CLASSICAL AND BAYESIAN STOCHASTIC ANALYSIS OF A TWO UNIT PARALLEL SYSTEM WITH WORKING AND REST TIME OF REPAIRMAN

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Abstract

The aim of the present paper is to deal with the analysis of the classical and Bayesian estimation of various measures of system effectiveness in a two non-identical unit parallel system. Each unit has two possible modes Normal (N) and total failure (F). A single repairman is always available with the system and after working for a random period he goes for rest for a random period. After taking complete rest he again starts the repair of the failed unit on a pre-emptive repeat basis. The system failure occurs when both the units are in (F-mode). The distributions of failure time as well as working and rest time of repairman are assumed to be exponential whereas repair time and rest time distribution of repairman are taken as general. A simulation study is also conducted for analysing the considered system model both in Classical and Bayesian setups. Bayesian estimates of various measure of system effectiveness are also obtained by taking different priors. The comparative study is made to judge the performance of Maximum likelihood estimation and Bayesian estimation methods. A simulation study at the end exhibits the behaviour of such a system. The Monte-Carlo technique is employed to draw observations for this simulation study. To obtain various interesting measure of system effectiveness technique have used the Regenerative point technique, MCMC technique and Gibbs sampler technique. From the graphs and tables we have drawn various important conclusions such that a smaller value of failure rate α_1 introduces a larger value of Maximum likelihood estimate and Bayes estimates for fixed value of the parameter of the repairman rest time distribution β. Moreso, when the value of the failure rate α_1 increases the mean time to system failure and net expected profit are also decreases. To compare the performance of asymptotic confidence interval and highest posterior density interval with the maximum likelihood estimates technique, it has been observed that width of the highest posterior density interval is less than the width of an asymptotic confidence interval.

Keywords: Transition probabilities, Mean sojourn time, mean time to system failure, Pre-emptive repeat repair, Regenerative point, Bayesian estimation, highest posterior density intervals, Maximum likelihood estimation, Gibbs sampler.

1. Introduction

In the planning, design, and operation of different stages of complex systems, the evaluation of high reliability is an important criterion. A two-component redundant system is frequently used to improve reliability as well as availability with maximum expected revenue. A large number of authors have analyzed the two identical and non-identical unit parallel system models in respect of their classical estimates of various measures of system effectiveness. Gupta et.al. [6] analyzed a two non-identical unit parallel system with two independent repairman-skilled and ordinary. A failed unit is first attended to by a skilled repairman to perform first phase repair and then it goes for second phase repair by an ordinary repairman. Both types of repair discipline are FCFS. Chaudhary et.al. [2] analyzed a two non-identical unit parallel system model assuming that an administrative delay occurs in getting the repairman available with the system whenever needed. Chandra et. Al. [1] performed the reliability and cost-benefit analysis of the two identical and non-identical unit parallel system modes by using the Semi-Markov process in the regenerative point technique. A study of comparison is made between the reliability characteristics for both the system models under study.

Realistic situations may arise when a repairman can't repair a failed unit continuously for a long period due to his tiredness/ fatigue as after some time, the working efficiency of the repairman may reduce and he needs to rest for some time. In view of this K. Murari et. al. [9] has analyzed a 2-unit parallel system with a single repairman assuming the working as well as rest period of a repairman. Gupta et. al. [7] has analyzed a single-server two-unit (priority and ordinary) cold standby system with two modes—normal and total failure. The priority unit gets preference both for operation and repair. After working for a random amount of time, the operator of the system needs to rest for a random amount of time and during the rest period of the operator, the system becomes down but not failed. The system failure occurs when both the units are in total failure mode.

Kishan et. Al. [8] analyzed of reliability characteristics of a two-unit parallel system under classical and Bayesian setups. They assumed that the system consists of two non-identical units arranged in a parallel configuration. System failure occurs when both the units stop functioning. Gupta et.al. [5] Performed the cost-benefit and reliability analysis of a two-unit cold standby system assuming that a failed unit enters into the fault detection to identify whether the failed unit needs minor or major repair with fixed known probabilities. Keeping the above idea in view, the present study deals with the analysis of two non-identical units in a parallel system model assuming that the working and rest time of the repairman are uncorrelated random variables. A single repairman is always available with the system to repair a failed unit with priority given to one of the units. The repairman also goes for rest after some time as he is unable to work continuously for a long time. After taking complete rest he again starts the repair of a failed unit assuming that the time already spent in the repair of the failed unit goes to waste. For a more concrete study of the system model, a simulation study is also carried out:

The probability density function (PDF) of exponential is given by:

$$f(t) = \theta e^{-\theta x}; \quad x \geq 0$$

In addition, it has been seen in practice that lifetime experiments take a long period because the ambient circumstances cannot be the same during the trial. As a result, random variables are considered for parameters characterizing the system/dependability unit's characteristics. Therefore, a simulation study is conducted for analysing the considered system model both in classical and Bayesian setups. The Monte Carlo simulation technique has been used in conducting the numerical study in a classical setup, the maximum likelihood estimate of the parameters involved in the model and reliability characteristics along with their standard errors and width of confidence intervals are obtained. In the Bayesian setup, Bayes estimates of the parameters and reliability characteristics along with their posterior standard error and width of highest posterior density intervals are computed. In the end, the comparative conclusions are drawn to judge the performance of the maximum likelihood and Bayes estimates.

The following economic related measures of system effectiveness are obtained by using regenerative point technique:

- Transition probabilities and mean sojourn times in various states.
- Reliability and mean time to system failure.
- Point-wise and steady state availability of the system as well as expected up time of the system during interval (0, t).
- Expected busy period of the repairman during time interval (0, t).
- Net expected profit earned by the system during a finite and interval and in steady state.

2. Model Description and Assumptions

- The system consists of two non-identical units –unit 1 and unit 2. Both the units work in parallel configuration.
- Each unit has two possible modes: Normal (N) and total failure (F).
- A single repairman is always available with the system. After working for a random period, he goes for rest due to his tiredness as after some time, the working efficiency of the repairman may reduce and he needs rest for some time. After taking complete rest the repairman again starts the repair of the failed unit.
- The failure time distribution, as well as the working time of the repairman, are assumed to be exponential whereas the repair time distribution of each unit and rest time of the repairman is taken as general. The repair discipline for the repair of units is FCFS.
- The system failure occurs when both the unit are in total F-mode.
- A repaired unit always works as good as new.

The transition diagram of the system model is shown in Figure [1].

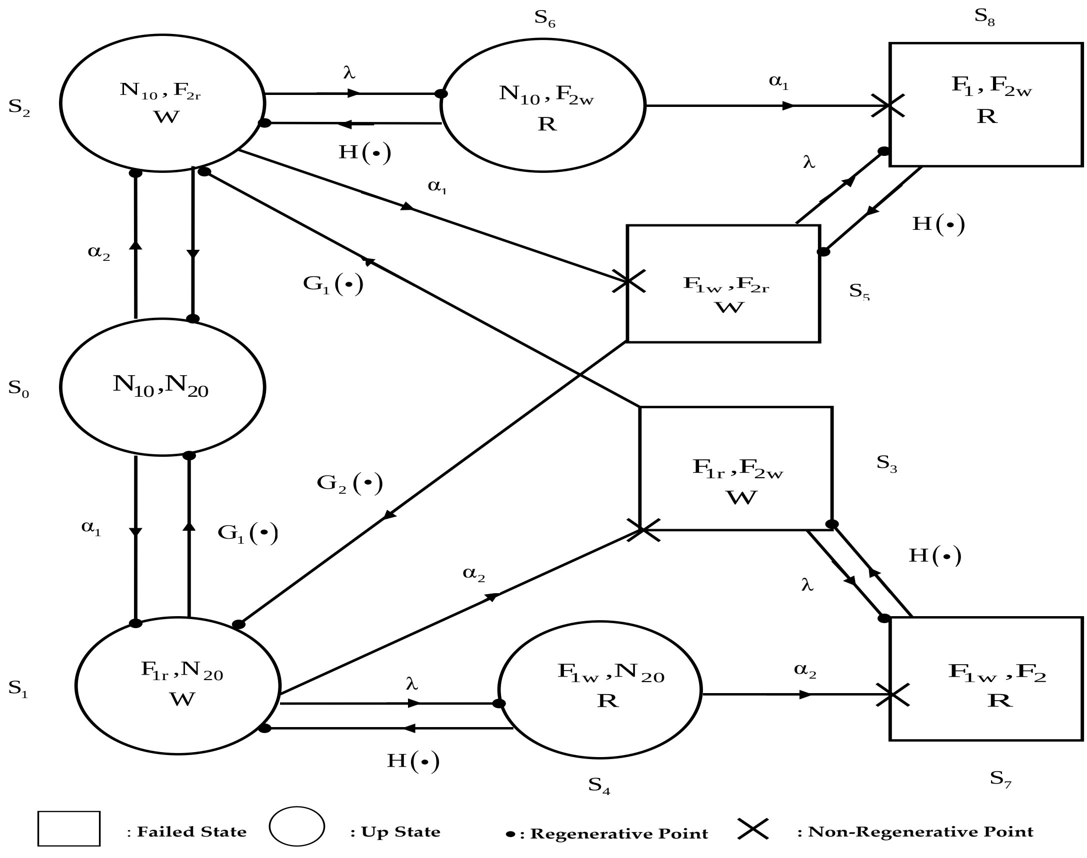

Figure. 1 Transition diagram

3. Notations and Sates of the system

3.1. Notations

α_1, α_2	:	Constant failure rates of first and second type of unit.
$G_1(\bullet), G_2(\bullet)$	:	c.d.f of time to complete repair for first and second unit respectively.
λ	:	Constant rate by which a repairman goes for rest.
$H(\bullet)$	:	General c.d.f rate which a repairman goes to working position after rest.

3.2. Symbols for the state of the system

N_{10}, N_{20}	:	Unit 1 and Unit 2 are N-mode and operative mode.
F_{1r}, F_{2r}	:	Unit 1 and Unit 2 are in F-mode and under repair respectively.
F_{1w}, F_{2w}	:	Unit 1 and Unit 2 are waiting for repair respectively.
F_1, F_2	:	Unit 1 and Unit 2 are failed respectively.

Considering the above symbols, we have the following states of the system:-

Up states: $S_0=(N_{10},N_{20})$, $S1=(F_{1r},N_{20})$, $S2=(N_{10},F_{2r})$, $S_4=(F_{1w},N_{20})$,
$S_6=(N_{10},F_{2w})$

Failed states: $S_3=(F_{1r},F_{2w})$, $S_5=(F_{1w},F_{2r})$, $S_7=(F_{1w},F_2)$, $S_8=(F_1,F_{2w})$

4. Transition Probabilities and Sojourn times

The non-zero elements of one and more steps steady state transition probabilities from state S_i to S_j are as follows-

$$p_{ij} = \lim_{t\to\infty} Q_{ij}(t) \qquad p_{ij}^{(k)} = \lim_{t\to\infty} Q_{ij}^{(k)}(t)$$

$$p_{01} = \int \alpha_1 e^{-(\alpha_1+\alpha_2)t} dt = \frac{\alpha_1}{(\alpha_1+\alpha_2)} \qquad p_{02} = \int \alpha_2 e^{-(\alpha_1+\alpha_2)t} dt = \frac{\alpha_2}{(\alpha_1+\alpha_2)}$$

$$p_{10} = \tilde{G}_1(\lambda+\alpha_2) \qquad p_{14} = \frac{\lambda}{(\lambda+\alpha_2)}\left[1-\tilde{G}_1(\lambda+\alpha_2)\right]$$

$$p_{12}^{(3)} = \frac{\alpha_2}{(\lambda+\alpha_2)} - \left[\tilde{G}_1(\lambda)-\tilde{G}_1(\lambda+\alpha_2)\right] \qquad p_{17}^{(3)} = \tilde{G}_1(\lambda)-\tilde{G}_1(\lambda+\alpha_2)$$

$$p_{20} = \tilde{G}_2(\lambda+\alpha_1) \qquad p_{26} = \frac{\lambda}{(\lambda+\alpha_1)}\left[1-\tilde{G}_2(\lambda+\alpha_1)\right]$$

$$p_{21}^{(5)} = \frac{\lambda}{(\lambda+\alpha_1)} - \left[\tilde{G}_2(\lambda)-\tilde{G}_2(\lambda+\alpha_1)\right] \qquad p_{28}^{(5)} = \tilde{G}_1(\lambda)-\tilde{G}_1(\alpha_1+\lambda)$$

$$p_{32} = \tilde{G}_1(\lambda) \qquad p_{37} = \left[1-\tilde{G}_1(\lambda)\right]$$

$$p_{41} = \tilde{H}(\alpha_2)$$

$$p_{51} = \tilde{G}_2(\lambda) \qquad p_{58} = \left[1-\tilde{G}_2(\lambda)\right]$$

$$p_{73} = p_{85} = 1$$

It can be easily verified that

$P_{01}+P_{02}=1,$ $P_{10}+P_{14}+P_{12}^{(3)}+P_{17}^{(3)}=1$

$P_{20}+P_{26}+P_{21}^{(5)}+P_{28}^{(5)}=1,$ $P_{32}+P_{37}=1$

$P_{41}+P_{43}^{(7)}=1,$ $P_{51}+P_{58}=1$

$P_{62}+P_{65}^{(8)}=1,$ $P_{73}=P_{85}=1$

Mean sojourn time ψ_i in state S_i is defined as the expected time taken by the system in state S_i before transition to any other state.

$$\psi_0=\int P(T_0>t)\,dt=\int e^{-(\alpha_1+\alpha_2)t}dt=\frac{1}{(\alpha_1+\alpha_2)}$$

Similarly,

$$\psi_1=\int e^{-(\lambda+\alpha_2)t}\,\bar{G}_1(t)\,dt, \qquad \psi_2=\int e^{-(\lambda+\alpha_1)t}\,\bar{G}_2(t)\,dt,$$

$$\psi_3=\int e^{-\lambda t}\bar{G}_1(t)\,dt, \qquad \psi_4=\int e^{-\alpha_2 t}\,\bar{H}(t)\,dt$$

$$\psi_5=\int e^{-\lambda t}\,\bar{G}_2(t)\,dt \qquad \psi_6=\int e^{-\alpha_1 t}\,\bar{H}(t)\,dt$$

$$\psi_7=\psi_8=\int \bar{H}(t)\,dt=m$$

5. Methodology for Developing Equations

To obtain various interesting measures of system effectiveness some techniques are available such as the semi-Markov process and regenerative-point technique [1-18]. The present study deals with the technique of regenerative point as it is easy to handle the problem when the behavior of the system at some epochs of entrance into the states is Non-Markovian. We develop the recurrence relations for reliability, availability, and busy period of repairman as follows-

5.1. Reliability of the system

Here we define $R_i(t)$ as the probability that the system does not fail up to t epochs 0,1, 2,…..(t-1) when it is initially started from up state S_i. To determine it, we regard the failed state S_3, S_5, S_7, S_8 as absorbing states. Now, for the expressions of $R_i(t)$; i=0, 1,2,4,6. By simple probabilistic arguments, the value of $R_0(t)$ in terms of its Laplace Transform (L.T.) is given by,

$$R_0^*(s)=\frac{\begin{array}{c}\left(1-q_{26}^*q_{62}^*-q_{14}^*q_{41}^*+q_{14}^*q_{41}^*q_{26}^*q_{62}^*\right)Z_0^*+\left(q_{01}^*-q_{01}^*q_{26}^*q_{62}^*\right)Z_1^*+\left(q_{02}^*-q_{02}^*q_{14}^*q_{41}^*\right)Z_2^*+\\ \left(q_{01}^*q_{14}^*+q_{01}^*q_{14}^*q_{26}^*q_{62}^*\right)Z_4^*+\left(q_{02}^*q_{26}^*-q_{14}^*q_{41}^*q_{26}^*q_{62}^*\right)Z_6^*\end{array}}{(1-q_{26}^*q_{62}^*-q_{14}^*q_{41}^*+q_{14}^*q_{41}^*q_{26}^*q_{62}^*)+q_{10}^*\left(q_{01}^*-q_{01}^*q_{26}^*q_{62}^*\right)-q_{20}^*\left(q_{02}^*-q_{02}^*q_{14}^*q_{41}^*\right)} \tag{1}$$

We have omitted the arguments from $q_{ij}^*(s)$ and $Z_i^*(s)$ for brevity, $Z_i^*(s)$; i=0,1,2,4,6 are the L.T. of

$$Z_0(t)=e^{-(\alpha_1+\alpha_2)t}, \qquad Z_1(t)=e^{-(\lambda+\alpha_2)t}\,\bar{G}_1(t),$$

$$Z_2(t)=e^{-(\lambda+\alpha_1)t}\,\bar{G}_2(t), \qquad Z_4(t)=e^{-(\alpha_2)t}\,\bar{H}(t),$$

$$Z_6(t)=e^{-(\alpha_1)t}\,\bar{H}(t),$$

Taking the inverse Laplace transform of [1], one can get the reliability of the system when system initially starts from S_0. The MTSF is given by

$$E(T_0)=\lim_{s\to 0}R_0^*(s)=\frac{N_1(0)}{D_1(0)}$$

$$R_0^*(s)=\frac{\begin{array}{c}(1-p_{26}p_{62}-p_{14}p_{41}+p_{14}p_{41}p_{26}p_{62})\psi_0+(p_{01}-p_{01}p_{26}p_{62})\psi_1+(p_{02}-p_{02}p_{14}p_{41})\psi_2+\\(p_{01}p_{14}+p_{01}p_{14}p_{26}p_{62})\psi_4+(p_{02}p_{26}-p_{14}p_{41}p_{26}p_{62})\psi_6\end{array}}{(1-p_{26}p_{62}-p_{14}p_{41}+p_{14}p_{41}p_{26}p_{62})+p_{10}(p_{01}-p_{01}p_{26}p_{62})-p_{20}(p_{02}-p_{02}p_{14}p_{41})}$$

5.2. Availability of the system

Let $A_i(t)$ be the probability that the system is up at epoch t, when initially it starts operation from state $S_i \in E$. Using Regenerative point technique and the tools of Laplace Transform, one can obtain the value of in terms of its Laplace Transforms i.e. $A_0^*(s)$ given as follows:

$$A_0^*(s)=\frac{N_2(s)}{D_2(s)}$$

The steady state values of equation (44) is given by

$$A_0=\lim_{t\to\infty}A_0(t)=\lim_{s\to 0}sA_0^*(s)=\lim_{s\to 0}s\frac{N_2(s)}{D_2(s)}$$

Where,

$$N_2(s)=U_0\psi_0+U_1\psi_1+U_2\psi_2+U_3\psi_3+U_4\psi_4+U_5\psi_5+U_6\psi_6$$

Where,

$$U_0=\left[\begin{array}{c}(1-p_{37})(1-p_{58})+p_{26}p_{62}(1-p_{37})(1-p_{58})-p_{21}^{(5)}p_{17}p_{32}(1-p_{58})-p_{21}^{(5)}p_{14}p_{43}p_{32}(1-p_{58})-p_{14}p_{41}(1-p_{37})(1-p_{58})+\\p_{26}p_{62}p_{14}(1-p_{37})(1-p_{58})-\ p_{17}^{(3)}p_{32}p_{26}p_{65}^{(8)}+p_{14}p_{43}^{(7)}p_{32}p_{26}p_{65}\end{array}\right]$$

$$U_1=\left[-p_{01}(1-p_{37})(1-p_{58})+p_{26}p_{62}(1-p_{37})(1-p_{58})-p_{02}p_{21}^{(5)}(1-p_{37})(1-p_{58})+p_{02}p_{26}p_{65}^{(8)}p_{51}(1-p_{37})\right]$$

$$U_2=\left[-p_{01}p_{17}^{(3)}p_{32}(1-p_{58})-p_{01}p_{14}p_{43}^{(7)}p_{32}(1-p_{58})-p_{02}(1-p_{37})(1-p_{58})-p_{14}p_{41}p_{02}(1-p_{37})(1-p_{58})\right]$$

$$U_3=\left[\begin{array}{c}-p_{01}p_{17}^{(3)}(1-p_{58})-p_{01}p_{14}p_{43}^{(7)}(1-p_{58})+p_{01}p_{17}^{(3)}p_{26}p_{62}(1-p_{58})+p_{01}p_{14}p_{43}^{(7)}p_{26}p_{62}(1-p_{58})+p_{02}p_{21}^{(5)}p_{17}^{(3)}(1-p_{58})\\-p_{02}p_{26}p_{65}^{(8)}p_{51}p_{17}^{(3)}-p_{02}p_{26}\ p_{65}^{(8)}p_{43}^{(7)}\end{array}\right]$$

$$U_4=\left[-p_{01}p_{14}(1-p_{37})(1-p_{58})-p_{01}p_{14}p_{26}p_{62}(1-p_{37})(1-p_{58})-p_{02}p_{21}^{(5)}p_{14}(1-p_{37})(1-p_{58})\right]$$

$$U_5=\left[p_{01}p_{17}^{(3)}p_{32}p_{26}p_{65}^{(8)}+p_{01}p_{14}p_{43}^{(7)}p_{32}p_{26}p_{65}^{(8)}-p_{02}p_{26}p_{65}^{(8)}(1-p_{37})\right]$$

$$U_6=\left[p_{01}p_{17}^{(3)}p_{32}p_{26}(1-p_{58})+p_{01}p_{14}p_{43}^{(7)}p_{32}p_{26}(1-p_{58})+p_{02}p_{26}(1-p_{37})(1-p_{58})-p_{14}p_{41}p_{02}p_{26}(1-p_{37})(1-p_{58})\right]$$

We observe that

$$D_2(0)=0$$

Therefore by using L. Hospital rule, we get

$$A_0=\lim_{s\to 0}\frac{N_2(s)}{D_2'(s)}=\frac{N_2(0)}{D_2'(0)} \tag{2}$$

Thus we have,

$$D_2'(0)=U_0\psi_0+(U_1+U_3)\psi_3+U_2\psi_5+(U_4+U_6)m+U_5\psi_5 \tag{3}$$

Using the relation $N_2(0)$ and $D_2'(0)$ in equation [2-3], we get the expression for A_0.

The expected up time of the system in interval (0, t) is given by

$$\mu_{up}(t)=\int_0^t A_0(u)\,du$$

So that

$$\mu_{up}^{*}(s)=\frac{A_0^{*}(s)}{s}$$

5.3. Busy Period analysis

Let us define $B_i(t)$ be the probability that the repairman is busy in the repair of a failed unit at epoch t, when the system starts form state .Here by using the basic probabilistic arguments, we have the following relations for B_i (t-1), i=0,1,2,3,4,5,6,7,8. The dichotomous variable δ taken value 0 and 1.

In the long run, fraction of time for which the system is under repair, starting from state S_0 is given by,

$$B_0^{*}=\lim_{t\to\infty}B_0(t)=\lim_{s\to 0}sB_0^{*}(s)=\lim_{s\to 0}s\frac{N_3(s)}{D_2(s)}$$

$D_2(0)=0$, therefore by L Hospital rule, we have

$$B_0=\lim_{s\to 0}\frac{N_3(s)}{D_2'(s)}=\frac{N_3(0)}{D_2'(0)}$$

$$N_3(s)=U_1\psi_1+U_2\psi_2-U_3\psi_3-U_5\psi_5$$

And $D_2'(0)$ is same as given in section [b].

Now by using $N_3(0)$ and $D_2'(0)$, the expression of B_0, can be obtained. The expected busy period of the repairman in repairing in time interval (0, t) is given by

$$\mu_b(t)=\int_0^t B_0(u)\,du$$

So that,

$$\mu_b^{*}(s)=\frac{B_0^{*}(s)}{s}$$

5.4. Profit function analysis

Let us define

K_0 = per-unit up time revenue by the system due to the operation of any unit.

K_1 = repair cost per-unit of time when a unit is under repair.

Here we assume that repair cost of the unit-1 and unit-2 are same.
Then, the net expected total cost incurred in time interval (0, t) is given by

$$P(t)=K_0\mu_{up}(t)-K_1\mu_b(t)$$

The expected total cost incurred in unit interval of time is

$$\frac{P(t)}{t}=K_0\frac{\mu_{up}(t)}{t}-K_1\frac{\mu_b(t)}{t}$$

The expected total cost per-unit time in steady state is given by-

$$\begin{aligned}P&=\lim_{t\to\infty}\frac{P(t)}{t}\\&=K_0\lim_{t\to\infty}\frac{\mu_{up}(t)}{t}-K_1\lim_{t\to\infty}\frac{\mu_b(t)}{t}\\&=K_0\lim_{s\to 0}s^2\mu_{up}^{*}(s)-K_1\lim_{s\to 0}s^2\mu_b^{*}(s)\\&=K_0\lim_{s\to 0}s^2\frac{A_0^{*}(s)}{s}-K_1\lim_{s\to 0}s^2\frac{B_0^{*}(s)}{s}\end{aligned}$$

$$= K_0 A_0 - K_1 B_0$$

Where A_0 and B_0 have been already defined.

6. Estimation of Parameters, MTSF and Profit Function

6.1. Classical Estimation

In this section, we consider the classical estimation of the model parameters. Suppose that the failure, repair, repairman rest time and working time distribution are independently distributed as Exponential with respective PDF defined in section [1]. Let

$$\underset{\sim}{T}_1 = \left(t_{11}, t_{12}, ...t_{1n_1}\right), \underset{\sim}{T}_2 = \left(t_{21}, t_{22}, ...t_{2n_2}\right), \underset{\sim}{T}_3 = \left(t_{31}, t_{32}, ...t_{3n_3}\right),$$
$$\underset{\sim}{T}_4 = \left(t_{41}, t_{42}, ...t_{4n_4}\right), \underset{\sim}{T}_5 = \left(t_{51}, t_{52}, ...t_{5n_5}\right) \text{and } \underset{\sim}{T}_6 = \left(t_{61}, t_{62}, ...t_{6n_6}\right)$$

Be the random samples respectively drawn from their respective PDF. Then the joint likelihood function is

$$\text{L=L}\left(\underset{\sim}{T}_1, \underset{\sim}{T}_2, \underset{\sim}{T}_3, \underset{\sim}{T}_4, \underset{\sim}{T}_5, \underset{\sim}{T}_6 / \alpha_1, \lambda_1, \beta, \alpha_2, \theta, \lambda\right) = \alpha_1^{n_1} e^{-\alpha_1 \sum x_1} * \lambda_1^{n_2} e^{-\lambda_1 \sum x_2} * \beta^{n_3} e^{-\beta \sum x_3} * \alpha_2 * \theta^{n_5} e^{-\theta \sum x_5} * \lambda^{n_6} e^{-\lambda \sum x_6}$$

6.2. Maximum Likelihood Estimation

The log-likelihood function is

$$\begin{aligned}\log L =& n_1 \log \alpha_1 - \alpha_1 \sum x_1 + n_2 \log \lambda_1 - \lambda_1 \sum x_2 + n_3 \log \beta - \beta \sum x_3 + n_4 \log \alpha_2 - \alpha_2 \sum x_4 + n_5 \log \theta - \theta \sum x_5 \\ &+ n_6 \log \lambda - \lambda \sum x_6\end{aligned} \tag{4}$$

By applying the Maximum likelihood approach, the maximum likelihood estimators (MLEs) $\left(\hat{\alpha}_1, \hat{\lambda}_1, \hat{\beta}, \hat{\alpha}_2, \hat{\theta}, \hat{\lambda}\right)$ of the parameters ($\alpha_1, \lambda_1, \beta, \alpha_2, \theta, \lambda$) can be obtained as

$$\hat{\alpha}_1 = \left(1+T_1\right)^{-1}, \ \hat{\lambda}_1 = \left(1+T_2\right)^{-1}, \ \hat{\beta} = \left(1+T_3\right)^{-1}, \ \hat{\alpha}_2 = \left(1+T_4\right)^{-1}, \ \hat{\theta} = \left(1+T_5\right)^{-1}, \ \hat{\lambda} = \left(1+T_6\right)^{-1}$$

Where,

$$\hat{\alpha}_1 = \frac{n_1}{\sum x_1}, \ \hat{\lambda}_1 = \frac{n_2}{\sum x_2}, \ \hat{\beta} = \frac{n_3}{\sum x_3}, \ \hat{\alpha}_2 = \frac{n_4}{\sum x_4}, \ \hat{\theta} = \frac{n_5}{\sum x_5}, \ \hat{\lambda} = \frac{n_6}{\sum x_6}$$

The asymptotic distribution of $\left(\hat{\alpha}_1 - \alpha_1, \hat{\lambda}_1 - \lambda_1, \hat{\beta} - \beta, \hat{\alpha}_2 - \alpha_2, \hat{\theta} - \theta, \hat{\lambda} - \lambda\right) \sim N_6\left(0, I^{-1}\right)$ is 6-variate normal distribution, where I is the Fisher information matrix with diagonal elements as

$$i_{11} = E\left[-\frac{\partial^2 \log L}{\partial \alpha_1^2}\right] = \frac{n_1}{\alpha_1^2}, \ i_{22} = E\left[-\frac{\partial^2 \log L}{\partial \lambda_1^2}\right] = \frac{n_2}{\lambda_1^2}, \ i_{33} = E\left[-\frac{\partial^2 \log L}{\partial \beta}\right] = \frac{n_3}{\beta^2},$$
$$i_{44} = E\left[-\frac{\partial^2 \log L}{\partial \alpha_2^2}\right] = \frac{n_4}{\lambda_4^2}, \ i_{55} = E\left[-\frac{\partial^2 \log L}{\partial \theta^2}\right] = \frac{n_5}{\theta^2}, \quad i_{66} = E\left[-\frac{\partial^2 \log L}{\partial \lambda^2}\right] = \frac{n_6}{\lambda^2}$$

All off- diagonal elements of I are zero. The maximum likelihood estimator $\hat{M}$ and $\hat{P}$ of MTSF and Profit function can be obtained on using the invariance property of MLE. Also, the asymptotic distribution of $\left(\hat{M} - M\right) \sim N_6\left(0, \Delta_1' I^{-1} \Delta_1\right)$ and $\left(\hat{P} - P\right) \sim N_6\left(0, \Delta_2' I^{-1} \Delta_2\right)$ are respectively,

Where,

$$\Delta_1 = \left(\frac{\partial M}{\partial \alpha_1}, \frac{\partial M}{\partial \lambda_1}, \frac{\partial M}{\partial \beta}, \frac{\partial M}{\partial \alpha_2}, \frac{\partial M}{\partial \theta}, \frac{\partial M}{\partial \lambda}\right) \text{ And } \Delta_2 = \left(\frac{\partial P}{\partial \alpha_1}, \frac{\partial P}{\partial \lambda_1}, \frac{\partial P}{\partial \beta}, \frac{\partial P}{\partial \alpha_2}, \frac{\partial P}{\partial \theta}, \frac{\partial P}{\partial \lambda}\right)$$

6.3. Bayesian Estimation

The Bayesian estimation is used to measure the impact of prior information along with the sample information. Therefore, in this section, the Bayesian method of estimation is also considered for estimating the model parameter. As discussed above, the Bayesian method of the estimation considers the parameters involved in the model as random variable. Here, we estimate the unknown parameters considering the prior distribution of gamma with respective PDFs

$$\alpha_1 \sim \text{Gamma}(a_1,b_1); \qquad (\alpha_1,a_1,b_1)>0, \tag{5}$$

$$\lambda_1 \sim \text{Gamma}(a_2,b_2); \qquad (\lambda_1,a_2,b_2)>0, \tag{6}$$

$$\beta \sim \text{Gamma}(a_3,b_3); \qquad (\beta,a_3,b_3)>0, \tag{7}$$

$$\alpha_2 \sim \text{Gamma}(a_4,b_4); \qquad (\alpha_2,a_4,b_4)>0, \tag{8}$$

$$\theta \sim \text{Gamma}(a_5,b_5); \qquad (\theta,a_5,b_5)>0, \tag{9}$$

$$\lambda \sim \text{Gamma}(a_6,b_6); \qquad (\lambda,a_6,b_6)>0, \tag{10}$$

Here a_i and b_i (i=1, 2, 3, 4, 5, 6) respectively denote the scale and shape parameters.

Now by using the likelihood in [4] and the priors in [5-10], the posterior distribution of the parameters $\alpha_1,\lambda_1,\beta,\alpha_2,\theta,\lambda$ given data are:

$$w_1(\alpha_1|x_1,\lambda_1,\beta,\alpha_2,\theta,\lambda) \sim \text{Gamma}(n_1+a_1,b_1+x_1), \tag{11}$$

$$w_2(\lambda_1|x_1,\alpha_1,\beta,\alpha_2,\theta,\lambda) \sim \text{Gamma}(n_2+a_2,b_2+x_2), \tag{12}$$

$$w_3(\beta|x_1,\alpha_1,\lambda_1,\alpha_2,\theta,\lambda) \sim \text{Gamma}(n_3+a_3,b_3+x_3), \tag{13}$$

$$w_4(\alpha_2|x_1,\alpha_1,\lambda_1,\beta,\theta,\lambda) \sim \text{Gamma}(n_4+a_4,b_4+x_4), \tag{14}$$

$$w_5(\theta|x_1,\alpha_1,\lambda_1,\beta,\alpha_2,\lambda) \sim \text{Gamma}(n_5+a_5,b_5+x_5), \tag{15}$$

$$w_6(\lambda|x_1,\alpha_1,\lambda_1,\beta,\alpha_2,\theta) \sim \text{Gamma}(n_6+a_6,b_6+x_6), \tag{16}$$

All the prior parameters (also known as hyper parameters) are assumed to be known. These parameters are those whose values are set before the Bayesian learning process start. We utilized the Markov Chain Monte Carlo (MCMC) techniques available that can be used to simulate draws from the posterior distribution and also use the Gibbs sampler, a well-known MCMC algorithm proposed by [1]. It allows us to generate posterior samples for all the parameters using their full conditional posterior distributions.

Now we proceed as follows:

- Simulate α_1 from $w_1(\alpha_1|x_1,\lambda_1,\beta,\alpha_2,\theta,\lambda)$, given in equation [11].
- Simulate λ_1 from $w_2(\lambda_1|x_1,\alpha_1,\beta,\alpha_2,\theta,\lambda)$, given in equation [12].
- Simulate β from $w_3(\beta|x_1,\alpha_1,\lambda_1,\alpha_2,\theta,\lambda)$, given in equation [13].
- Simulate α_2 from $w_4(\alpha_2|x_1,\alpha_1,\lambda_1,\beta,\theta,\lambda)$, given in equation [14].
- Simulate θ from $w_5(\theta|x_1,\alpha_1,\lambda_1,\beta,\alpha_2,\lambda)$, given in equation [15].
- Simulate λ from $w_6(\lambda|x_1,\alpha_1,\lambda_1,\beta,\alpha_2,\theta)$, given in equation [16].

Repeat steps 1-6, N times and record the sequence of $\alpha_1,\lambda_1,\beta,\alpha_2,\theta,\lambda$ after discarding the burn-in-sampler of size, say N_0, from the sample so that the effect of the initial values is neutralised.

Under the squared error loss function, Bayes estimates of $\alpha_1,\lambda_1,\beta,\alpha_2,\theta,\lambda$ are, respectively, the means of posterior distribution given in equations [11-16] and as follows:

$$\hat{\alpha}_1=\frac{n_1}{\sum x_1},\ \hat{\lambda}_1=\frac{n_2}{\sum x_2},\ \hat{\beta}=\frac{n_3}{\sum x_3},\ \hat{\alpha}_2=\frac{n_4}{\sum x_4},\ \hat{\theta}=\frac{n_5}{\sum x_5},\ \hat{\lambda}=\frac{n_6}{\sum x_6}$$

7. Simulation Study

In this section, a simulation study is carried out to investigate the behavior of an assumed system in steady state. Estimates for the parameters of interest as well as the reliability measures were obtained using both the classical maximum likelihood estimation technique and the Bayesian approach. For simulation purposes, random samples of size n were generated for every iteration from the assumed distribution after setting n1=n2=n3=n4=30, 50, 100, 150 to obtain the maximum likelihood estimate and Bayes estimator of the parameters using an expression in 6.1, 6.2 & 6.3 respectively. Then using the asymptotic convergence of maximum likelihood estimate of the standard errors (SE) and the confidence intervals for the maximum likelihood estimate of the reliability and profit have been obtained. For studying the posterior performance of the MTSF and Profit function, the values of hyper-parameters have been so chosen that $E(\alpha_1)=a_1/b_1$, $E(\lambda_1)=a_2/b_2$, $E(\beta)=a_3/b_3$, $E(\alpha_2)=a_4/b_4$, $E(\theta)=a_5/b_5$, $E(\lambda)=a_6/b_6$ to generate the samples from posterior distribution of the parameters. Here simulated results have been obtained by 10,000 iterations for the priors. These simulated posterior values have been used to obtain the posterior estimates of the reliability measures as the means of these simulated values and their posterior standard errors (PSEs). Also, highest posterior density [HPD] for mean time to system failure MTSF and Profit of the assumed system have been obtained using these posterior values.

The True values, maximum likelihood estimates MLE's, and Bayes estimates of mean time to system failure MTSF and Profit function for fixed repair rate α_1 and repairman rest time distribution β respectively, and varying failure rates α_1 of mean time to system failure and profit function have been plotted in fig.[2-3]and [4-5] and also shown in tables [1-4]. The 95% asymptotic confidence intervals and highest posterior density [HPD] intervals of mean time to system failure MTSF and Profit function are plotted in fig. [6-7] and [8-9].

8. Concluding Remarks

From the simulation results in Table [1-4] and various figures [2-9], it is observed that:

- Table [1] and Table [2] give the maximum likelihood estimates and Bayes estimates of mean time to system failure MTSF for various values of failure rate fixed repairman rest time β =0.04 and 0.09, its clear that a smaller value of failure rate α_1 introduces a larger value of maximum likelihood estimates and Bayes estimates for the given value of repairman rest time distribution β . As the value of the failure rate increases the MTSF decreases which shows in fig [2] and fig [3].
- Comparing table [1] and table [2], the value of a parameter of rest time distribution β increased from 0.04 to 0.09, which results in the decay of mean time to system failure MTSF for the given value and which shows in fig. [2-3].
- Table [3] and Table [4] observed that C and Bayes estimates of Profit function for the various value of failure rate α_1 and rest time distribution β= 0.04 and 0.09, it observed that the value of failure rate α_1 increases so the estimate of the true value of Profit function is decreased which shows in fig. [3-4].
- Comparing Table [3] and Table [4] shows that an increment in the value of the parameter of rest time distribution β from 0.04 to 0.09 results in a decay of the Profit function.
- To compare the performance of asymptotic confidence interval and higher posterior density [HPD] interval with the maximum likelihood estimates technique, from all the tables [1-4] and fig. [6-9] it has been seen that width of the highest posterior density [HPD] interval is less than the width of an asymptotic confidence interval.

- Fig [6-9] gives the posterior distribution of reliability measure to show the performance of mean time to system failure MTSF and Profit function at different value of Failure rate α_1 and fixed value of rest time distribution β .
- Comparing fig [6-7] shows that when sample size n=30, 50,100,150 i.e. increases then highest posterior density [HPD] intervals of mean time to system failure MTSF are less than the asymptotic confidence interval of mean time to system failure MTSF. The same trend can see in the highest posterior density [HPD] interval and Confidence interval of the Profit function in fig [8-9].

Graphs

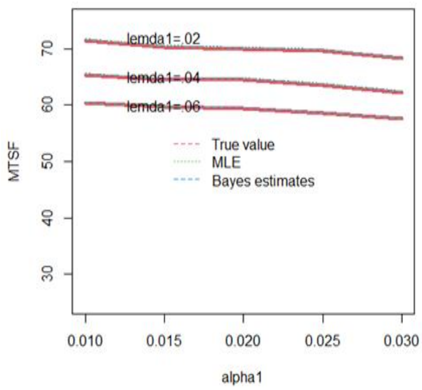

Fig.2: *Plot of MTSF for fixed beta=.04*

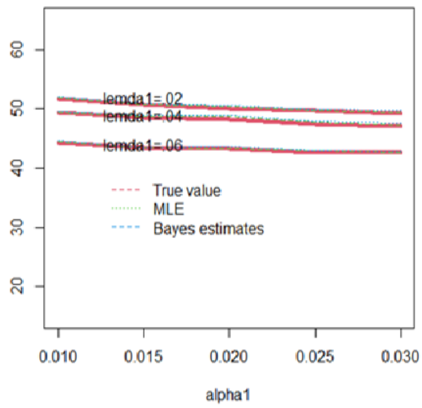

Fig.3: *Plot of MTSF for fixed beta=.09*

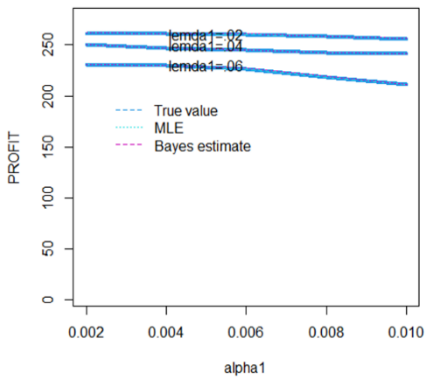

Fig.4: *Plot of PROFIT for fixed beta=0.04*

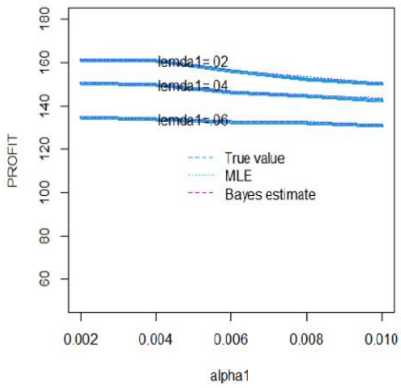

Fig.5: *Plot of PROFIT for fixed beta=0.09*

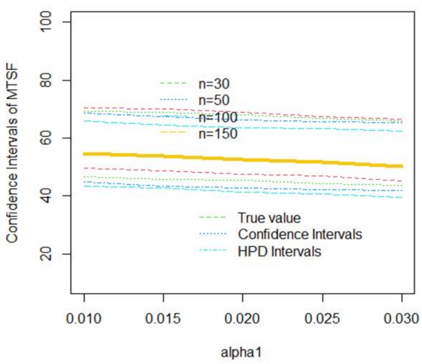

Fig.6: *Plot of CI for MTSF*

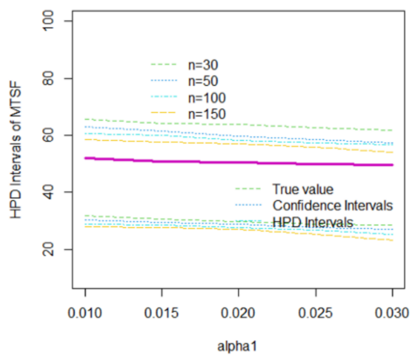

Fig.7: *Plot of HPD for MTSF*

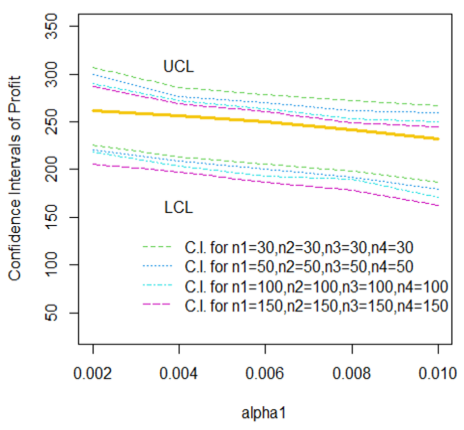

Fig.8: *Plot of CI for Profit*

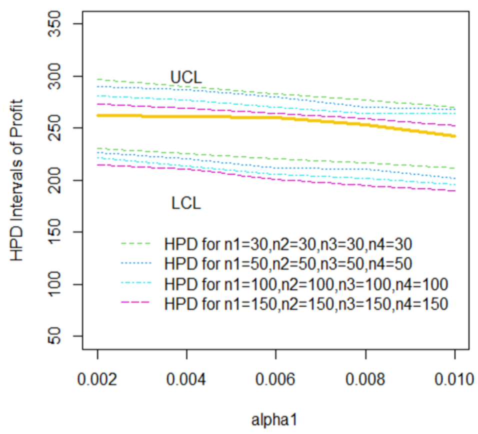

Fig.9: *Plot of HPD Intervals for Profit*

Tables

Table 1. *Various estimates of MTSF for fixed β=0.04 and varying α_1*

α_1	0.01	0.015	0.02	0.025	0.03
True Value	71.46	70.29	70.02	69.67	68.26
ML Estimates	71.58	70.45	70.1	69.74	68.3
Bayes Estimates	71.69	70.66	70.25	69.98	68.49
C.I. Width	22.25	21.14	20.24	19.89	17.04
HPD Width	21.76	20.53	20.07	19.23	16.65
MSE	0.5427	0.4284	0.3721	0.3486	0.2453

Table 2. *Various estimates of MTSF for fixed β=0.09 and varying α_1*

α_1	0.01	0.015	0.02	0.025	0.03
True Value	51.75	50.64	50.02	49.67	49.26
ML Estimates	51.89	50.76	50.3	49.74	49.4
Bayes Estimates	51.95	50.87	50.53	49.8	49.76
C.I. Width	21.56	21.15	20.47	19.43	18.51
HPD Width	20.43	19.65	19.54	18.08	17.22
MSE	0.4329	0.3897	0.3237	0.2983	0.1984

Table 3. *Various estimate of Profit for fixed β=0.04 and varying α_1*

α_1	0.002	0.004	0.006	0.008	0.01
True Value	261.39	260.74	260.19	258.27	256.11
ML Estimates	261.75	260.82	260.52	258.54	256.54
Bayes Estimates	261.83	260.95	260.67	258.67	256.35
C.I. Width	50.05	43.86	42.84	41.34	40.67
HPD Width	44.85	42.53	41.79	39.05	38.49
MSE	1.5678	1.5232	1.5137	1.5108	1.5099

Table 4. *Various estimate of Profit for fixed β=0.09 and varying α_1*

α_1	0.002	0.004	0.006	0.008	0.01
True Value	161.47	160.74	156.19	152.27	150.11
ML Estimates	161.8	160.89	156.52	152.23	150.56
Bayes Estimates	161.93	160.97	156.84	153.56	150.75
C.I. Width	50.05	43.86	42.84	41.34	40.67
HPD Width	52.85	51.53	50.27	49.05	48.82
MSE	1.2784	1.2643	1.262	1.2539	1.2487

9. Conclusion

This study report investigates the current work's value in several dimensions. First, this work investigates the steady-state reliability measures MTSF and Profit function of the subjected system, which are produced under failure time and repairman rest time distribution of the system, making the study applicable to a wide range of real-world circumstances. Second, the system's unknown parameters and reliability measures were estimated using the ML technique, along with their appropriate asymptotic confidence ranges. In addition, the Bayesian technique to estimate allows practitioners to use any prior information for better results, which is immediately seen when gamma priors are applied. Third, the supposed system's behaviour is evaluated using the Monte Carlo simulation approach to gain a better understanding of the system's behaviour. As a result, this study proves that the Bayesian method with appropriate prior is extremely utilitarian and simple to implement for analysing the redundant repairable system waiting for a repair facility.

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