A MODIFIED INVERSE WEIBULL DISTRIBUTION USING KM TRANSFORMATION

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Abstract

In the subject of reliability engineering and statistics, a new reliability model is proposed, where survival analysis or life time data analysis is of major importance in the current scenario. The goal of this study is to introduce a new model that has applications to real data sets from the field of survival analysis. Deriving out the new model there are various methods to propose a new model, one of them is by using the method of transforming a variable to the variable of interest and there are numerous transformation methods which are in use right now. The newly proposed model is achieved by using the transformation method known as KM Transformation where it does not require any additional parameters to the baseline distribution which absolutely is an advantage. The model considered in this paper as baseline model is Inverse Weibull distribution with two parameters, one is a scale and other is a shape parameter. Inverse Weibull distribution is a continuous probability distribution which presently has great applications in real life phenomenon as well as so many modifications and advanced studies are introduced in this distribution from various fields. A proper study on the newly proposed model is done by deriving out its various functions and statistical properties such as Probability density function, Cumulative distribution function, Hazard rate function, Moments, Moment generating function, Characteristic function, Quantile function, Order statistics, etc. along with its Probability density function plot and Hazard rate function plot which have both upside-down and decreasing curves. Focusing on the inference procedures, the estimation of the parameters involved in this model is done by using the method of Maximum likelihood estimation. A simulation study for valuing the parameter consistency using two parameter combinations is carried out as well as a data analysis on an actual data set is also conducted. A comparison of the newly proposed model with other popular well-known models such as Inverse Weibull distribution (IW), KME distribution and KMW distribution using R programming language yielded that the new model is a better fit for the real data considered in this paper. The results and conclusions achieved throughout the paper are also mentioned at the last.

Keywords: KM Transformation, Inverse Weibull distribution, upside-down curve, decreasing curve

1. Introduction

In the current scenario of model building and real life data analysis there are so many lifetime distributions that are in use for which [14] and [15] explains the basic ideas and concepts. Among these well-known distributions, Weibull distribution and its various modified Weibull distributions plays a great role in fitting real data sets. Inverse Weibull (IW) distribution has great applications in reliability engineering, where [3] and [4] studies the various inference procedures in the distribution of consideration. A theoretical analysis of IW distribution is done by [12] and order statistics with inference is done by [16]. [5] studies the generalized modified weibull distribution along with applications to two real data sets and [2] applied the IW distribution to model the wind speed data. Extended studies are done in this distribution where [13] studies the Bayesian inference and prediction of the IW distribution for type-II censored data and [10] produced a new model, generalized IW distribution.

Deriving out a new model with a baseline distribution using a transformation is common in practice but finding a model that best fits for a real data set than other related models are of great relavance. There are so many transformations in existance where the transformation named KM Transformation that studied in [11] is a recent development in the subject in which Exponential distribution and Weibull distribution are used as baseline models to fit real data sets. In this paper, we consider two-parameter Inverse Weibull distribution (IW) as baseline distribution to perform KM Transformation.

This paper is about the KM-IW Model which consists of the introduction to the model with its probability density function (pdf), cumulative distribution function (cdf) and hazard function in section 2. The basic properties of a model such as Moments, Moment generating function, Quantile function and Distribution of order Statistic are studied in Section 3. In section 4, estimation of the model parameters are done by using the method of maximum likelihood estimation. Section 5 gives the results on the simulation study of the model and section 6 gives the results on the fitting of KM-IW model to a real data set. And the conclusions achieved throughout the paper is mentioned in Section 7.

2. KM-IW Distribution

Let X be the random variable of interest. Then using the KM Transformation method the pdf, cdf and hazard function of X is obtained by the formulae given below respectively.

$$f(x) = \frac{e}{e-1}\, g(x)\, e^{-G(x)}$$

$$F(x) = \frac{e}{e-1}\, [1 - e^{-G(x)}]$$

and

$$h(x) = \frac{g(x)\, e^{1-G(x)}}{e^{1-G(x)} - 1}$$

where, g(x) and G(x) are the pdf and cdf of IW distribution which are as given below.

$$g(x) = \frac{\alpha}{\beta} \left(\frac{x}{\beta}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}, \, x > 0, \, \alpha, \, \beta > 0$$

and

$$G(x) = e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}, \, x > 0, \, \alpha, \, \beta > 0$$

Then the following are being the pdf, cdf and hazard function of our new model KM-IW distribution.

$$f(x) = \frac{e}{e-1} \frac{\alpha}{\beta} \left(\frac{x}{\beta}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta}\right)^{-\alpha}} e^{-e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}}, \, x > 0, \, \alpha, \, \beta > 0 \tag{1}$$

$$F(x) = \frac{e}{e-1} \left[1 - e^{-e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}}\right], \, x > 0, \, \alpha, \, \beta > 0 \tag{2}$$

and

$$h(x) = \frac{\frac{\alpha}{\beta} \left(\frac{x}{\beta}\right)^{-(\alpha+1)} e^{1-\left(\frac{x}{\beta}\right)^{-\alpha} - e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}}}{e^{1-e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}} - 1}, \, x > 0, \, \alpha, \, \beta > 0 \tag{3}$$

This newly proposed KM-IW Distribution has the pdf and hazard function plots as given below which are obtained using the R programming language.

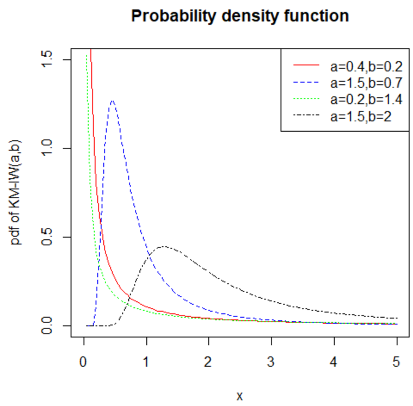

Figure 1: *Probability Density Function of KM-IW Distribution*

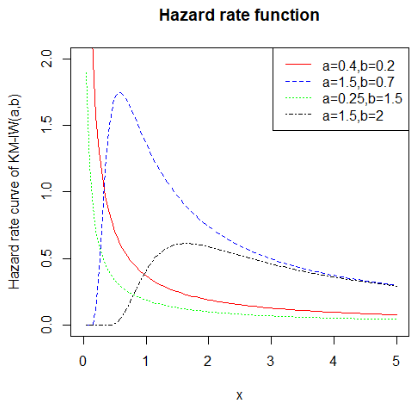

Figure 2: *Hazard Rate Function of KM-IW Distribution*

As seen in the plots, both pdf and hazard rate function have upside-down and decreasing curves for different combinations of α and β.

3. Properties of KM-IW Distribution

Here we discuss some statistical properties of the new model. They are Moments of the distribution, Moment generating function, Characteristic function, Quantile function and Order Statistic.

3.1. Moments of the Distribution

The r^{th} raw moment about origin of the distribution is obtianed as below.

$$\begin{aligned}
\mu_r' &= E\left(X^r\right) \\
&= \int_0^\infty x^r f(x)\,dx \\
&= \int_0^\infty x^r \frac{e}{e-1}\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta}\right)^{-\alpha}} e^{-e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}}\,dx
\end{aligned}$$

The exponential term is expanded, so we get,

$$\mu_r' = \frac{\alpha\,\beta^\alpha\,e}{e-1}\int_0^\infty x^r\,x^{-(\alpha+1)}e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}\sum_{m=0}^{\infty}\frac{\left[-e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}\right]^m}{m!}\,dx$$

On proper substitution and simplification, we get,

$$\mu_r' = \frac{e}{e-1}\sum_{m=0}^{\infty}\frac{(-1)^m\,\Gamma\left(-\frac{r}{\alpha}+1\right)(m+1)^{\frac{r}{\alpha}}\beta^r}{(m+1)!} \tag{4}$$

Putting $r=1$, we get the 1^{st} raw moment about origin (mean of the distribution) and is given by,

$$\mu_1' = E\,(X)$$

i.e.,

$$\mu_1' = \frac{e}{e-1}\sum_{m=0}^{\infty}\frac{(-1)^m\,\Gamma\left(-\frac{1}{\alpha}+1\right)(m+1)^{\frac{1}{\alpha}}\beta}{(m+1)!} \tag{5}$$

3.2. Moment Generating Function

The moment generating function (mgf) of the distribution is obtianed as below.

$$\begin{aligned} M_X(t) &= E\left(e^{tX}\right) \\ &= \int_0^\infty e^{tx} f(x)\,dx \\ &= \int_0^\infty e^{tx}\frac{e}{e-1}\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}\,e^{-e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}}\,dx \end{aligned}$$

on expanding the exponential term we get,

$$\begin{aligned} M_X(t) &= \frac{\alpha\,\beta^\alpha\,e}{e-1}\int_0^\infty e^{tx}\,x^{-(\alpha+1)}e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}\sum_{m=0}^{\infty}\frac{\left[-e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}\right]^m}{m!}\,dx \\ &= \frac{\alpha\,\beta^\alpha\,e}{e-1}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\int_0^\infty x^{-(\alpha+1)}e^{-(m+1)\left(\frac{x}{\beta}\right)^{-\alpha}}\sum_{n=0}^{\infty}\frac{(tx)^n}{n!}\,dx \end{aligned}$$

after simplification, we get,

$$M_X(t) = \frac{e}{e-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m\,t^n\,\Gamma\left(-\frac{n}{\alpha}+1\right)(m+1)^{\frac{n}{\alpha}}\beta^n}{(m+1)!n!} \tag{6}$$

Differentiating the mgf will yield the moments about origin of the distribution.
Similarly, the characteristic function is obtianed as

$$\phi_X(t) = \frac{e}{e-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m\,(it)^n\,\Gamma\left(-\frac{n}{\alpha}+1\right)(m+1)^{\frac{n}{\alpha}}\beta^n}{(m+1)!n!} \tag{7}$$

where, $i^2=-1$

3.3. Quantile Function

For obtaining the p^{th} quantile function (Q(p)) of KM-IW distribution, we solve $F(Q(p)) = p$, where $0 < p < 1$. And is obtained as,

$$Q(p) = \beta\left[-\log\left[-\log\left(1-\frac{p(e-1)}{e}\right)\right]\right]^{\frac{-1}{\alpha}} \tag{8}$$

3.4. Order statistic

Let the random sample of size n, $X_1, X_2, ..., X_n$ be from the new distribution. Then $X_{(1)}, X_{(2)}, ..., X_{(n)}$ are the order statistics respectively. The pdf and cdf of the r^{th} order statistic $f_r(x)$ and $F_r(x)$ are given by

$$f_r(x) = \frac{n!}{(r-1)!(n-r)!} F^{r-1}(x) \left[1 - F(x)\right]^{n-r} f(x)$$

and

$$F_r(x) = \sum_{j=r}^{n} \binom{n}{j} F^j(x) \left[1 - F(x)\right]^{n-j}$$

And those of our newly proposed model are,

$$\begin{aligned} f_r(x) &= \frac{n!\, \alpha\, \beta^\alpha}{(r-1)!(n-r)!(e-1)^n} x^{-(\alpha+1)} e^{\left[r - \left(\frac{x}{\beta}\right)^{-\alpha} - e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}\right]} \\ &\quad \left(1 - e^{-e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}}\right)^{r-1} \left(e^{1-e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}} - 1\right)^{n-r} \end{aligned} \tag{9}$$

and

$$\begin{aligned} F_r(x) &= \sum_{j=r}^{n} \binom{n}{j} \frac{e^j}{(e-1)^n} \left(1 - e^{-e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}}\right)^{j} \\ &\quad \left(e^{1-e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}} - 1\right)^{n-j} \end{aligned} \tag{10}$$

respectively.

4. Estimation of parameters

This section is about the estimates of the parameters involved in the distribution. Here we are using the maximum likelihood estimation method.
The likelihood function of KM-IW distribution is found by,

$$L(x, \alpha, \beta) = \prod_{i=1}^{n} f\left(x_i, \alpha, \beta\right)$$

i.e., we will have,

$$L(x) = \left(\frac{e}{e-1}\right)^n \frac{\alpha^n}{\beta^n} \left[\prod_{i=1}^{n} \left(\frac{x_i}{\beta}\right)^{-(\alpha+1)}\right] e^{-\sum_{i=1}^{n} \left(\frac{x_i}{\beta}\right)^{-\alpha}} e^{-\sum_{i=1}^{n} e^{-\left(\frac{x_i}{\beta}\right)^{-\alpha}}}$$

So that the log-likelihood function becomes,

$$\begin{aligned} \log L &= n \log e - n \log(e-1) + n \log \alpha + n\alpha \log \beta - (\alpha+1) \sum_{i=1}^{n} \log x_i \\ &\quad - \sum_{i=1}^{n} \left(\frac{x_i}{\beta}\right)^{-\alpha} - \sum_{i=1}^{n} e^{-\left(\frac{x_i}{\beta}\right)^{-\alpha}} \end{aligned} \tag{11}$$

The partial derivatives of log L with respect to the unknown parameters α and β are obtianed. Equating these non-linear equations to zero gives the MLEs of α and β.

$$\frac{n}{\alpha} + n \log \beta - \sum_{i=1}^{n} \log x_i + \sum_{i=1}^{n} \left(\frac{x_i}{\beta}\right)^{-\alpha} \log\left(\frac{x_i}{\beta}\right) - \sum_{i=1}^{n} e^{-\left(\frac{x_i}{\beta}\right)^{-\alpha}} \left(\frac{x_i}{\beta}\right)^{-\alpha} \log\left(\frac{x_i}{\beta}\right) = 0 \tag{12}$$

$$\frac{\alpha}{\beta}\left[n-\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{-\alpha}+\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{-\alpha}e^{-\left(\frac{x_i}{\beta}\right)^{-\alpha}}\right]=0 \tag{13}$$

These equations 12 and 13 cannot be solved analytically and by using statistical softwares it can be possible.

5. Simulation Study

For different combinations of α and β, samples of sizes 25, 50, 100, 500 and 1000 are generated from the KM-IW model.
We calculate the bias and mean square error (MSE)s of the estimates. Simulation is conducted for two different combinations of parameter values which are $\alpha = 2$, $\beta = 0.5$ and $\alpha = 0.5$, $\beta = 1.5$. As the sample size increases, the mse value decreases (see Table 1 and 2). The bias of the estimates approaches to zero as n increases. And the estimate values approaches to the true parameter values.

Table 1: *Simulation study at $\alpha = 2$ and $\beta = 0.5$*

n	Estimated value of Parameters	Bias	MSE
25	$\hat{\alpha} = 2.0849208$	0.1165587	0.1566783
	$\hat{\beta} = 0.5670579$	0.005577362	0.002953267
50	$\hat{\alpha} = 1.8655174$	0.0514391	0.06337958
	$\hat{\beta} = 0.5121194$	0.00162314	0.001382758
100	$\hat{\alpha} = 2.000336$	0.0207338	0.02929086
	$\hat{\beta} = 0.513538$	0.001644057	0.0006789977
500	$\hat{\alpha} = 1.9363471$	-0.01219562	0.004949019
	$\hat{\beta} = 0.5115159$	0.0007829622	0.0001313705
1000	$\hat{\alpha} = 2.021294$	-0.01847309	0.002710569
	$\hat{\beta} = 0.488996$	0.0008567411	6.075921×10^{-05}

Table 2: *Simulation study at $\alpha = 0.5$ and $\beta = 1.5$*

n	Estimated value of Parameters	Bias	MSE
25	$\hat{\alpha} = 0.5212292$	0.02913895	0.009792336
	$\hat{\beta} = 2.4815462$	0.1813135	0.6622222
50	$\hat{\alpha} = 0.4663787$	0.01285905	0.003961199
	$\hat{\beta} = 1.6508171$	0.07065759	0.2386087
100	$\hat{\alpha} = 0.5000835$	0.005182735	0.00183067
	$\hat{\beta} = 1.6691755$	0.04467576	0.1099399
500	$\hat{\alpha} = 0.4840861$	-0.003049672	0.0003093182
	$\hat{\beta} = 1.6430402$	0.01415897	0.01952109
1000	$\hat{\alpha} = 0.5053225$	-0.004619055	0.0001694182
	$\hat{\beta} = 1.3722554$	0.0124857	0.008962368

6. Real Data Analysis

Here we use KM-IW(α, β) distribution to fit a real data set and compare the results with IW distribution, KME distribution and KMW distribution. The data-set, considered here, represents survival times in Days, from a Two-Arm Clinical Trial considered by [8] and [18]. The survival time in days for the 31 patients from Arm B are:

Table 3: *survival time in days for the 31 patients from Arm B*

37	84	92	94	110	112	119	127	130
133	140	146	155	159	173	179	194	195
209	249	281	319	339	432	469	519	633
725	817	1557	1776					

The analysis is performed by using R programming language. Table 4 gives the estimates of the model parameters, AIC (Akaike information criterion) and the BIC (Bayesian information criterion) values.

$$AIC = -2l + 2k \tag{14}$$

$$BIC = -2l + k \log n \tag{15}$$

where, l denotes the log-likelihood function, k is the number of parameters and n is the sample size. And Kolmogrov-Simnorov (K-S) test is also performed and the p-value is used for comparison.

Table 4: *Results of the Data Analysis*

Model	**Estimates**	**AIC**	**BIC**	**KS statistic**	**p-value**
KM-IW	$\hat{\alpha} = 1.1949$	417.3394	420.2074	0.0847	0.9657
	$\hat{\beta} = 190.7256$				
IW	$\hat{\alpha} = 1.3375$	417.4484	420.3164	0.0897	0.9452
	$\hat{\beta} = 150.4226$				
KME	$\hat{\lambda} = 0.0022$	426.1304	427.5644	0.2112	0.1085
KMW	$\hat{\alpha} = 1.1864$	426.3770	429.2450	0.1779	0.2493
	$\hat{\beta} = 464.2267$				

From the Table 4, we can see that our model KM-IW has better AIC, BIC values, KS statistic value and p-value than IW, KMW and KME distributions. So we can conclude that the newly proposed model is a better fit for the data taken than the other models.

7. Conclusions

A new model is introduced by modifying the Inverse Weibull distribution using the KM Transformation. Its statistical properties are studied along with the estimation of parameters. A simulation study is carried out for 2 parameter combinations and a data set is fitted by the KM-IW model which yielded a better AIC, BIC, KS-statistic values and p-value than the models compared in the study.

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