(M, MAP)/(PH, PH)/1 queue with Nonpreemptive Priority, Working Interruption and Protection

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Abstract

In this paper we consider a (M,MAP)/(PH,PH)/1 queue with nonpreemptive priority, working interruption and protection. Two types of priority classes of customers where type I customers arrive according to a Poisson process and type II customers arrive according to Markovian Arrival Process are considered. Service time of both type I and type II customers follow mutually independent phase type distributions. The number of type I customers in the system is restricted to a maximum of L. Also type I customers are assumed to have a non-preemptive priority over type II customers. Customer services are subject to interruption by a self induced mechanism. The interruptions occur according to Poisson process. Instead of stopping service completely, the service continues at slower rate during interruption. Also we assume that an interruption occurring while customer is already under interruption will not affect the customer. The server continues to serve at this lower rate until interruption is fixed. The duration of interruption is assumed to be exponentially distributed. A protection mechanism to diminish the effect of interruptions on type I customers service is arranged. The protection for the service of type I customers is provided at the epoch of realization of the clock which starts ticking up the moment a type I customer is taken for service. Type II customers are not provided protection against interruption during their service. Also we assume that type I customers get service at a faster rate starting from the epoch of providing service protection. We analyse the distribution of service time duration of both type I and type II customers and the distribution of a p-cycle. Also we provide LSTs of busy cycle, busy period of type I customers generated during the service time of a type II customer and LSTs of waiting time distributions of type I and type II customers. Also we compute the expected number of interruptions during a type I and a type II service. We perform numerical computations to evaluate important system characteristics and also optimal system cost using a cost function.

Keywords: (M,MAP)/(PH,PH)/1 queue, nonpreemptive priority, working interruption, protection

1 Introduction

Queues with interruption play an important role in day to day life. We encounter different kinds of interruptions in various activities like internet browsing, banking, medical check ups, in supermarkets etc. The works so far reported in the literature discuss about interruptions such as server induced, customer induced, environment dependent service interruptions, server vacations,
vacation interruptions and arrival of a priority customer. The first reported work on queues with service interruption is by White and Christie in 1958 in which they considered a two-priority single server system with the low priority customer in service pre-empted on arrival of a high priority customer. Even in the case of single class customer system, the customer in service has to wait whenever a system breakdown occurs. The interrupted service starts from the very beginning (repeat) or from where it got interrupted (resumption) on completion of interruption. These two cases are separately considered in Keilson [2], Gaver [4] and by several other researchers. Fiems et al. [3] introduced probability measures for repeat/resumption on completion of interruption without assigning any rule. Krishnamoorthy et al. [6] are the first to give a specific rule for resumption/repetition of service. We refer the review paper by Krishnamoorthy et al. [5] for details on queueing models with system induced service interruption (priority queues not included).

Varghese et al. [12] introduced a new type of interruption called customer induced interruption in which a customer interrupts own service. They considered an infinite capacity queueing system with a single server in which customers arrive according to a Poisson process with the service time following an exponential distribution. The interruptions occur according to a Poisson process and the duration of each interruption follows an exponential distribution. The self-interrupted customers enter into a finite buffer of size K. Any interrupted customer, finding the buffer full, is considered lost. Those interrupted customers who complete their interruptions move into another buffer of same size and are given a nonpreemptive priority over new customers. They evaluated several performance measures. Numerical illustrations of the system behavior are also provided and also discussed an optimization problem through an illustrative example. Krishnamoorthy et al. [7] extended this to a multi-server queueing system. They investigated the behavior of the queueing system, several performance measures are evaluated and numerical illustrations of the system behavior are provided. Also an optimization problem to maximize the revenue with respect to number of servers is employed and optimal buffer size for the self-interrupted customers are discussed through two illustrative examples. Dudin et al. [13] extended these to MMAP/PH(PH)/c queue with negative arrivals. Varghese and Krishnamoorthy [8] considered a single-server retrial queue with infinite capacity of the primary buffer and finite capacity of the orbit to which customers arrive according to a Poisson process, and the service time follows phase-type distribution. The customer-induced interruption occurs according to a Poisson process. The self-interrupted customers enter into orbit. Any interrupted customer, finding the orbit full, is considered lost. The interrupted customers retries for service after the interruption is completed. Several performance measures were evaluated and some numerical illustrations of the system behavior were provided.

In this paper we consider a single server queueing model with two priority classes of customers where the type I customers are assumed to have a non-preemptive priority over type II customers. We consider customer induced interruption during own service. Instead of stopping service completely, the service continues at slower rate during interruption. The protection for the service of type I customers is provided at the epoch of realization of the clock which starts at the epoch at which the type I customer is taken for service. The rest of the paper is arranged as follows. The mathematical formulation is given in section 2. Section 3 provides steady state analysis of the model. Waiting time analysis of type I and type II customers are discussed in sections 4 and 5 respectively. Some other performance measures are discussed in section 7. A related cost function is discussed in section 8. Some numerical results are discussed in section 9. Proofs of two theorems stated in section 4, are given in appendix.

Notations and abbreviations used in the sequel:

• e(a) = Column vector of 1’s of order a
• e = Column vector of 1’s of appropriate order.
• **CTMC**: Continuous time Markov chain.
• \( I_a \) = identity matrix of order \( a \).
• \( e_a(b) \) = column vector of order \( b \) with 1 in the \( a \)-th position and the remaining entries zero.
• **MAP**: Markovian Arrival Process
• **LST**: Laplace-Steiltjes Transform
• **LIQBD**: Level independent Quasi-Birth and-Death
• **WI**: Working Interruption

### Parameters:
- \( \lambda \): arrival rate of type I customers,
- \( \gamma \): arrival rate of interruptions,
- \( \eta \): parameter of exponential duration of interruption,
- \( \delta \): parameter of exponential protection clock.

## 2 Mathematical formulation

We consider a single server queue with two priority classes of customers type I and type II with the former arriving according to a Poisson process of rate \( \lambda \) and the latter according to Markovian Arrival Process with representation \((D_0, D_1)\). Service time of both types follow distinct phase type distributions with representations \( \text{PH}(\alpha, T) \) of order \( m_1 \) and \( \text{PH}(\beta, S) \) of order \( m_2 \) respectively. The number of type I customers in the system is restricted to a maximum of \( L \). Also type I customers are assumed to have a non-preemptive priority over type II customers. Customer services are subject to interruption by a self induced mechanism. While in interruption arrival of another interruption does not affect the customer. The interruptions occur according to Poisson process with rate \( \gamma \). Instead of stopping the service of that customer completely, it continues at slower rate during interruption. That is, the service time of type I and type II, during an interruption follow phase type distributions with representation \( \text{PH}(\alpha, \theta T) \) and \( \text{PH}(\beta, \theta' S) \), \( 0 < \theta, \theta' < 1 \) respectively. Thus \( \mu = [\alpha (-T)^{-1} e]^{-1} \) is the normal service rate and \( \theta \mu \) is the interrupted service rate of type I customers and \( \mu' = [\beta (-S)^{-1} e]^{-1} \) and \( \theta' \mu' \) are respectively the corresponding rates of normal and interrupted services of type II customers. The server continues to serve at this lower rate until a random clock expires. The duration of interruption is assumed to be exponentially distributed with parameter \( \eta \). A protection mechanism to diminish the effect of interruptions on type I customers service is arranged. An exponential random clock with mean \( \frac{1}{\delta} \) is started simultaneously with each type I service. The protection for the service of type I customers is provided at the epoch of realization of this clock. Type II customers are not provided protection against interruption during their service. Also we assume that the service time of type I customers on activation of protection clock, follows phase type distribution with representation \( \text{PH}(\alpha, \phi T) \), \( \phi > 1 \) and finite.

Let \( Q^* = D_0 + D_1 \) be the generator matrix of the type II arrival process and \( \pi^* \) be its stationary probability vector. Hence \( \pi \) is the unique (positive) probability vector satisfying \( \pi^* Q^* = 0 \), \( \pi^* e = 1 \). The constant \( \beta^* = \pi^* D_1 e \), referred to as fundamental rate, gives the expected number of type II arrivals per unit of time in the stationary version of the MAP. It is assumed that the two arrival processes are mutually independent and are also independent of the service time distributions.

### 2.1 The QBD process

The model described in section 1 can be studied as a LIQBD process. First we introduce the following notations:

At time \( t \):
- \( N_1(t) \): number of type II customers in the system,
- \( N_2(t) \): number of type I customers in the system.
\[ J(t) = \begin{cases} 
0, & \text{if the type I customer in service is unprotected/type II customer is in service} \\
1, & \text{if the type I customer in service is protected} 
\end{cases} \]

\[ K(t) = \begin{cases} 
0, & \text{if the server provides service to type I customer in WI} \\
1, & \text{if the server provides service to type II customer in WI} \\
2, & \text{if the server provides normal service to type I customer} \\
3, & \text{if the server provides normal service to type II customer} 
\end{cases} \]

S(t): the phase of service when the server is busy

\[ M(t) : \text{the phase of arrival of the type II customer.} \]

It is easy to verify that \((N_1(t),N_2(t),J(t),K(t),S(t),M(t)) : t \geq 0)\] is a LIQBD with state space

\[ l(0) = \{(0,k)/1 \leq k \leq n \} \cup \{(0,i_2,0,j_2,k_1,k_2)/1 \leq i_2 \leq L,j_2 = 0 \text{ or } 2,1 \leq k_1 \leq m_1,1 \leq k \leq n \} \cup \{(0,i_2,1,2,k_1,k_2)/1 \leq i_2 \leq L,1 \leq k_1 \leq m_1,1 \leq k_2 \leq n \} \]

For \(i_1 \geq 1, \)

\[ \{(i_1,0,0,j_2,k_1,k_2)/j_2 = 1 \text{ or } 3,1 \leq k_1 \leq m_1,1 \leq k_2 \leq n \} \cup \{(i_1,i_2,0,j_2,k_1,k_2)/1 \leq i_2 \leq L,j_2 = 0 \text{ or } 2,1 \leq k_1 \leq m_1,1 \leq k_2 \leq n \} \cup \{(i_1,i_2,1,2,k_1,k_2)/1 \leq i_2 \leq L,1 \leq k_1 \leq m_1,1 \leq k_2 \leq n \} \]

The infinitesimal generator of this CTMC is

\[ Q_1 = \begin{bmatrix} 
B_0 & C_0 & & & \\
B_1 & A_1 & A_0 & & \\
A_2 & A_1 & A_0 & & \\
& & & & 
\end{bmatrix} \]

where \(B_0\) contains transitions within the level 0; \(C_0\) represents transitions from level 0 to level 1; \(B_1\) represents transitions from level 1 to level 0; \(A_0\) represents transitions from level \(g\) to level \(g+1\) for \(g \geq 1, A_1\) represents transitions within the level \(g\) for \(g \geq 1\) and \(A_2\) represents transitions from level \(g\) to \(g-1\) for \(g \geq 2\). The boundary blocks \(B_0, C_0, B_1\) are of orders \(n(1+3m_1L) \times n(1+3m_1L), n(1+3m_1L) \times (2m_2n+ (3m_1+ 2m_2)nL), (2m_2n+ (3m_1+ 2m_2)nL) \times n(1+3m_1L)\) respectively. \(A_0, A_1, A_2\) are square matrices of order \(2m_2n+ (3m_1+ 2m_2)nL\). Define the entries of \(B_0^{(h_1,i_1,j_1,k_1,l_1)}C_0^{(h_1,i_1,j_1,k_1,l_1)}B_1^{(h_1,i_1,j_1,k_1,l_1)}\) as transition submatrices which contains transitions of the form

\[ (0,h_1,i_1,j_1,k_1,l_1) \rightarrow (0,h_2,i_2,j_2,k_2,l_2) \]

\[ (0,h_1,i_1,j_1,k_1,l_1) \rightarrow (1,h_2,i_2,j_2,k_2,l_2) \text{ and } (1,h_1,i_1,j_1,k_1,l_1) \rightarrow (0,h_2,i_2,j_2,k_2,l_2) \text{ respectively.} \]

Define the entries of \(B_0^{(h_1,i_1,j_1,k_1,l_1)}A_1^{(h_2,i_2,j_2,k_2,l_2)}B_1^{(h_2,i_2,j_2,k_2,l_2)}\) as transition submatrices which contains transitions of the form

\[ (g,h_1,i_1,j_1,k_1,l_1) \rightarrow (g+1,h_2,i_2,j_2,k_2,l_2) \]

where \(g \geq 1, (g,h_1,i_1,j_1,k_1,l_1) \rightarrow (g,h_2,i_2,j_2,k_2,l_2)\), where \(g \geq 1, (g,h_1,i_1,j_1,k_1,l_1) \rightarrow (g-1,h_2,i_2,j_2,k_2,l_2)\), where \(g \geq 2\) respectively. Since none or one event alone could take place in a short interval of time with positive probability, in general, a transition such as \((g_1,h_1,i_1,j_1,k_1,l_1) \rightarrow (g_2,h_2,i_2,j_2,k_2,l_2)\) has positive rate only for exactly one of \(g_1,h_1,i_1,j_1,k_1,l_1\) different from \(g_2,h_2,i_2,j_2,k_2,l_2\).
\[
A^{(b_2, l_2, j_2, k_2, l_2)}_{(h_1, l_1, i_1, k_1, l_1)} = \begin{cases}
\lambda l_{m,n} & 1 \leq h_1 \leq L - 1, h_2 = h_1 + 1; i_1 = i_2 = 0; j_1 = j_2 = 0 or 2; \\
l_{m,n} & 0 \leq h_1 \leq L - 1, h_2 = h_1 + 1; i_1 = i_2 = 0; j_1 = j_2 = 1 or 3; \\
l_{m,n} & 1 \leq k_1, k_2 \leq m_1; 1 \leq l_1, l_2 \leq n \\
l_{m,n} & 1 \leq h_1 \leq L - 1, h_2 = h_1 + 1; i_1 = i_2 = 1; j_1 = j_2 = 2; \\
l_{m,n} & 1 \leq k_1, k_2 \leq m_1; 1 \leq l_1, l_2 \leq n \\
h \in N & h_1 = h_2 = 0; i_1 = i_2 = 0; j_1 = j_2 = 1 or 3; 1 \leq k_1, k_2 \leq m_2; \\
S^0 \beta \otimes I_n & h_1 = h_2 = 0; i_1 = i_2 = 0; j_1 = j_2 = 3; 1 \leq k_1, k_2 \leq m_2; \\
S^0 \alpha \otimes I_n & h_1 = h_2 = 0; i_1 = i_2 = 0; j_1 = j_2 = 3; 1 \leq k_1, k_2 \leq m_2; \\
\end{cases}
\]
3 Steady State Analysis

Let \( \pi = (\pi_0, \pi_1, ..., \pi_L) \) denote the steady state probability vector of the generator

\[
A = A_0 + A_1 + A_2 =
\begin{bmatrix}
F_0 & F_1 \\
F_2 & F_3 & \lambda I \\
& \ddots & \ddots & \ddots \\
& & F_4 & F_3 & \lambda I \\
& & & F_4 & F_3 & \lambda I
\end{bmatrix}, \quad \text{ie,} \quad \pi A = 0, \pi e = 1. \tag{1}
\]

In the above,

\[
F_0(k, l) = \begin{cases} 
\theta S \otimes D_0 - (\lambda + \eta) I_{m_2 n} + I_{m_2} \otimes D_1, & k = 1, l = 1 \\
\eta I_{m_2 n} + \theta' S \otimes I_n, & k = 1, l = 2 \\
\gamma I_{m_2 n}, & k = 2, l = 1
\end{cases}, \quad F_1(k, l) =
\]

\[
\begin{cases} 
\lambda I_{m_2 n}, & k = 1, l = 2 \\
\lambda I_{m_2 n}, & k = 2, l = 4 \\
0, & \text{otherwise,}
\end{cases}
\]

\[
F_2(k, l) = \begin{cases} 
\theta T \otimes D_0 - (\lambda + \eta + \delta) I_{m_1 n} + I_{m_1} \otimes D_1, & k = 1, l = 1 \\
\eta I_{m_1 n}, & k = 1, l = 3 \\
\delta I_{m_1 n}, & k = 1, l = 5 \\
\theta' S \otimes D_0 - (\lambda + \eta) I_{m_2 n} + I_{m_2} \otimes D_1, & k = 2, l = 2 \\
\theta' S \otimes I_n, & k = 2, l = 3 \\
\eta I_{m_1 n}, & k = 2, l = 4 \\
\gamma I_{m_1 n}, & k = 3, l = 1 \\
T \otimes D_0 - (\lambda + \gamma + \delta) I_{m_1 n} + I_{m_1} \otimes D_1, & k = 3, l = 3 \\
\delta I_{m_1 n}, & k = 3, l = 5 \\
\gamma I_{m_1 n}, & k = 4, l = 2 \\
S \otimes D_0 - (\lambda + \gamma) I_{m_2 n} + I_{m_2} \otimes D_1, & k = 4, l = 4 \\
\phi T \otimes D_0 - \lambda I_{m_1 n} + I_{m_1} \otimes D_1 & k = 5, l = 5 \\
0, & \text{otherwise}
\end{cases}
\]

\[
F_4(k, l) = \begin{cases} 
\theta T \otimes I_n, & k = 1, l = 3 \\
T \otimes I_n, & k = 3, l = 3 \\
\phi T \otimes I_n, & k = 5, l = 3 \\
0, & \text{otherwise}
\end{cases}
\]
\[
\begin{align*}
\theta T \otimes D_0 - (\eta + \delta)I_{m_1 n} + I_{m_1} \otimes D_1 & \quad k = 1, l = 1 \\
\eta I_{m_1 n} & \quad k = 1, l = 3 \\
\delta I_{m_1 n} & \quad k = 1, l = 5 \\
\theta S \otimes D_0 - \eta I_{m_2 n} + I_{m_2} \otimes D_1 & \quad k = 2, l = 2 \\
\theta' S \otimes I_n & \quad k = 2, l = 3 \\
\eta I_{m_2 n} & \quad k = 2, l = 4 \\
\gamma I_{m_2 n} & \quad k = 3, l = 1 \\
T \otimes D_0 - (\gamma + \delta)I_{m_1 n} + I_{m_1} \otimes D_1 & \quad k = 3, l = 3 \\
\delta I_{m_1 n} & \quad k = 3, l = 5 \\
\gamma I_{m_2 n} & \quad k = 4, l = 2 \\
S \otimes D_0 - \gamma I_{m_2 n} + I_{m_2} \otimes D_1 & \quad k = 4, l = 4 \\
\phi T \otimes D_0 + I_{m_1} \otimes D_1 & \quad k = 5, l = 5 \\
0 & \quad \text{otherwise}
\end{align*}
\]

with dimensions of \( F_0, F_1, F_2 \) be \( 2m_2 n \times 2m_2 n, \ 2m_2 n \times (3m_1 + 2m_2)n, (3m_1 + 2m_2)n \times 2m_2 n \) respectively. \( F_0, F_1 \) and \( F_2 \) are square matrices of order \((3m_1 + 2m_2)n\). The \( LIQBD \) description of the model indicates that the queueing system is stable (see Neuts [9]) if and only if the left drift exceeds that of right drift. That is,

\[
\pi A_0 e < \pi A_2 e. \tag{2}
\]

The vector \( \pi \) cannot be obtained directly in terms of the parameters of the model. The inequality (2) is simplified in (5) below. From (1) we get

\[
\pi_i = \pi_{i-1} U_{i-1}, \ 1 \leq i \leq L \tag{3}
\]

where

\[
U_0 = -F_2(F_3 + U_1 F_4)^{-1}
\]

\[
U_i = \begin{cases} 
-\lambda(F_3 + U_{i+1} F_4)^{-1} & \text{for } 1 \leq i \leq L - 2 \\
-\lambda F_5^{-1} & \text{for } i = L - 1.
\end{cases}
\]

From the normalizing condition \( \pi e = 1 \) we have

\[
\pi_0 \left( \sum_{j=0}^{L-1} U_i e \right) e = 1. \tag{4}
\]

The inequality (2) gives the stability condition as

\[
\pi_0 \left[ (I_{(2m_2)} \otimes D_1) e + \sum_{j=0}^{L-1} U_j (I_{3m_1 + 2m_2} \otimes D_1) e \right] < \pi_0 \left[ e_1(2)(\theta S^0 \otimes I) + e_2(2)(S^0 \otimes I) e(m_2 n) + \sum_{j=0}^{L-1} U_j \left[ e_2(5)(\theta S^0 \otimes I) + e_4(5)(S^0 \otimes I) e(m_2 n) \right] \right]. \tag{5}
\]

Let \( \mathbf{x} \) be the steady state probability vector of \( Q \). We partition this vector as \( \mathbf{x} = (x_0, x_1, x_2, \ldots) \), where \( x_0 \) is of dimension \( n(1 + 3m_1 L) \) and \( x_1, x_2, \ldots \) are each of dimension \( n(2m_2 + (3m_1 + 2m_2)L) \). Under the stability condition, we have \( x_i = x_1 R^{i-1}, i \geq 2 \), where the matrix \( R \) is the minimal nonnegative solution to the matrix quadratic equation

\[
R^2 A_2 + R A_1 + A_0 = 0
\]

and the vectors \( x_0 \) and \( x_1 \) are obtained by solving the equations

\[
x_0 B_0 + x_1 B_1 = 0 \tag{6}
\]

\[
x_0 C_0 + x_1 (A_1 + R A_2) = 0 \tag{7}
\]

subject to the normalizing condition

\[
x_0 e + x_1 (I - R)^{-1} e = 1 \tag{8}
\]

3.1 Analysis of service time of a type I customer

The duration of service of a type I customer is a phase type distribution with
representation \((\alpha',S_1)\) where the underlying MC has state space \(((i,j,k): i = 0, j = 0 \text{ or } 2, 1 \leq k \leq m_1) \cup ((i,2,k): i = 1, 1 \leq k \leq m_1)\) where \(i\) denotes the status of the protection clock, \(j\) the status of the server, \(k\), the service phase and \(*\), the absorbing state indicating service completion. The infinitesimal generator is

\[
S_1 = \begin{bmatrix} S_1 & S_0 \\ 0 & 0 \end{bmatrix}, \text{where, } S_1 = \begin{bmatrix} \theta T - (\eta + \delta)I_{m_1} & \eta I_{m_1} \\ \gamma I_{m_1} & T - (\gamma + \delta)I_{m_1} \end{bmatrix} \text{ and } S_0 = \begin{bmatrix} \theta T^0 & 0 \\ 0 & \phi T^0 \end{bmatrix}
\]

The initial probability vector is \(\alpha' = [0 \alpha 0]\), where 0 is a zero matrix of order \(1 \times m_1\).

Thus the service time distribution of a type I customer is \(Ph(\alpha',S_1)\) of order \(3m_1n\).

### 3.2 Analysis of service time of a type II customer

The duration of service of a type II customer turns out to be a phase type distribution \((\beta',S_2)\) where the underlying MC has state space \(((i,j): i = 1 \text{ or } 3, 1 \leq j \leq m_2) \cup \{*\}\) where \(i\) denotes the status of the server, \(j\) the service phase and \(*\), the absorbing state indicating service completion. The infinitesimal generator is

\[
S_2 = \begin{bmatrix} S_2 & S_0 \\ 0 & 0 \end{bmatrix}, \text{where, } S_2 = \begin{bmatrix} \theta' S - \eta I_{m_2} & \eta I_{m_2} \\ \gamma I_{m_2} & S - \gamma I_{m_2} \end{bmatrix} \text{ and } S_0 = \begin{bmatrix} \theta' S^0 \\ S_0 \end{bmatrix}
\]

The initial probability vector is \(\beta' = [0 \alpha]\), where 0 is a zero matrix of order \(1 \times m_2\). Thus we have the service time distribution of a type II customer is \(Ph(\beta',S_2)\) of order \(2m_2n\).

### 4 Waiting time analysis

#### 4.1 Type I Customer

To find the waiting time of a type I customer who joins for service at time \(t\), we have to consider different possibilities depending on the status of server at that time. Let \(W(t)\) be the waiting time of a type I customer who arrives at time \(t\) and \(W^*(s)\) be the corresponding LST.

**Case I**

Suppose that \(E_1\) denote the event the system is in the state \((0,v), 1 \leq v \leq n\) when the tagged type I customer arrives. Let \(W^*(s/E_1)\) denote the corresponding LST. Then

\[
W^*(s/E_1) = 1
\]

**Case II**

\(E_2\) be the event that the system is in the state \((n_1,a,0,0,u,v), n_1 \geq 0, 1 \leq a \leq L - 1, 1 \leq u \leq m_1, 1 \leq v \leq n\) when the tagged customer arrives the system. In this case the waiting time is the sum of the residual service time of the type I customer in service when the tagged customer arrives and service time of \(a - 1\) remaining type I customers. Let \(W^*(s/E_2)\) represent the corresponding conditional LST. Then

\[
W^*(s/E_2) = (e_u(3m_1)(sl - S_1)^{-1}S_1^0)(\alpha'(sl - S_1)^{-1}S_1^0)^{a-1}.
\]

**Case III**

\(E_3\) denotes the event: the system is in the state \((n_2,a,0,2,u,v), n_2 \geq 0, 1 \leq a \leq L - 1, 1 \leq u \leq m_1, 1 \leq v \leq n\) when the tagged customer arrives the system. In this case the waiting time is the sum of the residual service time of the type I customer in service when the tagged customer arrives and service times of \(a - 1\) remaining type I customers. With \(W^*(s/E_3)\) as the corresponding conditional LST, we have

\[
W^*(s/E_3) = (e_{m_1+u}(3m_1)(sl - S_1)^{-1}S_1^0)(\alpha'(sl - S_1)^{-1}S_1^0)^{a-1}.
\]
Case IV

$E_4$ denotes the event: the system is in the state $(n_3, a, 1, 2, u, v), n_3 \geq 0, 1 \leq a \leq L - 1, 1 \leq u \leq m_1, 1 \leq v \leq n$, when the tagged customer arrives the system. In this case the waiting time is the sum of the residual service time of the type I customer in service when the tagged customer arrives and service times of $a - 1$ remaining type I customers. Let $W^*(s/E_4)$ represent the corresponding conditional LST. Then

$$W^*(s/E_4) = (e_{2m_2 + u}(3m_1)(sl - S_1)^{-1}S_2^0)(a'(sl - S_1)^{-1}S_1^0)^{a-1}.$$ 

Case V

$E_5$ denotes the event: the system is in the state $(n_1, a, 0, 1, u, v), n_1 \geq 1, 0 \leq a \leq L - 1, 1 \leq u \leq m_2, 1 \leq v \leq n$, when the tagged customer arrives the system. In this case the waiting time is the sum of the residual service time of the type II customer in service when the tagged customer arrives and service times of $a - 1$ remaining type I customers. Let $W^*(s/E_5)$ represent the corresponding conditional LST. Then

$$W^*(s/E_5) = (e_u(2m_2)(sl - S_2)^{-1}S_2^0)(a'(sl - S_1)^{-1}S_1^0)^a.$$ 

Case VI

$E_6$ denotes the event: the system is in the state $(n_1, a, 0, 3, u, v), n_1 \geq 1, 0 \leq a \leq L - 1, 1 \leq u \leq m_2, 1 \leq v \leq n$, when the tagged customer arrives the system. In this case the waiting time is the sum of the residual service time of the type II customer in service when the tagged customer arrives and service times of a remaining type I customers. Let $W^*(s/E_6)$ represent the corresponding conditional LST. Then

$$W^*(s/E_6) = (e_u(2m_2)(sl - S_2)^{-1}S_2^0)(a'(sl - S_1)^{-1}S_1^0)^a.$$ 

Thus the LST of the waiting time

$$W^*(s) = \frac{1}{d} \sum_{n=1}^{\infty} x_{0,n} + \sum_{n=1}^{\infty} \sum_{a=1}^{L-1} \sum_{u=1}^{m_1} \sum_{v=1}^{n} W^*(s/E_2) x_{n,a,0,0,u,v} + \sum_{n=1}^{\infty} \sum_{a=1}^{L-1} \sum_{u=1}^{m_1} \sum_{v=1}^{n} W^*(s/E_3) x_{n,a,0,2,u,v} + \sum_{n=1}^{\infty} \sum_{a=1}^{L-1} \sum_{u=1}^{m_1} \sum_{v=1}^{n} W^*(s/E_4) x_{n,a,1,2,u,v} + \sum_{n=1}^{\infty} \sum_{a=1}^{L-1} \sum_{u=1}^{m_1} \sum_{v=1}^{n} W^*(s/E_5) x_{n,a,0,1,u,v}$$

$$+ \sum_{n=1}^{\infty} \sum_{a=1}^{L-1} \sum_{u=1}^{m_1} \sum_{v=1}^{n} W^*(s/E_6) x_{n,a,0,0,3,u,v}$$

where, $d = \sum_{n=1}^{\infty} x_{0,n} + \sum_{n=1}^{\infty} \sum_{a=1}^{L-1} \sum_{u=1}^{m_1} \sum_{v=1}^{n} x_{n,a,0,0,u,v} + \sum_{n=1}^{\infty} \sum_{a=1}^{L-1} \sum_{u=1}^{m_1} \sum_{v=1}^{n} x_{n,a,0,2,u,v} + \sum_{n=1}^{\infty} \sum_{a=1}^{L-1} \sum_{u=1}^{m_1} \sum_{v=1}^{n} x_{n,a,1,2,u,v} + \sum_{n=1}^{\infty} \sum_{a=1}^{L-1} \sum_{u=1}^{m_1} \sum_{v=1}^{n} x_{n,a,0,1,u,v} + \sum_{n=1}^{\infty} \sum_{a=1}^{L-1} \sum_{u=1}^{m_1} \sum_{v=1}^{n} x_{n,a,0,0,1,u,v} + \sum_{n=1}^{\infty} \sum_{a=1}^{L-1} \sum_{u=1}^{m_1} \sum_{v=1}^{n} x_{n,a,0,0,3,u,v}.$

4.2 Type II customer

To find the LST of the waiting time distribution of a type II customer, we have to compute certain distributions. We proceed to such computations.

**Definition 1** Consider the duration of time with $p$ type I customers in the system at a service commencement epoch of type I customers until the number of type I customers become zero for the first time, we call this a $p$-cycle, denoted by $B_p$.

4.2.1 Distribution of a $p$-cycle

This is a phase type distribution with representation $(\gamma_p, T_1)$ where the underlying Markov chain has state space $\{(i, j, k, l): 1 \leq i \leq L, j = 0, k = 0 or 2, 1 \leq l \leq m_1 \} \cup \{(i, j, k, l): 1 \leq i \leq L, j = 1, k = 2, 1 \leq l \leq m_1 \} \cup \{\ast\}$ and $i, j, k, l$ and $\ast$ respectively denote the number of type I customers in the system, the status of the protection clock, the status of the server, the service phase and the absorbing state indicating that the number of type I customers become zero. The infinitesimal generator $T_1$ of $B_p(t)$ has the form
\[ T_1 = \begin{bmatrix} T_1 & T_1^0 \\ 0 & 0 \end{bmatrix}, \text{where} T_1 = \begin{bmatrix} E_1 & \lambda I_{m_1} \\ E_2 & E_1 & \lambda I_{m_1} \\ \vdots & \vdots & \ddots & \ddots \\ E_2 & E_1 & \lambda I_{m_1} \\ 0 \end{bmatrix} \]

\[ T_1 = \begin{bmatrix} E^0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \]

where

\[ E_1 = \begin{bmatrix} \beta T_1 & \lambda T_1 & \delta T_1 \\ \gamma T_1 & \delta T_1 & \gamma T_1 \\ 0 & \beta T_1^0 & 0 \\ 0 & \gamma T_1^0 & 0 \\ 0 & \lambda T_1^0 & 0 \end{bmatrix} \]

\[ E_2 = \begin{bmatrix} \beta T_2 & \lambda T_2 & \delta T_2 \\ \gamma T_2 & \delta T_2 & \gamma T_2 \\ 0 & \beta T_2^0 & 0 \\ 0 & \gamma T_2^0 & 0 \\ 0 & \lambda T_2^0 & 0 \end{bmatrix} \]

\[ E_3 = \begin{bmatrix} \beta T_3 & \lambda T_3 & \delta T_3 \\ \gamma T_3 & \delta T_3 & \gamma T_3 \\ 0 & \beta T_3^0 & 0 \\ 0 & \gamma T_3^0 & 0 \\ 0 & \lambda T_3^0 & 0 \end{bmatrix} \]

\[ E^0 = \begin{bmatrix} \beta T^0_1 \\ T^0 \phi^0 \end{bmatrix} \]

The initial probability vector is

\[ y_p = \begin{bmatrix} 0 & \ldots & 0 & \gamma' & 0 & \ldots & 0 \end{bmatrix}, 1 \leq p \leq L \]

where 0 is a zero matrix of order \(1 \times 3m_1\), with \(\gamma' = \begin{bmatrix} 0 & \alpha & 0 \end{bmatrix}, 1 \leq p \leq L \) is in the \(p\)th position and 0 is a zero matrix of order \(1 \times m_1\).

We can compute the LST of the length of the busy period as

\[ y_p (sI - T_1)^{-1} T_1^0 \]

4.2.2 LST of the busy cycle generated by type I customers arriving during the service time of a type II customer

**Theorem 1**

The LST of the busy cycle generated by type I customers arriving during the service time of a type II customer is given by

\[ \tilde{B}_c(s) = \beta'[(s + \lambda)I - S_2]^{-1} S_2^0 + \sum_{p=1}^{L} y_p (sI - T_1)^{-1} T_1^0 \beta'^{(p+1)}[(s + \lambda)I - S_2]^{-1} S_2^0 \]

\[ \gamma_L(sI - T_1)^{-1} T_1^0 \beta'[(s + \lambda)I - S_2]^{-1} I - \lambda[(s + \lambda)I - S_2]^{-1}[(s + \lambda)I - S_2]^{-1} S_2^0 \]

\[ \beta'[\lambda^{-1}((s + \lambda)I - S_2)]^{-1} I - \lambda[(s + \lambda)I - S_2]^{-1}[(s + \lambda)I - S_2]^{-1} S_2^0 \] (10)

**Proof.**

The proof is given in the appendix.

4.2.3 LST of the busy period of type I customers generated during the service time of a type II customer
Theorem 2

The LST of the busy period generated by type I customers arriving during the service time of a type II customer is given by

$$
\hat{B}_L(s) = \beta [\lambda I - S_2]^{-1} S_0 + \sum_{\alpha = 1}^\infty \gamma_\alpha (s l - T_1)^{-1} T_0 \beta' [\lambda I - S_2]^{-1} S_0
$$

(11)

Proof.

The proof is given in the appendix.

Now, to find the waiting time of a type II customer who joins for service at time $t$, we have to consider different possibilities depending on the status of server at that time. Let $W(t)$ be the waiting time of a type II customer who arrives at time $t$ and $W^*(s)$ be the corresponding LST.

Case I

Suppose that $F_1$ denotes the event the system is in the state $(0, v), 1 \leq v \leq n$ when the tagged customer arrives. Let $W^*(s/F_1)$ denote the corresponding LST. Then

$$W^*(s/F_1) = 1$$

Case II

$F_2$ be the event that the system is in one of the states $(b, a, 0, 0, u, v), b \geq 0, 1 \leq a \leq L, 1 \leq u \leq m_1, 1 \leq v \leq n$ when the tagged customer arrives. In this case, the waiting time is the length of the busy cycle generated by a type I customers starting from his arrival epoch plus lengths of busy cycles of type I customers generated during service times of each of the $b$ type II customers. Let $W^*(s/F_2)$ denote the corresponding LST. Then

$$W^*(s/F_2) = e_{(a-1)m_1 + u}(3l m_1)(s l - T_1)^{-1} T_0 (\hat{B}_L(s))^b$$

Case III

$F_3$ denotes the event the system is in one of the states $(b, a, 0, 2, u, v), b \geq 0, 1 \leq a \leq L, 1 \leq u \leq m_1, 1 \leq v \leq n$ when the tagged customer arrives. In this case, the waiting time is the length of the busy cycle generated by a type I customers starting from his arrival epoch plus lengths of busy cycles of type I customers generated during service times of each of the $b$ type II customers. Let $W^*(s/F_3)$ denote the corresponding LST. Then

$$W^*(s/F_3) = e_{(a-1)m_1 + m_1 + u}(3l m_1)(s l - T_1)^{-1} T_0 (\hat{B}_L(s))^b$$

Case IV

$F_4$ denotes the event the system is in one of the states $(b, a, 1, 2, u, v), b \geq 0, 1 \leq a \leq L, 1 \leq u \leq m_1, 1 \leq v \leq n$ when the tagged customer arrives. In this case, the waiting time is the length of the busy cycle generated by a type I customers starting from his arrival epoch plus lengths of busy cycles of type I customers generated during service times of each of the $b$ type II customers. Let $W^*(s/F_4)$ denote the corresponding LST. Then

$$W^*(s/F_4) = e_{(a-1)m_1 + 2m_1 + u}(3l m_1)(s l - T_1)^{-1} T_0 (\hat{B}_L(s))^b$$

Case V

$F_5$ denotes the event the system is in one of the states $(b, a, 0, 1, u, v), b \geq 1, 0 \leq a \leq L, 1 \leq u \leq m_2, 1 \leq v \leq n$ when the tagged customer arrives. In this case, the waiting time is the length of residual service time of the type II customer in service plus length of the busy period generated by type I customers arriving during the service time of the type II customer in service plus lengths of busy cycles of type I customers generated during service time of each of the $b - 1$ type II customers. Let $W^*(s/F_5)$ denote the corresponding LST. Then

$$W^*(s/F_5) = e_u(2m_2)(s l - S_2)^{-1} S_0 (\hat{B}_L(s) (\hat{B}_L(s)))^{b-1}$$

Case VI

$F_6$ denotes the event the system is in one of the states $(b, a, 0, 3, u, v)$ when the tagged customer arrives. In this case the waiting time is the length of residual service time of the type II customer in service plus the length of the busy period generated by type I customers arriving during the service time of the type II customer in service plus lengths of busy cycles of type I
customers generated during service time of each of the \( b-1 \) type II customers. Let \( W^*(s/F_b) \) denote the corresponding LST. Then

\[
W^*(s/F_b) = e^{m_2+u(2m_2)}(sI - S_2)^{-1}S_2^0\tilde{B}_L(s)(\tilde{B}_c(s))^{b-1}
\]

Thus the LST of the waiting time

\[
W^*(s) = \sum_{\nu=1}^n x_{0,\nu} + \sum_{\nu=0}^m L_{\nu-1} \sum_{\nu=1}^n \sum_{\nu=1}^n W^*(s/F_2) x_{b,a,0,0,u,v} + \sum_{\nu=0}^m L_{\nu-1} \sum_{\nu=1}^n \sum_{\nu=1}^n W^*(s/F_3) x_{b,a,0,2,u,v} + \sum_{\nu=0}^m L_{\nu-1} \sum_{\nu=1}^n \sum_{\nu=1}^n W^*(s/F_4) x_{b,a,0,1,2,u,v} + \sum_{\nu=0}^m L_{\nu-1} \sum_{\nu=1}^n \sum_{\nu=1}^n W^*(s/F_5) x_{b,a,0,3,u,v}
\]

(12)

5 Expected number of interruptions during a single type I service

5.1 Distribution of duration of time till interruptions occur during a single type I service

Consider the Markov process, \( \chi_1 = (N(t),J(t),K(t)) \), where \( N(t) \) denotes the number of interruptions up to time \( t \), \( J(t) \) status of the server (providing normal or interrupted service) and \( K(t) \) the service phase at time \( t \). The state space of the process is given by \{ \{(0,2,k)/1 \leq k \leq m_1\} \cup \{((i,j,k))/i \geq 1,j = 0 \text{ or } 2,1 \leq k \leq m_1 \} \cup \{s_1\} \cup \{s_2\} \} where \( s_1 \) denotes the absorbing state indicating the service completion and \( s_2 \) denotes the absorbing state indicating the realization of protection.

The infinitesimal generator of the process is given by

\[
U =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\delta e(m_1) & T^0 & T - (\gamma + \delta)I_{m_1} & \gamma I_{m_1} & 0 & 0 & \cdots \\
\delta e(m_2) & \theta T^0 & 0 & \theta T - (\eta + \delta)I_{m_2} & \eta I_{m_2} & 0 & \cdots \\
\delta e(m_3) & T^0 & 0 & 0 & T - (\gamma + \delta)I_{m_1} & \gamma I_{m_1} & 0 & \cdots \\
\delta e(m_4) & \theta T^0 & 0 & 0 & 0 & \theta T - (\eta + \delta)I_{m_1} & \eta I_{m_1} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]

5.2 Distribution of number of interruptions during a single type I service

Let \( y_k \) be the probability that the number of interruptions during a single type I service is \( k \). Then \( y_k \) is the probability that the absorption occurs from the level \( k \) for the process \( \chi_1 \). Hence \( y_k \) are given by

\[
y_0 = -\alpha(T - (\gamma + \delta)I)^{-1}(T^0 + \delta e)
\]

and for \( k = 1,2,3, \ldots \)

\[
y_k = \alpha(T - (\gamma + \delta)I)^{-1}\gamma I((\theta T - (\eta + \delta)I)\gamma I(T - (\gamma + \delta)I)^{-1}\gamma I)^{-1}((\theta T^0 + \delta e) - \eta I(T - (\gamma + \delta)I)^{-1}(T^0 + \delta e))
\]

Therefore, the expected number of interruptions during any particular type I customer service,

\[
E(i) = \sum_{k=0}^\infty k y_k = \alpha(T - (\gamma + \delta)I)^{-1}\gamma I((\theta T - (\eta + \delta)I)\gamma I(T - (\gamma + \delta)I)^{-1}\gamma I)^{-2}((\theta T - (\eta + \delta)I)^{-1}((\theta T^0 + \delta e) - \eta I(T - (\gamma + \delta)I)^{-1}(T^0 + \delta e)).
\]

6 Expected number of interruptions during a single type II service

6.1 Distribution of duration of time till interruptions occur during a single type II service

Consider the Markov process, \( \chi_2 = (N(t),J(t),K(t)) \), where \( N(t) \) denotes the number of
interruptions, $I(t)$, status of the server (providing normal or interrupted service) and $K(t)$, the service phase at time $t$. The state space of the process of the process is given by $((0,3,k), 1 \leq k \leq m_2) \cup \{(i, j, k)/i \geq 1, j = 1 \text{ or } 3, 1 \leq k \leq m_2\} \cup \{\star\}$ where $\star$ denotes the absorbing state indicating the service completion. The infinitesimal generator of the process is given by

$$
\begin{align*}
\mathcal{U} &= \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
S^0 & S - \gamma l_{m_2} & \gamma l_{m_2} & 0 & 0 & 0 & \ldots \\
\theta'S^0 & 0 & \theta'S - \eta l_{m_2} & \eta l_{m_2} & 0 & 0 & \ldots \\
S^0 & 0 & 0 & S - \gamma l_{m_2} & \gamma l_{m_2} & 0 & \ldots \\
\theta'S^0 & 0 & 0 & 0 & \theta'S - \eta l_{m_2} & \eta l_{m_2} & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{bmatrix}
\end{align*}
$$

6.2 Distribution of number of interruptions during a single type II service

Let $z_k$ be the probability that the number of interruptions during a single type II service is $k$. Then $z_k$ is the probability that the absorption occurs from the level $k$ for the process $\chi_k$. Hence $z_k$ are given by

$$z_0 = -a(S - \gamma I)^{-1}S^0$$

and for $k = 1, 2, 3, \ldots$

$$z_k = a(S - \gamma I)^{-1}S^0 (\theta'S - \eta I)^{-1}S^0 (\gamma I)^{-1}(\theta'S - \eta I)^{-1}S^0 (\gamma I)^{-1}(\theta'S - \eta I)^{-1}S^0 (\gamma I)^{-1} \ldots$$

Therefore, the expected number of interruptions during any particular type II customer service,

$$E(i) = \sum_{k=0}^{\infty} k z_k = a(S - \gamma I)^{-1}S^0 (\theta'S - \eta I)^{-1}S^0 (\gamma I)^{-1}(\theta'S - \eta I)^{-1}S^0 (\gamma I)^{-1} \ldots$$

7 Other Performance Measures

- The probability that the server is idle:

$$p_{idle} = \sum_{v=1}^{n} x_{0,v}$$

- Mean number of type I customers in the system:

$$E_{sh} = \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{L} \sum_{m_1=1}^{n_1} \sum_{m_2=1}^{n_2} n_1 x_{n_1,n_2,n_0,0,0,0} + \sum_{n_1=1}^{L} \sum_{n_2=1}^{m_1} \sum_{m_2=1}^{n_2} \sum_{v=1}^{1} n_2 x_{n_1,n_2,0,0,0,1,v} + \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{L} \sum_{m_1=1}^{n_1} \sum_{m_2=1}^{n_2} \sum_{v=1}^{1} n_2 x_{n_1,n_2,0,2,0,1,v} + \sum_{n_1=1}^{L} \sum_{n_2=1}^{m_1} \sum_{m_2=1}^{n_2} \sum_{v=1}^{3} n_2 x_{n_1,n_2,0,0,0,3,v} + \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{L} \sum_{m_1=1}^{n_1} \sum_{m_2=1}^{n_2} \sum_{v=1}^{1} n_2 x_{n_1,n_2,1,2,0,1,v}$$

- Mean number of type II customers in the system:

$$E_{sf} = \sum_{n_1=0}^{\infty} n_1 x_{n_1}$$

- The fraction of time during which the system is protected:
\[ T_p = \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{L} \sum_{m_1=1}^{n_1} \sum_{n_2=1}^{n_2} x_{n_1,n_2,1,2,u,v} \]

• The fraction of time the server is providing service to type I customers during WI:

\[ T_{ih} = \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{L} \sum_{m_1=1}^{n_1} \sum_{n_2=1}^{n_2} x_{n_1,n_2,0,0,u,v} \]

• The fraction of time the server is providing service to type II customers during WI:

\[ T_{ii} = \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{L} \sum_{m_2=1}^{n_1} \sum_{n_2=1}^{n_2} x_{n_1,n_2,0,1,u,v} \]

• The fraction of time the server is providing service to type I customers in normal mode:

\[ T_{nh} = \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{L} \sum_{m_1=1}^{n_1} \sum_{n_2=1}^{n_2} x_{n_1,n_2,0,2,u,v} \]

• The fraction of time the server provides service to type II customers in normal mode:

\[ T_{nl} = \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{L} \sum_{m_2=1}^{n_1} \sum_{n_2=1}^{n_2} x_{n_1,n_2,0,3,u,v} \]

8 Analysis of a cost function

We construct a cost function based on the above performance measures.

Let

\( C_h \): Holding cost for retaining a type I customer

\( C_l \): Holding cost for retaining a type II customer

\( C_p \): Unit time cost of providing service with protection

\( C_{ih} \): Unit time cost of providing service when the server is providing service to type I customer in WI

\( C_{il} \): Unit time cost of providing service when the server is providing service to type II customer in WI

\( C_{nh} \): Unit time cost of providing service when the server is providing service to type I customer in normal mode

\( C_{nl} \): Unit time cost of providing service when the server is providing service to type II customer in normal mode

Then the expected cost per unit time,

\[ C = E_{n_{sh}} \times C_h + E_{n_{sl}} \times C_l + T_p \times \phi C_p + T_{ih} \times \theta C_{ih} + T_{il} \times \theta' C_{il} + T_{nh} \times C_{nh} + T_{nl} \times C_{nl} \]

9 Numerical Results

For the arrival process of type II customers, we consider the following two sets of matrices for \( D_0 \) and \( D_1 \):

1. MAP with negetive correlation (MNA)

\[
D_0 = \begin{bmatrix}
-0.8101 & 0.8101 & 0 \\
0 & -1.3497 & 0 \\
0 & 0 & -40.5065 \\
\end{bmatrix}, \quad D_1 = \begin{bmatrix}
0 & 0 & 0 \\
0.0810 & 0 & 1.2687 \\
38.0761 & 0 & 2.4304 \\
\end{bmatrix}
\]
2. MAP with positive correlation (MPA)

\[
D_0 = \begin{bmatrix}
-0.8101 & 0.8101 & 0 \\
0 & -1.3497 & 0 \\
0 & 0 & -40.5065
\end{bmatrix}, \quad D_1 = \begin{bmatrix}
0 & 0 & 0 \\
1.2687 & 0 & 0.0810 \\
2.4304 & 0 & 38.0761
\end{bmatrix}
\]

These two MAP processes are normalized so as to have an arrival rate of 1. The arrival process labeled MNA has correlated arrivals with correlation between two successive interarrival times given by -0.4211 and the arrival process corresponding to the one labelled MPA has a positive correlation with value 0.4211.

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Table 1: Effect of \(\theta\): Fix \(L = 3, \theta' = 0.6, \lambda = 2, \eta = 0.5, \delta = 1, \gamma = 0.6\) and \(\phi = 4\)

Tables 1 to 6 contain the effect of different parameters on various performance measures and on the cost function when the arrival process of type II customer is MNA and tables 7 to 12 contain the effect of different parameters on various performance measures and on the cost function when the arrival process of type II customer is MPA.

Table 1 indicates the effect of the parameter \(\theta\) on various performance measures and the cost function. As \(\theta\) increases, type I customers get faster service during WI and hence \(E_{\text{nsh}}\) decreases. Then more number of type II customers also get service and hence \(E_{\text{nl}}\) also decreases. \(T_p\) and \(T_{th}\) also decreases since the expected number of type I customers during WI decreases. As \(\theta\) increases, \(T_{il}\) and \(T_{nl}\) remains fixed due to the diminished effect of \(\theta\) on type II customers and \(T_{nh}\) increases due to the fact that the system stays in WI serving type I customers for lesser time and hence it stays more in normal mode serving type I customers. As \(\theta\) increases, the system cost first decreases, reach an optimal value(30.9648) corresponding to \(\theta = 0.4\) and then increases.

<table>
<thead>
<tr>
<th>(\phi)</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
<th>4</th>
<th>4.5</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E_{\text{nsh}})</td>
<td>1.3572</td>
<td>1.1902</td>
<td>1.1112</td>
<td>1.0658</td>
<td>1.0366</td>
<td>1.0162</td>
<td>1.0013</td>
<td>0.9898</td>
<td>0.9808</td>
</tr>
<tr>
<td>(E_{\text{nl}})</td>
<td>1.1787 \times 10^4</td>
<td>12.1182</td>
<td>7.6300</td>
<td>6.1872</td>
<td>5.4634</td>
<td>5.0334</td>
<td>4.7941</td>
<td>4.5473</td>
<td>4.3968</td>
</tr>
<tr>
<td>(T_p)</td>
<td>0.1581</td>
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<td>0.0665</td>
<td>0.0557</td>
<td>0.0479</td>
<td>0.0420</td>
<td>0.0374</td>
<td>0.0337</td>
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<tr>
<td>(T_{th})</td>
<td>0.0482</td>
<td>0.0497</td>
<td>0.0504</td>
<td>0.0507</td>
<td>0.0509</td>
<td>0.0511</td>
<td>0.0512</td>
<td>0.0513</td>
<td>0.0513</td>
</tr>
<tr>
<td>(T_{il})</td>
<td>0.0863</td>
<td>0.0863</td>
<td>0.0863</td>
<td>0.0863</td>
<td>0.0863</td>
<td>0.0863</td>
<td>0.0863</td>
<td>0.0863</td>
<td>0.0863</td>
</tr>
<tr>
<td>(T_{al})</td>
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<td>0.3702</td>
<td>0.3751</td>
<td>0.3778</td>
<td>0.3795</td>
<td>0.3806</td>
<td>0.3814</td>
<td>0.3820</td>
<td>0.3824</td>
</tr>
<tr>
<td>(T_{nl})</td>
<td>0.3482</td>
<td>0.3482</td>
<td>0.3482</td>
<td>0.3482</td>
<td>0.3482</td>
<td>0.3482</td>
<td>0.3482</td>
<td>0.3482</td>
<td>0.3482</td>
</tr>
<tr>
<td>(C)</td>
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<td>34.1476</td>
<td>34.2737</td>
<td>34.2923</td>
<td>34.3216</td>
<td>34.3469</td>
<td>34.3673</td>
<td>34.3837</td>
<td>34.3969</td>
</tr>
</tbody>
</table>

Table 2: Effect of \(\phi\): Fix \(L = 3, \theta = 0.7, \theta' = 0.6, \lambda = 2, \eta = 0.5, \delta = 1.5\) and \(\gamma = 0.6\)

Table 2 indicates the effect of the parameter \(\phi\) on various performance measures and the cost function. As \(\phi\) increases, the type I customers in protected mode get faster service and hence
$E_{nh}$ decreases. As a result, $E_{nl}$ also decreases. As expected $T_p$ also decreases. As $\phi$ increases, $T_{lh}$ and $T_{nh}$ increase since $T_p$ decreases. $T_{il}$ and $T_{nl}$ remains unchanged since $\phi$ has only a small effect on low priority customers. As $\phi$ increases, the system cost first decreases, reach an optimal value ($34.2737$) corresponding to $\phi = 2$ and then increases.

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
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<th>0.8</th>
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<tbody>
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<td>1.2883</td>
<td>1.2562</td>
<td>1.2260</td>
<td>1.1975</td>
<td>1.1706</td>
<td>1.1452</td>
<td>1.1212</td>
<td>1.0985</td>
</tr>
<tr>
<td>$E_{nl}$</td>
<td>1071.6</td>
<td>57.3614</td>
<td>29.5220</td>
<td>19.9883</td>
<td>15.1618</td>
<td>12.2491</td>
<td>10.3021</td>
<td>8.9100</td>
<td>7.8661</td>
<td>7.0548</td>
</tr>
<tr>
<td>$T_p$</td>
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<td>0.0069</td>
<td>0.0102</td>
<td>0.0133</td>
<td>0.0164</td>
<td>0.0193</td>
<td>0.0222</td>
<td>0.0250</td>
<td>0.0276</td>
<td>0.0302</td>
</tr>
<tr>
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<td>0.0831</td>
<td>0.0798</td>
<td>0.0767</td>
<td>0.0737</td>
<td>0.0709</td>
<td>0.0683</td>
<td>0.0657</td>
<td>0.0633</td>
<td>0.0611</td>
</tr>
<tr>
<td>$T_{il}$</td>
<td>0.0863</td>
<td>0.0863</td>
<td>0.0863</td>
<td>0.0863</td>
<td>0.0863</td>
<td>0.0863</td>
<td>0.0863</td>
<td>0.0863</td>
<td>0.0863</td>
<td>0.0863</td>
</tr>
<tr>
<td>$T_{nh}$</td>
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<td>0.4674</td>
<td>0.4599</td>
<td>0.4526</td>
<td>0.4454</td>
<td>0.4384</td>
<td>0.4315</td>
<td>0.4247</td>
<td>0.4181</td>
<td>0.4116</td>
</tr>
<tr>
<td>$T_{nl}$</td>
<td>0.3482</td>
<td>0.3482</td>
<td>0.3482</td>
<td>0.3482</td>
<td>0.3482</td>
<td>0.3482</td>
<td>0.3482</td>
<td>0.3482</td>
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</tr>
<tr>
<td>$C$</td>
<td>129.7496</td>
<td>29.2871</td>
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<td>27.4443</td>
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<td>29.0719</td>
<td>29.7453</td>
<td>30.4286</td>
<td>31.1111</td>
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</table>

Table 3: Effect of $\phi$: Fix $L = 3, \theta = 0.7, \theta' = 0.6, \lambda = 2, \eta = 0.5, \gamma = 0.6$ and $\phi = 4$

Table 3 indicates the effect of the parameter $\delta$ on various performance measures and the cost function. As $\delta$ increases, protection clock realizes quickly and hence $T_p$ increases, so $T_{ih}$ and $T_{nh}$ decreases. But $T_{il}$ and $T_{nl}$ remains unchanged since $\delta$ has only a small effect on low priority customers. In this case also, as $\delta$ increases, the system cost first decreases, reach an optimal value ($27.4443$) corresponding to $\delta = 0.4$ and then increases.

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>0.1</th>
<th>0.2</th>
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<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{nh}$</td>
<td>1.1164</td>
<td>1.1112</td>
<td>1.1067</td>
<td>1.1025</td>
<td>1.0985</td>
<td>1.0948</td>
<td>1.0913</td>
<td>1.0880</td>
<td>1.0848</td>
<td>1.0819</td>
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<tr>
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<td>0.0302</td>
<td>0.0302</td>
<td>0.0302</td>
<td>0.0302</td>
<td>0.0302</td>
<td>0.0302</td>
<td>0.0302</td>
<td>0.0303</td>
<td>0.0303</td>
<td>0.0303</td>
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<tr>
<td>$T_{ih}$</td>
<td>0.0663</td>
<td>0.0649</td>
<td>0.0636</td>
<td>0.0623</td>
<td>0.0611</td>
<td>0.0599</td>
<td>0.0587</td>
<td>0.0576</td>
<td>0.0566</td>
<td>0.0555</td>
</tr>
<tr>
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<td>0.0958</td>
<td>0.0924</td>
<td>0.0893</td>
<td>0.0863</td>
<td>0.0836</td>
<td>0.0810</td>
<td>0.0785</td>
<td>0.0762</td>
<td>0.0740</td>
</tr>
<tr>
<td>$T_{nh}$</td>
<td>0.4058</td>
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<td>0.4089</td>
<td>0.4103</td>
<td>0.4116</td>
<td>0.4129</td>
<td>0.4141</td>
<td>0.4153</td>
<td>0.4165</td>
<td>0.4175</td>
</tr>
<tr>
<td>$T_{nl}$</td>
<td>0.3403</td>
<td>0.3425</td>
<td>0.3445</td>
<td>0.3464</td>
<td>0.3482</td>
<td>0.3499</td>
<td>0.3514</td>
<td>0.3529</td>
<td>0.3543</td>
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</table>

Table 4: Effect of $\eta$: Fix $L = 3, \theta = 0.7, \theta' = 0.6, \lambda = 2, \eta = 0.5, \gamma = 0.6$ and $\phi = 4$

Table 4 indicates the effect of the parameter $\eta$ on various performance measures and the cost function. As $\eta$ increases, the server turns to normal mode quickly. Hence $T_{nh}$ and $T_{nl}$ increase and $E_{nh}, E_{nl}, T_{ih}$ and $T_{il}$ decrease. $\eta$ has only a very small effect on $T_p$. The cost function decreases as $\eta$ increases.

<table>
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<th>$\eta'$</th>
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<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{nh}$</td>
<td>1.3367</td>
<td>1.1562</td>
<td>1.0535</td>
<td>0.9894</td>
<td>0.9467</td>
<td>0.9166</td>
<td>0.8945</td>
<td>0.8779</td>
<td>0.8649</td>
<td>0.8569</td>
</tr>
<tr>
<td>$T_p$</td>
<td>0.0464</td>
<td>0.0490</td>
<td>0.0504</td>
<td>0.0512</td>
<td>0.0517</td>
<td>0.0521</td>
<td>0.0523</td>
<td>0.0525</td>
<td>0.0527</td>
<td>0.0527</td>
</tr>
<tr>
<td>$T_{ih}$</td>
<td>0.0369</td>
<td>0.0389</td>
<td>0.0400</td>
<td>0.0406</td>
<td>0.0410</td>
<td>0.0413</td>
<td>0.0415</td>
<td>0.0417</td>
<td>0.0418</td>
<td>0.0418</td>
</tr>
<tr>
<td>$T_{il}$</td>
<td>1.2089</td>
<td>1.1576</td>
<td>1.1260</td>
<td>1.0949</td>
<td>1.0988</td>
<td>1.0785</td>
<td>1.0697</td>
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<td>1.0569</td>
</tr>
<tr>
<td>$T_{nh}$</td>
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<td>1.3357</td>
<td>1.3452</td>
<td>1.3508</td>
<td>1.3544</td>
<td>1.3568</td>
<td>1.3586</td>
<td>1.3599</td>
<td>1.3608</td>
<td>1.3608</td>
</tr>
<tr>
<td>$T_{nl}$</td>
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<td>1.3685</td>
<td>1.3622</td>
<td>1.3580</td>
<td>1.3551</td>
<td>1.3529</td>
<td>1.3512</td>
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<td>1.3488</td>
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<tr>
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</table>
Table 5 indicates the effect of the parameter $\theta'$ on various performance measures and the cost function. As expected, $T_{il}$ decreases and hence $E_{nst}$ and $E_{nsh}$ decrease, $T_{ih}$, $T_{nh}$ and $T_p$ increase since type I customers have high priority. As a result, $T_{nt}$ decreases. As $\theta'$ increases, the system cost first decreases, reach an optimal value (35.5315) corresponding to $\theta' = 0.2$ and then increases.

<table>
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<th>$\gamma$</th>
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<th>0.4</th>
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<th>0.8</th>
<th>0.9</th>
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<td>1.0085</td>
<td>1.1169</td>
<td>1.1349</td>
<td>1.1525</td>
<td>1.1697</td>
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<tr>
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<td>0.0301</td>
<td>0.0302</td>
<td>0.0302</td>
<td>0.0302</td>
<td>0.0303</td>
<td>0.0303</td>
<td>0.0303</td>
<td>0.0303</td>
<td>0.0303</td>
</tr>
<tr>
<td>$T_{il}$</td>
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<td>0.0220</td>
<td>0.0324</td>
<td>0.0423</td>
<td>0.0519</td>
<td>0.0611</td>
<td>0.0699</td>
<td>0.0784</td>
<td>0.0866</td>
<td>0.0945</td>
</tr>
<tr>
<td>$T_{nh}$</td>
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<td>0.0313</td>
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<td>0.0599</td>
<td>0.0734</td>
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<td>0.0988</td>
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<tr>
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<td>33.3148</td>
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</table>

Table 6: Effect of $\gamma$: Fix $L = 3, \theta = 0.7, \lambda = 2, \eta = 0.8, \delta = 2, \gamma = 0.6$ and $\phi = 4$

Table 6 indicates the effect of the parameter $\gamma$ on various performance measures and the cost function. As $\gamma$ increases, more interruptions occur during service and hence both $E_{nsh}$ and $E_{nst}$ increases. $T_p$ also increases in a slow rate. As $\gamma$ increases $T_{ih}$ and $T_{il}$ increase and $T_{nh}$ and $T_{nt}$ decrease since the system stays more time in interruption mode. As $\gamma$ increases, the cost function increases. Note the sharpness in decrease of the value of $E_{nst}$ is quite pronounced. However the trend is not seen in table 4 which gives the effect of $\eta$.

<table>
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<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
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</tr>
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<td>1.0109</td>
<td>1.0083</td>
<td>1.0207</td>
<td>1.0340</td>
<td>1.0609</td>
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<td>36.4713</td>
<td>35.5556</td>
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<td>33.8616</td>
<td>30.3784</td>
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<td>0.0324</td>
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<td>0.0313</td>
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<td>0.0296</td>
</tr>
<tr>
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<td>0.0863</td>
<td>0.0863</td>
<td>0.0863</td>
<td>0.0863</td>
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</tr>
<tr>
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<td>0.4134</td>
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</tr>
<tr>
<td>$T_{nt}$</td>
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<td>0.3482</td>
<td>0.3482</td>
<td>0.3482</td>
<td>0.3482</td>
<td>0.3482</td>
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<td>$C$</td>
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<td>43.4206</td>
<td>39.2737</td>
<td>37.5789</td>
<td>36.6882</td>
<td>36.1497</td>
<td>35.7932</td>
<td>35.5413</td>
<td>35.3548</td>
<td>35.2114</td>
</tr>
</tbody>
</table>

Table 7: Effect of $\theta$: Fix $L = 3, \theta' = 0.6, \lambda = 2, \eta = 0.5, \delta = 1, \gamma = 0.6$ and $\phi = 4$

Table 7 indicates the effect of the parameter $\theta$ on various performance measures and the cost function. In this case also $E_{nsh}$ and $E_{nst}$ decreases as $\theta$ increases. But the values of $E_{nst}$ is much high when the arrival process of type II customer is MPA. All other values are same as in the case of MNA. But the cost function decreases as $\theta$ increases.

<table>
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<tr>
<th>$\phi$</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
<th>4</th>
<th>4.5</th>
<th>5</th>
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<td>$E_{nsh}$</td>
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<td>1.1900</td>
<td>1.1141</td>
<td>1.0711</td>
<td>1.0436</td>
<td>1.0244</td>
<td>1.0104</td>
<td>0.9996</td>
<td>0.9911</td>
</tr>
<tr>
<td>$E_{nst}$</td>
<td>4.4374 $\times 10^4$</td>
<td>90.9874</td>
<td>58.6062</td>
<td>47.8674</td>
<td>42.5211</td>
<td>39.5245</td>
<td>37.1995</td>
<td>35.6852</td>
<td>34.5516</td>
</tr>
<tr>
<td>$T_p$</td>
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<td>0.0826</td>
<td>0.0665</td>
<td>0.0557</td>
<td>0.0479</td>
<td>0.0420</td>
<td>0.0374</td>
<td>0.0337</td>
</tr>
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<td>$T_{il}$</td>
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<td>0.0507</td>
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<td>0.0513</td>
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<td>0.0863</td>
<td>0.0863</td>
<td>0.0863</td>
<td>0.0863</td>
<td>0.0863</td>
<td>0.0863</td>
<td>0.0863</td>
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<td>$T_{nt}$</td>
<td>0.3591</td>
<td>0.3702</td>
<td>0.3751</td>
<td>0.3778</td>
<td>0.3795</td>
<td>0.3806</td>
<td>0.3814</td>
<td>0.3820</td>
<td>0.3824</td>
</tr>
<tr>
<td>$C$</td>
<td>4.4699 $\times 10^3$</td>
<td>42.3015</td>
<td>39.5705</td>
<td>38.4629</td>
<td>38.0309</td>
<td>37.7801</td>
<td>37.6169</td>
<td>37.5023</td>
<td>37.4175</td>
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</table>
Table 8: Effect of $\phi$: Fix $L = 3, \theta = 0.7, \theta' = 0.6, \lambda = 2, \eta = 0.5, \delta = 1.5$ and $\gamma = 0.6$

Table 8 indicates the effect of the parameter $\phi$ on various performance measures and the cost function. Both $E_{ns}$ and $E_{nsl}$ decrease as $\phi$ increases. The cost function and $E_{nsl}$ decreases sharply as $\phi$ increases from 1 to 1.5. However, with further increase in $\phi$ value does not produce that decrease in values of cost function and $E_{nsl}$.

<table>
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<tr>
<th>$\delta$</th>
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<th>0.4</th>
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<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{ns}$</td>
<td>1.3589</td>
<td>1.3214</td>
<td>1.2867</td>
<td>1.2545</td>
<td>1.2244</td>
<td>1.1965</td>
<td>1.1703</td>
<td>1.1458</td>
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<td>1.1013</td>
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<tr>
<td>$E_{nsl}$</td>
<td>7.5197 \times 10^3</td>
<td>411.6330</td>
<td>214.5271</td>
<td>146.4434</td>
<td>111.9502</td>
<td>91.1156</td>
<td>77.1729</td>
<td>67.1914</td>
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<td>0.0102</td>
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<td>0.0164</td>
<td>0.0193</td>
<td>0.0222</td>
<td>0.0250</td>
<td>0.0276</td>
<td>0.0302</td>
</tr>
<tr>
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<td>0.0865</td>
<td>0.0831</td>
<td>0.0798</td>
<td>0.0767</td>
<td>0.0737</td>
<td>0.0709</td>
<td>0.0683</td>
<td>0.0657</td>
<td>0.0633</td>
<td>0.0611</td>
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<td>0.0863</td>
<td>0.0863</td>
<td>0.0863</td>
<td>0.0863</td>
<td>0.0863</td>
<td>0.0863</td>
<td>0.0863</td>
<td>0.0863</td>
<td>0.0863</td>
</tr>
<tr>
<td>$T_{sl}$</td>
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<td>0.4674</td>
<td>0.4599</td>
<td>0.4526</td>
<td>0.4454</td>
<td>0.4384</td>
<td>0.4315</td>
<td>0.4247</td>
<td>0.4181</td>
<td>0.4116</td>
</tr>
<tr>
<td>$T_{ns}$</td>
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<td>0.3482</td>
<td>0.3482</td>
<td>0.3482</td>
<td>0.3482</td>
<td>0.3482</td>
<td>0.3482</td>
<td>0.3482</td>
<td>0.3482</td>
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<tr>
<td>$C$</td>
<td>774.5584</td>
<td>54.7362</td>
<td>45.9761</td>
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<td>36.3143</td>
<td>35.7588</td>
<td>35.5737</td>
<td>35.6123</td>
<td>35.7932</td>
</tr>
</tbody>
</table>

Table 9: Effect of $\delta$: Fix $L = 3, \theta = 0.7, \theta' = 0.6, \lambda = 2, \eta = 0.5, \gamma = 0.6$ and $\phi = 4$

Table 9 indicates the effect of the parameter $\delta$ on various performance measures and the cost function. Both $E_{ns}$ and $E_{nsl}$ decrease as $\delta$ increases. In this case, as $\delta$ increases, the system cost first decreases, reaches an optimal value (35.5737) corresponding to $\delta = 0.8$ and then increases. Both $E_{nsl}$ and the cost show sharp decrease in their values when $\delta$ moves from 0.1 to 0.2. Thereafter the decrease is not that pronounced.

<table>
<thead>
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<th>$\eta$</th>
<th>0.1</th>
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<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
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</thead>
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<tr>
<td>$E_{ns}$</td>
<td>1.1184</td>
<td>1.1136</td>
<td>1.1093</td>
<td>1.1051</td>
<td>1.1013</td>
<td>1.0976</td>
<td>1.0942</td>
<td>1.0910</td>
<td>1.0880</td>
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</tr>
<tr>
<td>$E_{nsl}$</td>
<td>58.6679</td>
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<td>54.8999</td>
<td>53.8616</td>
<td>52.9096</td>
<td>52.0337</td>
<td>51.2250</td>
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<td>0.0302</td>
<td>0.0302</td>
<td>0.0302</td>
<td>0.0302</td>
<td>0.0302</td>
<td>0.0302</td>
<td>0.0303</td>
<td>0.0303</td>
<td>0.0303</td>
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<td>0.0576</td>
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<td>0.0555</td>
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<td>$T_p$</td>
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<td>0.0785</td>
<td>0.0762</td>
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<tr>
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<td>0.4089</td>
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<td>0.4116</td>
<td>0.4129</td>
<td>0.4141</td>
<td>0.4153</td>
<td>0.4165</td>
<td>0.4175</td>
</tr>
<tr>
<td>$T_{ns}$</td>
<td>0.3403</td>
<td>0.3425</td>
<td>0.3445</td>
<td>0.3464</td>
<td>0.3482</td>
<td>0.3499</td>
<td>0.3514</td>
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<td>35.7932</td>
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<td>35.4062</td>
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<td>35.0634</td>
<td>34.9061</td>
</tr>
</tbody>
</table>

Table 10: Effect of $\eta$ Fix $L = 3, \theta = 0.7, \theta' = 0.6, \lambda = 2, \eta = 0.5, \gamma = 0.6$ and $\phi = 4$

Table 10 indicates the effect of the parameter $\eta$ on various performance measures and the cost function. Both $E_{ns}$ and $E_{nsl}$ decrease as $\eta$ increases. The cost function decreases as $\eta$ increases.

<table>
<thead>
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<th>0.5</th>
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<th>0.8</th>
<th>0.9</th>
<th>1</th>
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</thead>
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<td>$E_{ns}$</td>
<td>1.3380</td>
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<td>1.0642</td>
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<td>0.8947</td>
<td>0.8819</td>
<td>0.8716</td>
</tr>
<tr>
<td>$r_p$</td>
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<td>0.0512</td>
<td>0.0517</td>
<td>0.0521</td>
<td>0.0523</td>
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</tr>
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<td>$r_{sl}$</td>
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<td>0.0389</td>
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<td>0.0417</td>
<td>0.0418</td>
<td>0.0419</td>
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<td>$T_{sl}$</td>
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<td>0.0898</td>
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<td>0.0675</td>
<td>0.0597</td>
<td>0.0569</td>
<td>0.0521</td>
</tr>
<tr>
<td>$T_{ns}$</td>
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<td>0.3586</td>
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<td>0.3608</td>
<td>0.3616</td>
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</table>

Table 11: Effect of $\theta'$: Fix $L = 3, \theta = 0.7, \lambda = 2, \eta = 0.8, \delta = 2, \gamma = 0.6$ and $\phi = 4$
Table 11 indicates the effect of the parameter $\theta'$ on various performance measures and the cost function. Both $E_{nsb}$ and $E_{nsl}$ decrease as $\theta'$ increases. The cost function decreases as $\theta'$ increases, as it is to be expected. However, there is a sharp decrease in value of $E_{nsl}$ when $\theta'$ moves from 0.1 to 0.2. For higher values of $\theta'$, the initial sharpness in decrease is not seen.

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<th>0.9</th>
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<tbody>
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<td>$E_{nsl}$</td>
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<td>40.5924</td>
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<td>53.8616</td>
<td>59.6115</td>
<td>66.2942</td>
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<td>0.0303</td>
<td>0.0303</td>
<td>0.0303</td>
<td>0.0303</td>
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<td>0.4116</td>
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<td>0.3876</td>
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<td>37.0389</td>
<td>38.3530</td>
<td>39.7616</td>
<td>41.3003</td>
</tr>
</tbody>
</table>

Table 12: Effect of $\gamma$: Fix $L = 3$, $\theta = 0.7$, $\lambda = 2$, $\eta = 0.8$, $\delta = 2$, $\gamma = 0.6$ and $\phi = 4$

Table 12 indicates the effect of the parameter $\gamma$ on various performance measures and the cost function. Both $E_{nsb}$ and $E_{nsl}$ increase as $\gamma$ increases. As expected, the cost increases as $\gamma$ increases.

**Conclusion**

In this paper, we considered a (M,MAP)/(PH,PH)/1 queue with non preemptive priority, exponentially distributed working interruptions and protection. We analysed the distribution of service time of type I and type II customers and the distribution of a p-cycle. Also we provided LSTs of busy cycle, busy period of type I customers generated during the service time of a type II customer. For the waiting time distributions of type I and type II customers, we provided an analysis using LST and the matrix analytic method. We also performed some numerical experiments to evaluate some performance measures and also found optimal values using a cost function. Extension of the model discussed to multi-server is proposed to be taken up in a future study.

**References**


multiserver system, Neural, Parallel and Scientific Computations, 20, 153-172.


Appendix

Proof of Theorem 1

Proof. Let $B_{c_1}$ denote the length of the busy cycle generated by type I customers arriving during the service time of a type II customer, $\hat{B}_{c_1}(s)$ the LST of the length of the busy cycle and I the number of type I customers that arrive during service time of type II customer.

Then $B_{c_1} = X + B_1 + \cdots B_l$, where $X$ denote the service time of the type II customer in service, $B_l$ the busy period generated by $j$th type I customers that arrive during $X$, where $1 \leq j \leq l$.

\[ \hat{B}_{c_1}(s) = E(e^{-sB_{c_1}}) = \int_{x=0}^{\infty} e^{-sB_{c_1}} P(s \leq x) dx \]

Substituting (18) in (17), its third term

\[ \gamma_L(s) = \frac{\gamma_L(s - T_1)}{s - T_1} \frac{\gamma_L(s - T_1)}{s - T_1} \frac{\gamma_L(s - T_1)}{s - T_1} \sum_{p=1}^{\infty} \frac{\gamma_p}{s - T_1} \frac{\gamma_p}{s - T_1} \frac{\gamma_p}{s - T_1} \]

Substituting (19) in (17) gives

\[ \hat{B}_{c_1}(s) = B'(s + \lambda) I - S_2 - 1 \sum_{p=1}^{\infty} \frac{\gamma_p}{s - T_1} \frac{\gamma_p}{s - T_1} \frac{\gamma_p}{s - T_1} \]

\[ \frac{\gamma_L(s - T_1)}{s - T_1} \frac{\gamma_L(s - T_1)}{s - T_1} \frac{\gamma_L(s - T_1)}{s - T_1} \sum_{p=1}^{\infty} \frac{\gamma_p}{s - T_1} \frac{\gamma_p}{s - T_1} \frac{\gamma_p}{s - T_1} \]

Proof of theorem 2

Proof. Let $B_l$ denote the length of the busy period generated by type I customers arriving during the service time of a type II customer, $\hat{B}_l(s)$ the LST of the length of the busy period and I the number of type I customers that arrive during service time of type II customer.

Then $B_l = B_1 + \cdots B_l$, where $B_l$ denote the busy period generated by $j$th type I customers that arrive during $X$, where $1 \leq j \leq l$. Proceeding as in the above proof, we get the required result.