-parameter Weibull distribution and generated a three-parameter PEW distribution, several statistical and reliability measures of PEW distribution have been obtained.

In section 2, the pdf and the cdf of the novel method have been obtained and various general properties of this method have been discussed. In section 3, the method has been specialized on two-parameter Weibull distribution and its structural properties as well as reliability measures have been obtained. In section 4, estimates of unknown parameters and simulation study have been performed. In section 5, two real data sets were analyzed to illustrate the efficacy of the suggested model. In section 6, the conclusion is stated.

2. General properties of PET method

Let X be a continuous rv, then the cdf of PET for $x \in \mathbb{R}$, is defined as

$$F_{PET}(x) = \frac{\pi^{(F(x))^{\alpha}} - 1}{\pi - 1}\;; \qquad \alpha > 0. \tag{1}$$

Obviously, $F_{PET}(x)$ is a valid cdf only if $F(x)$ is a valid cdf. The corresponding pdf of PET for $x \in \mathbb{R}$, is defined as

$$f_{PET}(x) = \frac{\alpha log\pi}{\pi - 1}\pi^{(F(x))^{\alpha}}(F(x))^{\alpha-1}f(x)\;; \qquad \alpha > 0. \tag{2}$$

Clearly, $f_{PET}(x)$ is a weighted version of $f(x)$, the weight function is given by

$$v(x) = \pi^{(F(x))^{\alpha}}(F(x))^{\alpha-1}.$$

Therefore, $f_{PET}(x)$ can be written as

$$f_{PET}(x) = \frac{f(x)v(x)}{k}.$$

Where, $k = E[v(X)]$ is the normalizing constant.

By using the following power series

$$\alpha^u = \sum_{j=0}^{\infty} \frac{(log\alpha)^j}{j!} u^j, \tag{3}$$

the linear representation for the cdf and the pdf in (1) and (2) are respectively given by

$$F_{PET}(x) = \frac{1}{\pi - 1} \left[\sum_{j=0}^{\infty} a_j (F(x))^{\alpha j} - 1 \right]$$

and

$$f_{PET}(x) = b \sum_{j=0}^{\infty} a_j (F(x))^{\alpha(j+1)-1} f(x).$$

Where, $a_j = \frac{(log\alpha)^j}{j!}$ and $b = \frac{\alpha log\pi}{\pi-1}$.

The reliability function $R_{PET}(x)$ is given by

$$R_{PET}(x) = \frac{\pi}{\pi - 1} \left(1 - \pi^{(F(x))^\alpha - 1} \right) \ ; \qquad \alpha > 0. \tag{4}$$

The hazard rate function $h_{PET}(x)$ is given by

$$h_{PET}(x) = \frac{\alpha log\pi f(x) (F(x))^{\alpha-1}}{\pi^{1-(F(x))^\alpha} - 1} \ ; \qquad \alpha > 0. \tag{5}$$

If $h(x)$ and $R(x)$ are the hazard rate function and reliability function of f then the hazard rate $h_{PET}(x)$ is given by

$$h_{PET}(x) = \alpha log\pi h(x) R(x) \frac{(F(x))^{\alpha-1}}{\pi^{1-(F(x))^\alpha} - 1} \ ; \qquad \alpha > 0. \tag{6}$$

From (6), it is clear that

$$\lim_{x \to -\infty} h_{PET}(x) = \begin{cases} 0 & \forall\ \alpha > 1 \\ \frac{log\pi}{\pi-1} \lim_{x \to -\infty} h(x) & \forall\ \alpha = 1 \\ \infty & \forall\ \alpha < 1 \end{cases}$$

and

$$\lim_{x \to \infty} h_{PET}(x) = \lim_{x \to \infty} h(x).$$

If $F^{-1}(x)$ exists, then for $\alpha > 0$, a random sample from $F_{PET}(x)$ can be obtained as

$$X = F^{-1} \left\{ \left(\frac{log(1 + U(\pi - 1))}{log\pi} \right)^{\frac{1}{\alpha}} \right\}$$

where U is a uniform rv, $0 < u < 1$.

3. PEW distribution and its properties

A rv X has a three-parameter PEW distribution denoted by PEW(α, β, λ) with parameters α, β and λ, if the cdf and the pdf of X for $x > 0$, are respectively, given by

$$F_{PEW}(x) = \frac{\pi^{(1-e^{-\lambda x^\beta})^\alpha} - 1}{\pi - 1} \ ; \qquad \alpha, \beta, \lambda > 0 \tag{7}$$

and

$$f_{PEW}(x) = \frac{\alpha\lambda\beta log\pi}{\pi - 1} x^{\beta-1} e^{-\lambda x^{\beta}} \pi^{(1-e^{-\lambda x^{\beta}})^{\alpha}} (1 - e^{-\lambda x^{\beta}})^{\alpha-1} \; ; \qquad \alpha, \beta, \lambda > 0. \tag{8}$$

The linear representations for the cdf in (7)is given by (9).

$$F_{PEW} = \frac{1}{\pi - 1} \left(\sum_{k=0}^{\infty} a_m e^{-k\lambda x^{\beta}} - 1 \right). \tag{9}$$

Where

$$a_m = \sum_{j=0}^{\infty} (-1)^k \binom{j\alpha}{k} \frac{(log\pi)^j}{j!}.$$

The linear representations for the pdf in (8) is given by (10).

$$f_{PEW} = \sum_{k=0}^{\infty} b_m g(x). \tag{10}$$

Where

$$b_m = \sum_{j=0}^{\infty} \frac{(-1)^k \alpha (log\pi)^{j+1}}{(\pi - 1)(k+1) j!} \binom{\alpha(j+1) - 1}{k}$$

and

$$g(x) = (k+1)\lambda\beta x^{\beta-1} e^{-(k+1)\lambda x^{\beta}}.$$

Clearly, $g(x)$ is the Weibull distribution with scale parameter $(k+1)\lambda$ and shape parameter β.

The reliability and the hazard rate of PEW distribution for $x > 0$ are given by (11) and (12), respectively

$$R_{PEW}(x) = \frac{\pi}{\pi - 1} \left(1 - \pi^{(1-e^{-\lambda x^{\beta}})^{\alpha} - 1} \right) \; ; \qquad \alpha, \beta, \lambda > 0 \tag{11}$$

and

$$h_{PEW}(x) = \frac{\alpha\lambda\beta log\pi x^{\beta-1} e^{-\lambda x^{\beta}} (1 - e^{-\lambda x^{\beta}})^{\alpha-1}}{\pi^{1-(1-e^{-\lambda x^{\beta}})^{\alpha}} - 1} \; ; \; \alpha, \beta, \lambda > 0. \tag{12}$$

Figure 1 shows some PEW density graphs for various selected parameter values. Figure 2 depicts graphs of the hazard rate of the PEW distribution for different parameter values.

3.1. Simulation and Quantile

The PEW distribution can be simulated using inverse cdf method

$$X = \left\{ -\frac{1}{\lambda} log \left[1 - \left(\frac{log(1 + U(\pi - 1))}{log\pi} \right)^{\frac{1}{\alpha}} \right] \right\}^{\frac{1}{\beta}}.$$

Where U is a uniform rv, $0 < u < 1$. The q^{th} quantile of PEW distribution is given by

$$x_q = \left\{ -\frac{1}{\lambda} log \left[1 - \left(\frac{log(1 + q(\pi - 1))}{log\pi} \right)^{\frac{1}{\alpha}} \right] \right\}^{\frac{1}{\beta}}$$

The median can be obtained as

$$x_{0.5} = \left\{ -\frac{1}{\lambda} log \left[1 - \left(\frac{log(\frac{1}{2}(\pi + 1))}{log\pi} \right)^{\frac{1}{\alpha}} \right] \right\}^{\frac{1}{\beta}}$$

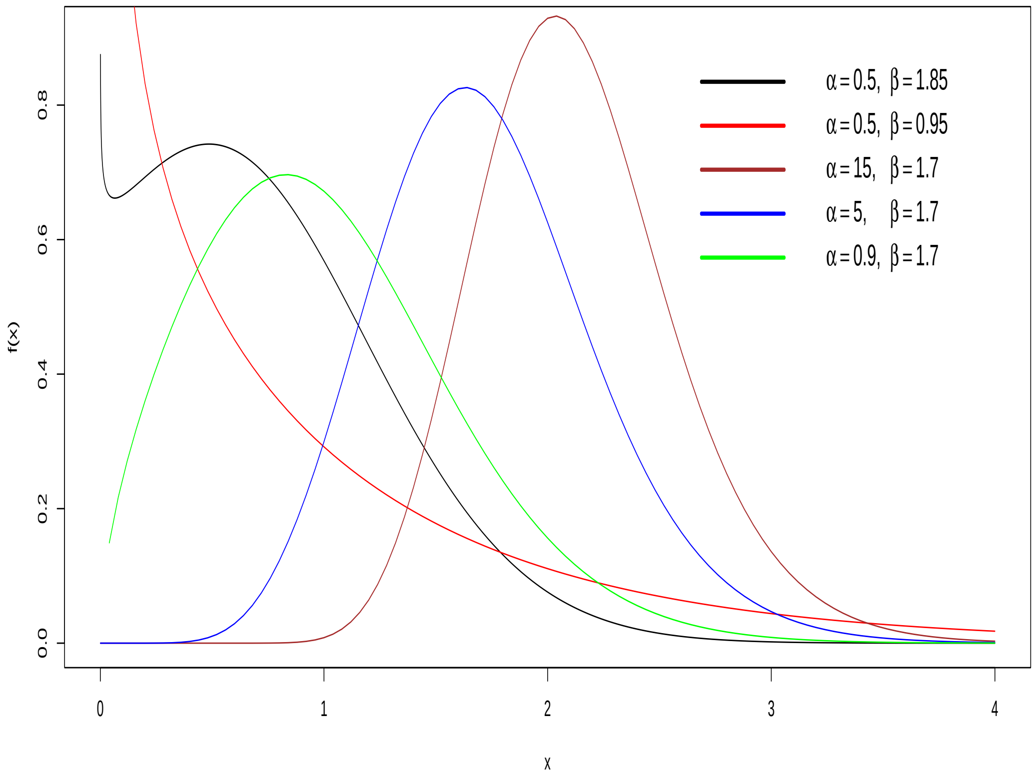

Figure 1: *Density plots of PEW for different combinations of* α, β *and* $\lambda = 1$.

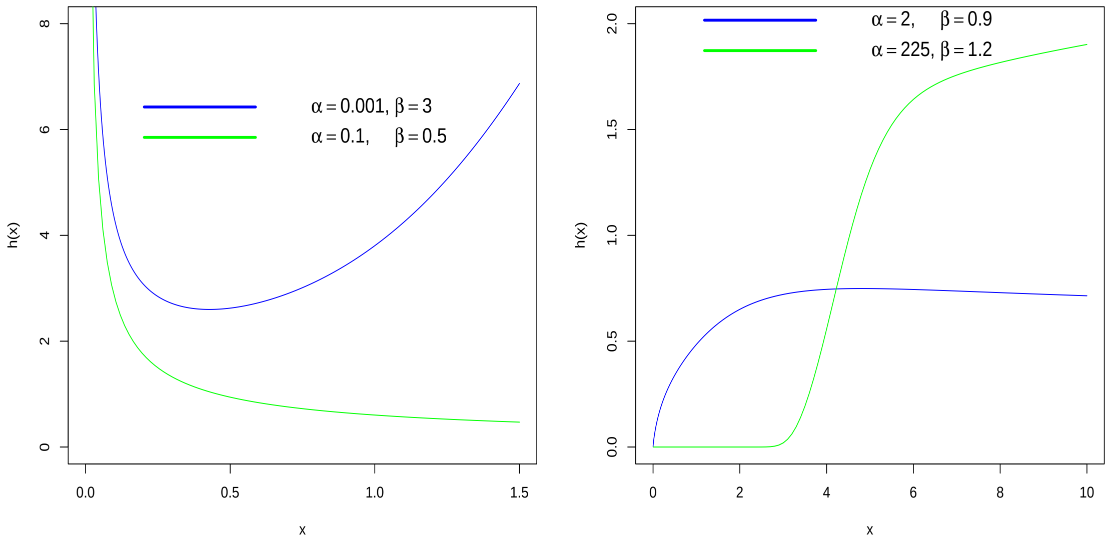

Figure 2: *Hazard rate plots of PEW for different combinations of* α, β *and* $\lambda = 1$.

3.2. Moments and generating function

The r^{th} moment of PEW distribution is obtained by using the following series representation.

$$\alpha^x = \sum_{k=0}^{\infty} \frac{(log\alpha)^k x^k}{k!} \tag{13}$$

$$(1-x)^{b-1} = \sum_{m=0}^{\infty} (-1)^m \binom{b-1}{m} x^m \,; \qquad |x| < 1,\ b > 0. \tag{14}$$

The r^{th} moment of X can be obtained as

$$\begin{aligned} E(X^r) &= \int_0^\infty x^r f(x) dx \\ &= \frac{\alpha\lambda\beta log\pi}{\pi-1} \int_0^\infty x^{r+\beta-1} e^{-\lambda x^\beta} \pi^{(1-e^{-\lambda x^\beta})^\alpha} (1-e^{-\lambda x^\beta})^{\alpha-1} dx. \end{aligned} \tag{15}$$

Using (13) and (14) in (15), we have

$$E(X^r) = \frac{\alpha\lambda\beta}{\pi-1} \sum_{a,m=0}^{\infty} \frac{(log\pi)^{a+1}(-1)^m}{a!} \binom{\alpha(a+1)-1}{m} \int_0^\infty x^{r+\beta-1} e^{-\lambda(m+1)x^\beta} dx. \tag{16}$$

By applying the transformation $x^\beta = y$ in (16), we get the final expression as

$$E(X^r) = \frac{\alpha}{\pi-1} \sum_{a,m=0}^{\infty} \frac{(-1)^m (log\pi)^{a+1}}{\lambda^{\frac{r}{\beta}} a!(m+1)^{\frac{r}{\beta}+1}} \binom{\alpha(a+1)-1}{m} \Gamma(\frac{r}{\beta}+1).$$

The moment generating function of PEW distribution is obtained as

$$M_X(t) = \int_0^\infty e^{tx} f(x) dx.$$

By using the same procedure as above, we get the final expression for moment generating function as

$$M_X(t) = \frac{\alpha}{\pi-1} \sum_{a,l,m=0}^{\infty} \frac{(-1)^m t^l (log\pi)^{a+1}}{\lambda^{\frac{l}{\beta}} l! a!(m+1)^{\frac{l}{\beta}+1}} \binom{\alpha(a+1)-1}{m} \Gamma(\frac{l}{\beta}+1)$$

3.3. The Mean residual life of PEW distribution

The mean residual life function, say $\mu(t)$ of PEW distribution can be obtained as

$$\mu(t) = \frac{1}{R(t)} \left(E(t) - \int_0^t x f(x) dx \right) - t. \tag{17}$$

Where

$$E(t) = \frac{\alpha}{\pi-1} \sum_{a,m=0}^{\infty} \frac{(-1)^m (log\pi)^{a+1}}{\lambda^{\frac{1}{\beta}} a!(m+1)^{\frac{1}{\beta}+1}} \binom{\alpha(a+1)-1}{m} \Gamma(\frac{1}{\beta}+1) \tag{18}$$

and

$$\begin{aligned} \int_0^t x f(x) dx =& \frac{\alpha}{\pi-1} \sum_{a,m=0}^{\infty} \frac{(-1)^m (log\pi)^{a+1}}{\lambda^{\frac{1}{\beta}} a!(m+1)^{\frac{1}{\beta}+1}} \\ & \times \binom{\alpha(a+1)-1}{m} \gamma\left(\lambda(m+1)t^\beta, \frac{1}{\beta}+1\right). \end{aligned} \tag{19}$$

Substituting (11), (18) and (19) in (17), we have

$$\mu(t) = \frac{\alpha}{\pi - \pi^{(1-e^{-\lambda x^{\beta}})^{\alpha}}} \sum_{a,m=0}^{\infty} \frac{(-1)^m (log\pi)^{a+1}}{\lambda^{\frac{1}{\beta}} a!(m+1)^{\frac{1}{\beta}+1}} \binom{\alpha(a+1)-1}{m}$$
$$\times \left[\Gamma\left(\frac{1}{\beta}+1\right) - \gamma\left(\lambda(m+1)t^{\beta}, \frac{1}{\beta}+1\right)\right] - t.$$

Where $\gamma(p,q) = \int\limits_0^p x^{q-1} e^{-x} dx$, is called lower incomplete gamma function.

The mean waiting time $\bar{\mu}(t)$ of PEW distribution, can be obtained as

$$\bar{\mu}(t) = t - \frac{1}{F(t)} \int\limits_0^t x f(x) dx. \tag{20}$$

Substituting (7) and (19) in (20), we get

$$\bar{\mu}(t) = t - \frac{\alpha}{\pi^{(1-e^{-\lambda x^{\beta}})^{\alpha}} - 1} \sum_{a,m=0}^{\infty} \frac{(-1)^m (log\pi)^{a+1}}{\lambda^{\frac{1}{\beta}} a!(m+1)^{\frac{1}{\beta}+1}}$$
$$\times \binom{\alpha(a+1)-1}{m} \gamma\left(\lambda(m+1)t^{\beta}, \frac{1}{\beta}+1\right)$$

3.4. Renyi Entropy

Renyi entropy of PEW distribution, say$RE_X(u)$ can be obtained as

$$RE_X(u) = \frac{1}{1-u} log\left(\int\limits_{-\infty}^{\infty} f(x)^u dx\right); \quad u > 0, \quad u \neq 1.$$
$$= \frac{1}{1-u} log\left(\int\limits_0^{\infty} \left(\frac{\alpha\lambda\beta log\pi}{\pi-1}\right)^u x^{u(\beta-1)} e^{-u\lambda x^{\beta}}\right.$$
$$\left. \times (1-e^{-\lambda x^{\beta}})^{u(\alpha-1)} \pi^{u(1-e^{-\lambda x^{\beta}})^{\alpha}} dx\right). \tag{21}$$

Using (13) in (21), we have

$$RE_X(u) = \frac{u}{1-u} log\left(\frac{\alpha\lambda log\pi}{\pi-1}\right) - log(\beta) + log\left(\sum_{a=0}^{\infty} \frac{(ulog\pi)^a}{a!}\right.$$
$$\left. \times \int\limits_0^{\infty} \beta x^{u(\beta-1)} e^{-u\lambda x^{\beta}} (1-e^{-\lambda x^{\beta}})^{\alpha(a+u)-u} dx\right). \tag{22}$$

Using (14) and applying the transformation $y = x^{\beta}$ in (22) , then the final expression for $RE_X(u)$ is given by

$$RE_X(u) = \frac{u}{1-u} log\left(\frac{\alpha\lambda log\pi}{\pi-1}\right) - log(\beta) + log\left(\sum_{a,m=0}^{\infty} \frac{(-1)^m (ulog\pi)^a}{a!}\right.$$
$$\left. \times \binom{\alpha(a+u)-u}{m} \frac{\Gamma\left(u+\frac{1-u}{\beta}\right)}{(\lambda(m+u))^{u+\frac{1-u}{\beta}}}\right)$$

3.5. Order Statistics

Let $X_1, X_2, ..., X_n$ be a random sample of size n, and let $X_{r:n}$ denote the r^{th} order statistic, then, the pdf of $X_{r:n}$, say $f_{r:n}(x)$ is given by

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} F(x)^{r-1} f(x)(1-F(x))^{n-r}. \tag{23}$$

Substituting (7) and (8) in (23) and using(14), we get

$$\begin{aligned} f_{r:n}(x) =& \frac{\alpha\lambda\beta log\pi}{B(r,n-r+1)} \sum_{a=0}^{n-r} \frac{(-1)^a \binom{n-r}{a}}{(\pi-1)^{a+r}} \left(\pi^{(1-e^{-\lambda x^\beta})^\alpha} - 1\right)^{a+r-1} \\ & \times x^{\beta-1} e^{-\lambda x^\beta} \pi^{(1-e^{-\lambda x^\beta})^\alpha} (1-e^{-\lambda x^\beta})^{\alpha-1}. \end{aligned}$$

Where $B(a,m)$ is a beta function.

3.6. Stress Strength Reliability

If $X_1 \sim PEW(\alpha_1, \lambda_1, \beta)$ and $X_2 \sim PEW(\alpha_2, \lambda_2, \beta)$, where X_1 and X_2 are independent strength and stress rv's respectively, then, the stress strength reliability $P(X_1 > X_2)$, say SSR, can be obtained as

$$SSR = \int_{-\infty}^{\infty} f_1(x) F_2(x) dx. \tag{24}$$

Using (7) and (8) in (24), we have

$$\begin{aligned} SSR = \int_0^\infty & \left(\frac{\alpha_1\lambda_1\beta log\pi}{(\pi-1)^2} x^{\beta-1} e^{-\lambda_1 x^\beta} \pi^{(1-e^{-\lambda_1 x^\beta})^{\alpha_1}} \right. \\ & \left. \times (1-e^{-\lambda_1 x^\beta})^{\alpha_1 - 1} \pi^{(1-e^{-\lambda_2 x^\beta})^{\alpha_2}} \right) dx - \frac{1}{\pi-1}. \end{aligned} \tag{25}$$

Using (13), (14) and applying the transformation $y = x^\beta$ in (25), then the final expression for SSR is given by

$$SSR = \frac{1}{\pi-1} \left(\frac{\alpha_1\lambda_1}{(\pi-1)} \sum_{a,b=0}^{\infty} \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n} (log\pi)^{a+b+1}}{a!b!(\lambda_1(1+m)+n\lambda_2)} \binom{\alpha_1(a+1)-1}{m} \binom{b\alpha_2}{n} - 1 \right)$$

4. Estimation

4.1. Maximum Likelihood Estimation

Let $x_1, x_2, ..., x_n$ be a random sample from PEW distribution, then the logarithm of the likelihood function is

$$\begin{aligned} l =& nlog(\alpha\lambda\beta) + nlog\left(\frac{log\pi}{\pi-1}\right) + (\beta-1)\sum_{i=1}^n x_i - \lambda \sum_{i=1}^n x_i^\beta \\ & + log\pi \sum_{i=1}^n \left(1 - e^{-\lambda x_i^\beta}\right) + (\alpha-1)\sum_{i=1}^n log\left(1-e^{-\lambda x_i^\beta}\right). \end{aligned} \tag{26}$$

The MLEs of α, λ and β are obtained by partially differentiating (26) with respect to the corresponding parameters and equating to zero, we have

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} log\left(1 - e^{-\lambda x_i^{\beta}}\right)\left(log\pi(1 - e^{-\lambda x_i^{\beta}})^{\alpha} + 1\right) = 0 \tag{27}$$

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} x_i - \lambda \sum_{i=1}^{n} x_i^{\beta} log x_i + \sum_{i=1}^{n} \frac{\lambda x_i^{\beta} log x_i}{e^{-\lambda x_i^{\beta}} - 1}\left(\alpha + \alpha log\pi(1 - e^{-\lambda x_i^{\beta}})^{\alpha} - 1\right) = 0 \tag{28}$$

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i^{\beta} + \sum_{i=1}^{n} \frac{x_i^{\beta}}{e^{-\lambda x_i^{\beta}} - 1}\left(\alpha + \alpha log\pi(1 - e^{-\lambda x_i^{\beta}})^{\alpha} - 1\right) = . \tag{29}$$

Since, the above equations (27), (28) and (29) are not in closed form and are difficult to solve analytically. As a result, it is difficult to calculate the estimates of the parameters α, β and λ. However, R software can be used to solve the equations numerically.

4.2. Simulation study

The simulation study has been conducted using R Software to demonstrate the behaviour of the MLEs in terms of the sample size. Two sets of sample (n=50, n=100) each repeated 1000 times with different combinations of parameters $\lambda = (1, 2)$, $\alpha = (0.5, 1.5, 3)$ and $\beta = (0.5, 1.5, 3, 5)$ were achieved from PEW. In each setting, the average values of MLEs and the corresponding empirical mean squared errors (MSEs) were obtained. The simulation results are presented in tables 1 and 2. Tables 1 and 2 show that the estimates are stable and reasonably close to the true parameter values. As the sample size increases the MSE decreases in all the cases.

5. Applications

In this section, we examine two data sets in order to describe the significance and flexibility of PEW distribution. The first data set has been taken from (Cordeiro and Brito [6]), consist of 48 rock samples from a petroleum reservoir. The dataset corresponds to twelve core samples from petroleum reservoirs that were sampled by four cross-sections. Each core sample was measured for permeability and each cross-section has the following variables: the total area of pores, the total perimeter of pores and shape. We analyze the shape perimeter by squared (area) variable. The observations are: 0.0903296, 0.2036540, 0.2043140, 0.2808870, 0.1976530, 0.3286410, 0.1486220, 0.1623940, 0.2627270, 0.1794550, 0.3266350, 0.2300810, 0.1833120, 0.1509440, 0.2000710, 0.1918020, 0.1541920, 0.4641250, 0.1170630, 0.1481410, 0.1448100, 0.1330830, 0.2760160, 0.4204770, 0.1224170, 0.2285950, 0.1138520, 0.2252140, 0.1769690, 0.2007440, 0.1670450, 0.2316230, 0.2910290, 0.3412730, 0.4387120, 0.2626510, 0.1896510, 0.1725670, 0.2400770, 0.3116460, 0.1635860, 0.1824530, 0.1641270, 0.1534810, 0.1618650, 0.2760160, 0.2538320, 0.2004470.

The second set of data is taken from (Aydin [2]) representing a random sample of average daily wind speed data for March, collected in 2015 from the Turkish Meteorological Services for Sinop, Turkey.The data are recorded as follows
2.8, 1.8, 3.2, 5.0, 2.4, 4.8, 2.9, 2.9, 2.3, 3.2, 2.3, 2.0, 1.9, 3.3, 4.4, 6.7, 4.3, 1.9, 2.2, 3.3, 2.1, 4.0, 2.0, 3.1, 3.8, 3.1, 3.2, 3.4, 2.8, 2.1, 3.1.

We compare the fit of the proposed PEW distribution with its sub-model Weibull (W) (see [20]) and a number of other competing models, namely Alpha Power Weibull (APW) (see [13]), Alpha Power Inverse Weibull (APIW) (see [3]), Modified Weibull (MW) (see [18]), Transmuted

Table 1: *mean values of ML estimates and their corresponding mean square errors(n=50).*

Parameter			MLE			MSE		
λ	α	β	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$
1	0.5	0.5	1.10179	0.50379	0.50421	0.33514	0.01926	0.01998
		1.5	1.09164	0.50369	1.48989	0.29139	0.02022	0.07911
		3	1.09261	0.50384	2.96632	0.29186	0.01920	0.27951
		5	1.10208	0.50390	4.93128	0.33049	0.01722	0.77969
	1.5	0.5	1.10144	1.48760	0.50812	0.34598	0.06202	0.02153
		1.5	1.10143	1.48689	1.49968	0.34594	0.05021	0.05021
		3	1.10492	1.47913	3.07588	0.35744	0.07182	0.30989
		5	1.09812	1.48823	4.97413	0.34323	0.06181	0.92916
	3	0.5	1.06841	2.92528	0.51901	0.34669	0.26838	0.02320
		1.5	1.06708	2.92446	1.53312	0.35017	0.26879	0.11011
		3	1.05536	2.92289	3.06026	0.27696	0.26986	0.39581
		5	1.06038	2.92293	5.08217	0.27756	0.26961	1.06087
2	0.5	0.5	2.05707	0.50408	0.50614	0.58601	0.01735	0.01989
		1.5	2.0553	0.50405	1.49456	0.58328	0.02217	0.07999
		3	2.05262	0.50411	2.97686	0.58133	0.01935	0.28283
		5	2.06155	0.50419	4.94529	0.58885	0.01855	0.76788
	1.5	0.5	2.07548	1.48263	0.51078	0.47035	0.06455	0.02193
		1.5	2.07572	1.48192	1.52755	0.47079	0.06453	0.09875
		3	2.07602	1.48205	3.10344	0.46824	0.06445	0.35757
		5	2.06563	1.48288	5.190317	0.39804	0.06414	0.93515
	3	0.5	2.08168	2.92146	0.51719	0.50571	0.27205	0.02432
		1.5	2.08232	2.92217	1.5266	0.50634	0.27417	0.12172
		3	2.06791	2.92024	3.04978	0.44313	0.27343	0.43695
		5	2.06542	2.92144	5.07546	0.44743	0.27186	1.19552

Table 2: *mean values of ML estimates and their corresponding mean square errors(n=100).*

Parameter			MLE			MSE		
λ	α	β	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$
1	0.5	0.5	1.0506	0.50247	0.50289	0.23614	0.01735	0.01694
		1.5	1.05062	0.50287	1.49297	0.21722	0.01631	0.05371
		3	1.04942	0.50287	2.97447	0.20015	0.01601	0.17807
		5	1.04802	0.50289	4.96282	0.19792	0.01496	0.47121
	1.5	0.5	1.08086	1.49116	0.50075	0.28492	0.05099	0.01989
		1.5	1.09562	1.49818	1.501702	0.26724	0.04504	0.08107
		3	1.08072	1.48961	3.06019	0.27247	0.06647	0.30124
		5	1.09101	1.49015	4.98213	0.28726	0.05136	0.76879
	3	0.5	1.06714	2.95807	0.51053	0.23006	0.19014	0.02037
		1.5	1.06075	2.95628	1.50918	0.19142	0.19102	0.08285
		3	1.04935	2.95675	3.00808	0.19031	0.19078	0.29455
		5	1.04988	2.95522	5.01794	0.14934	0.19213	0.77231
2	0.5	0.5	2.01161	0.50271	0.50106	0.46357	0.01587	0.01686
		1.5	2.01208	0.50251	1.49633	0.46451	0.01733	0.05311
		3	2.01008	0.50253	2.98083	0.46424	0.01524	0.17549
		5	2.01285	0.50246	4.96023	0.45973	0.01634	0.46167
	1.5	0.5	2.01167	1.48439	0.49989	0.39586	0.05389	0.01959
		1.5	2.01133	1.48356	1.49515	0.39526	0.05386	0.07749
		3	2.01735	1.48402	2.98175	0.36287	0.05382	0.27414
		5	2.01093	1.48444	4.92742	0.36888	0.05417	0.72391
	3	0.5	2.06175	2.95508	0.51010	0.31644	0.19261	0.02031
		1.5	2.05964	2.95508	1.50872	0.31542	0.19263	0.08399
		3	2.06125	2.95504	3.00326	0.31317	0.19257	0.29708
		5	2.05041	2.95512	5.00925	0.28678	0.19251	0.77898

Weibull (TW) (see [1]), Odd Weibull (OW) (see [4]), Lindley Weibull (LW) (see [5]), Alpha Power Within Weibull Quantile (APWQ) (see [17]), Marshall Olkin Weibull (MOW) (see [14]) and Alpha Power exponential (APE) ([13]). The corresponding density functions for $x > 0$ are presented in

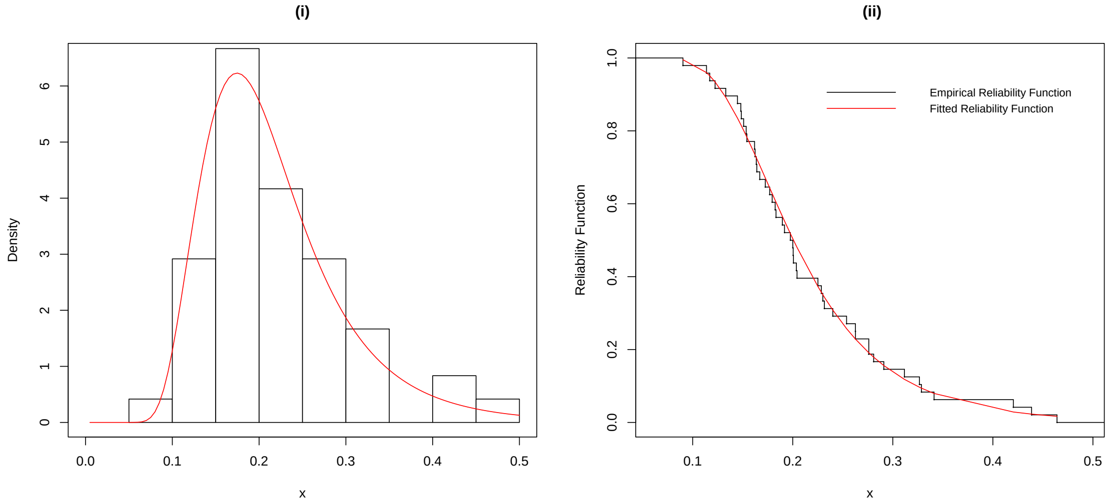

Figure 3: *(i) Fitted PEW density & relative histogram. (ii) Fitted PEW reliability & empirical reliability for first data set.*

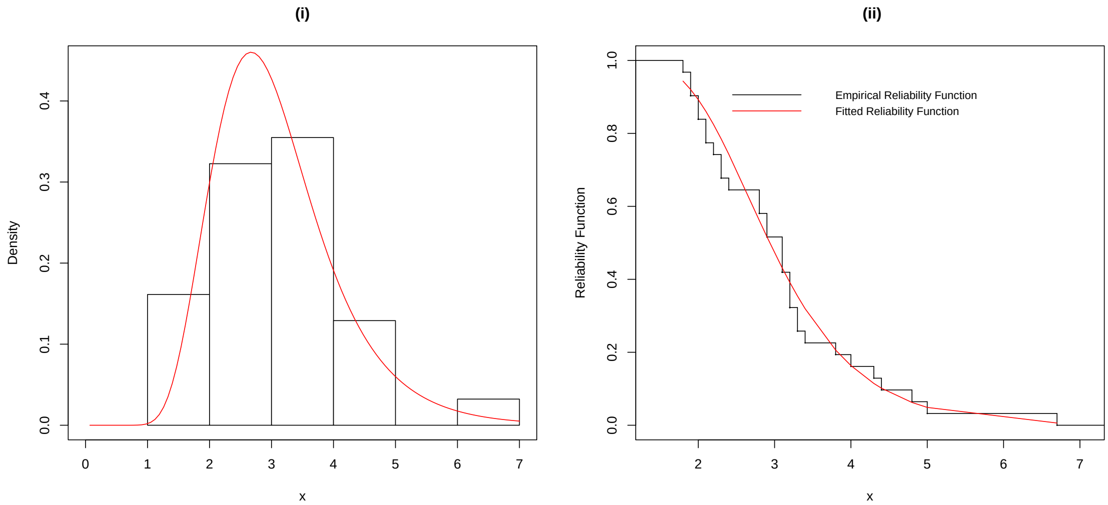

Figure 4: *(i) Fitted PEW density & relative histogram. (ii) Fitted PEW reliability & empirical reliability for second data set.*

the Appendix.

Tables 3, 4, 5 and 6 show that the PEW distribution has the minimum $-2l(\hat{\beta})$, AIC, AICC, BIC and K-S values, as well as the greatest p-value, of all the competing models. As a result, the suggested model fits both the data sets better than the other competitive models. Also the Figures 3, 4, 5 and 6 definitely confirm the conclusions presented in Tables 3, 4, 5,& 6.

6. Conclusion

In this manuscript, a novel method known as PET has been presented. The PET approach has been applied to the Weibull distribution, and a new three-parameter PEW distribution is

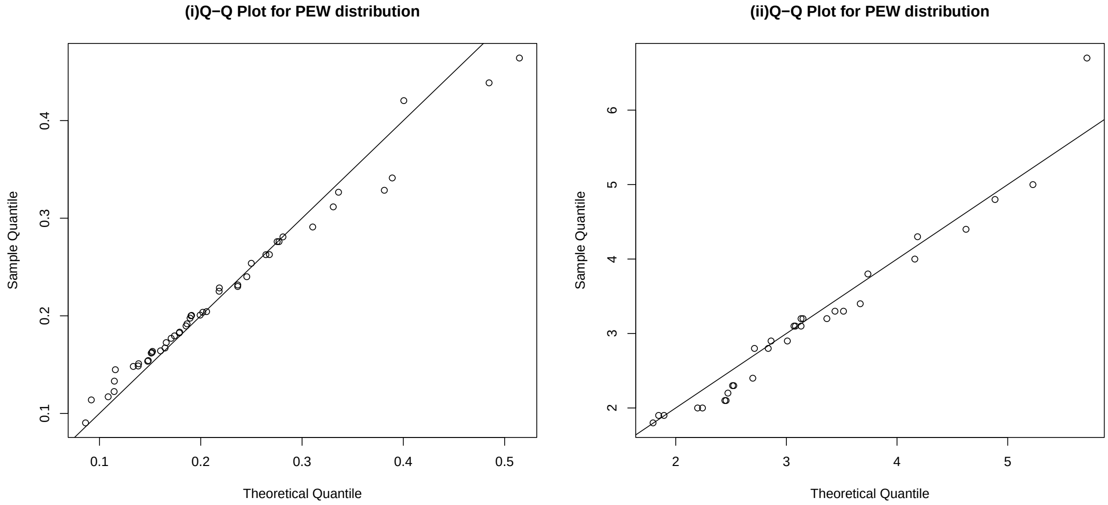

Figure 5: *q-q plot for first and second data set.*

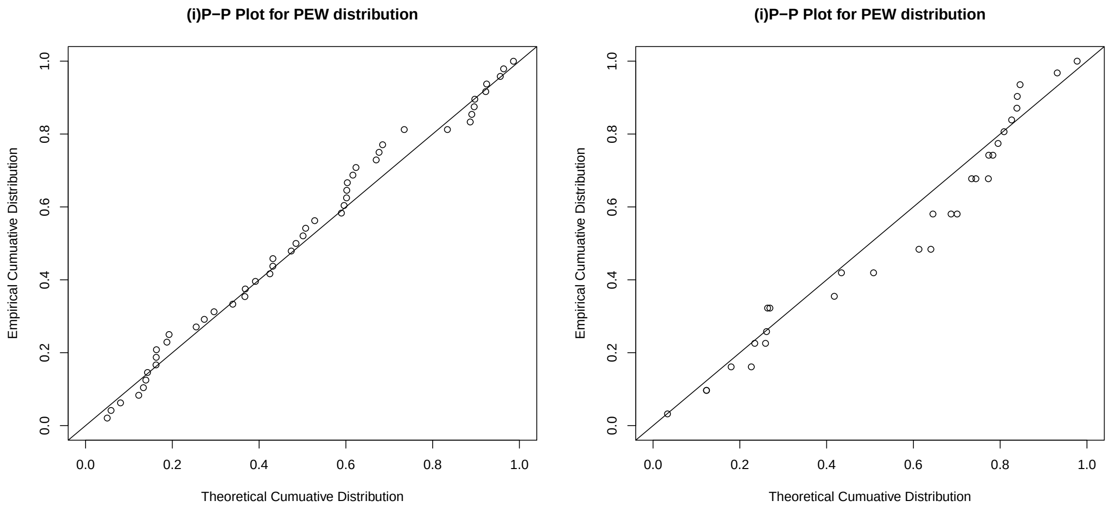

Figure 6: *p-p plot for first and second data set.*

established. Various structural properties as well as reliability measures of the PEW distribution have been highlighted. The reason for adopting this method is that its cdf has a closed form and can represent data with monotone and non-monotone failure rates. It has been revealed that the three-parameter PEW distribution offers more flexibility in respect of hazard rate function and the density function. The suggested model is fitted to two distinct real-life data sets, and the figures demonstrate that it fits both data sets better than any other competing models.

Table 3: *Estimates (standard errors) and kolmogorov smirnov test statistic for the first data set.*

Model	Estimates			Statistics	
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	K-S	p-value
PEW	358.7757 (24.4872)	0.5380 (0.3304)	15.8364 (2.0124)	0.08433	0.8844
APW	0.0320 (0.0508)	3.4096 (0.3606)	4.4898 (2.4429)	0.12804	0.4108
APIW	4.5086 (2.2779)	3.0823 (0.4720)	0.0029 (0.0017)	0.10264	0.6927
MW	0.0010 (2.0711)	2.7475 (0.3700)	47.5555 (7.9292)	0.14985	0.2313
TW	0.6464 (0.2711)	3.0077 (0.3111)	0.2796 (0.0213)	0.14075	0.2976
OW	27.13668 (15.5796)	0.1312 (0.0737)	3.2941 (5.1657)	0.08862	0.8452
LW	17.0146 (22.4843)	2.7406 (0.2854)	1.4788 (1.4712)	0.15011	0.2296
APWQ	64.6499 (9.0106)	6.8937 (0.2609)	65.4380 (0.7298)	0.17289	0.1134
MOW	0.0224 (0.0362)	4.8044 (0.6295)	2.2389 (9.9652)	0.09189	0.8124
APE	100.4597 (16.7779)	-	15.4005 (0.8223)	0.10423	0.6741
W	-	2.7475 (0.2844)	47.5560 (17.9142)	0.14990	0.2310

Table 4: *Information measures for the first data set.*

Model	$-2l(\hat{\beta})$	AIC	AICC	BIC
PEW	-116.4881	-110.4881	-109.9427	-104.8745
APW	-110.56961	-104.56961	-104.02416	-98.95601
APIW	-113.1797	-107.1797	-106.6342	-101.5661
MW	-105.4775	-99.4775	-98.9321	-93.8639
TW	-107.8930	-101.8930	-101.3476	-96.2794
OW	-114.7898	-108.7898	-108.2443	-103.1762
LW	-105.42378	-99.42378	-98.87832	-93.81017
APWQ	-111.7091	-105.7091	-105.1636	-100.0955
MOW	-115.3954	-109.3954	-108.8500	-103.7818
APE	-111.3370	-107.3370	-106.7915	-103.5946
W	-105.48441	-101.48441	-101.21774	-97.74201

Appendix

$$\text{APW} \quad f(x) = \frac{log\alpha}{\alpha-1}\lambda\beta\alpha^{1-e^{-\lambda x^{\beta}}}x^{\beta-1}e^{-\lambda x^{\beta}}$$

$$\text{APIW} \quad f(x) = \frac{log\alpha}{\alpha-1}\lambda\beta x^{-(\beta+\alpha)}e^{-\lambda x^{-\beta}}\alpha^{e^{-\lambda x^{-\beta}}}$$

$$\text{MW} \quad f(x) = (\alpha+\lambda\beta x^{\beta-1})e^{-\alpha x-\lambda x^{\beta}}$$

$$\text{TW} \quad f(x) = \frac{\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1}e^{-\left(\frac{x}{\lambda}\right)^{\beta}}\left(1-\alpha+2\alpha e^{-\left(\frac{x}{\lambda}\right)^{\beta}}\right)$$

$$\text{OW} \quad f(x) = \frac{\alpha\beta}{x}\left(\frac{x}{\lambda}\right)^{\beta}e^{\left(\frac{x}{\lambda}\right)^{\beta}}\left(e^{\left(\frac{x}{\lambda}\right)^{\beta}}-1\right)^{\alpha-1}\left[1+\left(e^{\left(\frac{x}{\lambda}\right)^{\beta}}-1\right)^{\alpha}\right]^{-2}$$

$$\text{LW} \quad f(x) = \frac{\beta\alpha^2}{\alpha+1}\lambda^{\beta}x^{\beta-1}+\lambda^{2\beta}x^{2\beta-1}e^{-\alpha(\lambda x)^{\beta}}$$

Table 5: *Estimates (standard errors) and kolmogorov smirnov test statistic for the second data set.*

Model	Estimates			Statistics	
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	K-S	p-value
PEW	48.9866 (71.8227)	0.8570 (0.2715)	1.8620 (1.1181)	0.10299	0.8974
APW	0.5344 (2.3730)	1.2214 (1.5742)	7.8545 (0.1492)	0.10759	0.8655
APIW	2.6751 (4.6311)	4.0481 (0.7247)	32.1024 (16.6057)	0.13443	0.6297
MW	0.0010 (0.2108)	2.9427 (0.4632)	0.0255 (0.0249)	0.16492	0.3680
TW	0.7341 (0.2973)	3.2334 (0.4079)	4.0055 (0.3343)	0.14982	0.4897
OW	56.6837 (34.145)	6.9111 (4.0662)	5.8591 (1.8199)	0.10573	0.8789
LW	0.0146 (0.0189)	2.1105 (0.2638)	3.0945 (2.1745)	0.15024	0.4860
APWQ	7.9249 (7.7298)	3.7269 (0.3934)	0.0047 (0.0034)	0.16588	0.3611
MOW	0.0139 (0.0249)	5.3051 (0.8008)	0.1529 (0.0459)	0.10472	0.8859
APE	183.6176 (22.3726)	-	1.0341 (0.7071)	0.12275	0.7385
W	-	2.9413 (0.3668)	0.0256 (0.0140)	0.16544	0.3642

Table 6: *Information measures for the second data set.*

Model	$-2l(\hat{\beta})$	AIC	AICC	BIC
PEW	83.70147	89.70147	90.59036	94.00343
APW	84.90786	90.90786	91.79675	95.20982
APIW	85.51669	91.51669	92.40557	95.81865
MW	92.26848	98.26848	99.15737	102.57044
TW	90.26095	96.26095	97.14984	100.56291
OW	85.10416	91.10416	91.99305	95.40612
LW	89.30294	95.30294	96.19183	99.60490
APWQ	90.44442	96.44442	97.33331	100.74638
MOW	84.89439	90.89439	91.78328	95.19636
APE	88.34464	92.34464	93.23353	95.21262
W	92.19582	96.19582	96.62439	99.06379

$$\text{APWQ} \;\; f(x) = \frac{(\alpha-1)\lambda\beta x^{\beta-1}e^{-\lambda x^{\beta}}}{log\alpha\left(1+(\alpha-1)(1-e^{-\lambda x^{\beta}})\right)}$$

$$\text{MW} \;\; f(x) = \frac{\alpha\lambda\beta(\lambda x)^{\beta-1}e^{-(\lambda x)^{\beta}}}{1-(1-\alpha)e^{-(\lambda x)^{\beta}}}$$

$$\text{APE} \;\; f(x) = \frac{log\alpha}{\alpha-1}\lambda e^{-\lambda x}\alpha^{1-e^{-\lambda x}}$$

where $\alpha,\beta,\lambda > 0$ and $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$ is the gamma function.

Declaration

Conflict of interest: The authors declare that they have no Conflict of interest.

References

[1] Gokarna R Aryal and Chris P Tsokos. Transmuted weibull distribution: A generalization of theweibull probability distribution. *European Journal of pure and applied mathematics*, 4(2):89–102, 2011.

[2] Demet Aydin. The new weighted inverse rayleigh distribution and its application. *Mathemtics and Informatics*, 34(3):511–523, 2019.

[3] Abdulkareem M Basheer. Alpha power inverse weibull distribution with reliability application. *Journal of Taibah University for Science*, 13(1):423–432, 2019.

[4] Kahadawala Cooray. Generalization of the weibull distribution: the odd weibull family. *Statistical Modelling*, 6(3):265–277, 2006.

[5] Gauss M Cordeiro, Ahmed Z Afify, Haitham M Yousof, Selen Cakmakyapan, and Gamze Ozel. The lindley weibull distribution: properties and applications. *Anais da Academia Brasileira de Ciências*, 90:2579–2598, 2018.

[6] Gauss Moutinho Cordeiro and Rejane dos Santos Brito. The beta power distribution. *Brazilian journal of probability and statistics*, 26(1):88–112, 2012.

[7] Nicholas Eugene, Carl Lee, and Felix Famoye. Beta-normal distribution and its applications. *Communications in Statistics-Theory and methods*, 31(4):497–512, 2002.

[8] Anwar Hassan, I. H Dar, and M. A Lone. A novel family of generating distributions based on trigonometric function with an application to exponential distribution. *Journal of Scientific Research*, 65(5), 2021.

[9] Dinesh Kumar, U Singh, and Umesh Singh. Life time distributions: Derived from some minimum guarantee distribution. *Sohag Journal of Mathematics*, 4(1):7–11, 2017.

[10] Nikolay Kyurkchiev. A new transmuted cumulative distribution function based on the verhulst logistic function with application in population dynamics. *Biomath Communications*, 4(1), 2017.

[11] MA Lone, IH Dar, and TR Jan. A new method for generating distributions with an application to weibull distribution. *Reliability: Theory & Applications*, 17(1 (67)):223–239, 2022.

[12] Murtiza Ali Lone, Ishfaq Hassain Dar, and TR Jan. An innovative method for generating distributions: Applied to weibull distribution. *Journal of Scientific Research*, 66(3), 2022.

[13] Abbas Mahdavi and Debasis Kundu. A new method for generating distributions with an application to exponential distribution. *Communications in Statistics-Theory and Methods*, 46(13):6543–6557, 2017.

[14] Albert W Marshall and Ingram Olkin. A new method for adding a parameter to a family of distributions with application to the exponential and weibull families. *Biometrika*, 84(3):641–652, 1997.

[15] Sandeep K Maurya, Arun Kaushik, Rajwant K Singh, Sanjay K Singh, and Umesh Singh. A new method of proposing distribution and its application to real data. *Imperial Journal of Interdisciplinary Research*, 2(6):1331–1338, 2016.

[16] Govind S Mudholkar and Deo Kumar Srivastava. Exponentiated weibull family for analyzing bathtub failure-rate data. *IEEE transactions on reliability*, 42(2):299–302, 1993.

[17] M Nassar, A Alzaatreh, O Abo-Kasem, M Mead, and M Mansoor. A new family of generalized distributions based on alpha power transformation with application to cancer data. *Annals of Data Science*, 5(3):421–436, 2018.

[18] Ammar M Sarhan and Mazen Zaindin. Modified weibull distribution. *APPS. Applied Sciences*, 11:123–136, 2009.

[19] William T Shaw and IR Buckley. The alchemy of probability distributions: Beyond gram-charlier & cornish-fisher expansions, and skew-normal or kurtotic-normal distributions. *Submitted, Feb*, 7:64, 2007.

[20] Waloddi Weibull. A statistical distribution function of wide applicability. *Journal of applied mechanics*, 1951.