ON SENSITIVITY OF RELIABILITY SYSTEMS OPERATING IN RANDOM ENVIRONMENT TO SHAPE OF THEIR INPUT DISTRIBUTIONS

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ABSTRACT

Removable double redundant reliability system operating in Markov environment is considered. The problem of the system steady state probabilities insensitivity to shape of its elements life and repair time distributions is studied.

1 Introduction

Stability of systems behavior to small changing some of their parameters, initial states or exterior factors are central problem for all natural sciences. The stability of complex stochastic systems often can be represented in terms of sensitivity of their output characteristics to the shape of some input distributions. One of the earliest result concerning insensitivity of systems' characteristics to the shape of service time distribution has been done by B. Sevast'yanov [13], who proved insensitivity of the Erlang's formulas to the shape of service time distribution with fixed mean value for loss queueing systems with Poisson input.

I. Kovalenko [8] found the necessary and sufficient condition for insensitivity of stationary characteristics of reliability systems with exponential their elements life time to the shape of their elements general repair time distribution. It consists in possibility immediately begin to repair any failed element. The sufficiency of this condition for general life and repair time distributions has been found by V. Rykov in [9] with the help of multi-dimensional alternative processes theory. From another side as it is very known, in the case of limited possibilities for restoration these results are not true (see, for example in [7]). However as it was shown in the papers of B. Gnedenko and A. Solov'ev [2, 3, 14, 4] and others under quick restoration the reliability function of the system tends to exponential independently of their elements life and repair time distributions.

There are many papers, where reliability systems are studied in stable environment (see, for example, Cui & Hawkes [1]). However, the very important for applications problem of reliability characteristics insensitivity or character of their sensitivity to the shapes of some input distributions and the influence of environment randomness to their behavior are not enough studied yet. There are some papers devoted to operating of queueing systems in random environment. Some review of the previous and modern investigations on this topic one can find in [6]. Some previous results about sensitivity of reliability characteristics of the system operating in stable environment to the shape of their elements life and repair time distributions have been done in [10, 11]. In the paper these results are generalized to the system operating in random environment.

The paper is organized as follows. After the problem set and notations in the next section the steady state reliability characteristics of a double redundant systems with one repair unit, which is operate in Markov environment will be considered in two next sections. The numerical investigation of the model will be represented in the section 5. The paper ends with conclusion and some problems description.

2 The problem set and notations

Consider a simple cold double redundant repairable system that operates in a random environment. Throughout the paper we will use a generalization of Kendall's notation [5] for queueing systems. In this notation a closed queueing system, which operates in a Random Environment will be denoted by $< GI_n|GI|m(RE) >$, where the symbols "$< >$" stand for closed systems, or systems with finite population, where the flow of customers is generated by a finite number n of sources that is shown by index at the first position; GI means "General Independent" and at the first position of this notation specifies the general distributions of units' independent life times and at the second one – the general distributions of their independent repair times, at least the number "m" means the number of repair servers, and symbol "(RE)" means "Random Environment". These symbols can be changed by M for exponential distributions in a Markov case.

In the paper for simplicity and due to some calculation reasons we limited ourselves by studying of the sensitivity problem for simple cold double redundant models, operating in a two-state Markov Random Environment, where only one of distributions (life or repair time) distinguish from exponential, namely $< M_2|GI|1(ME) >$ and $< GI_2|M|1(ME) >$.

The units' random life time when the system operates in i-th environment is denoted by A_i and its random repair time is denoted by B_i. Their cumulative distribution functions (c.d.f.) are denoted respectively by $A_i(x)$ and $B_i(x)$. It is supposed the existence of the corresponding probability density functions (p.d.f.), which are denoted by $a_i(x)$ and $b_i(x)$. The mean life and repair times, the relative repair rate as well as the failure and repair hazard rate functions are denoted by

$$\begin{aligned} \bar{a}_i &= \int_0^\infty (1 - A_i(x))\,dx, \quad \bar{b}_i = \int_0^\infty (1 - B_i(x))\,dx \quad \text{and} \quad \rho_i = \frac{\bar{a}_i}{\bar{b}_i}; \\ \alpha_i(x) &= \frac{a_i(x)}{1 - A_i(x)} \quad \text{and} \quad \beta_i(x) = \frac{b_i(x)}{1 - B_i(x)}. \end{aligned}$$

Define also moment generation functions (m.g.f.) of life A_i and repair B_i times, the Laplace-Stiltjes transforms (LST) of their distribution by following expressions,

$$\tilde{a}_i(s) = \int_0^\infty e^{-sx} a_i(x)\,dx \quad \text{and} \quad \tilde{b}_i(s) = \int_0^\infty e^{-sx} b_i(x)\,dx, \quad Re[s] \geq 0.$$

3 Cold redundancy $< M_2|GI|1(ME) >$ system, operating in Markov environment

Consider a two units cold redundant system $< M_2|GI|1(ME) >$ with one repair server, operating in two-state Markov environment. The elements have an exponential life time distributions with parameter α_i and general repair time distribution $B_i(t)$, when environment is in state i, and the environment change their states with intensities λ_i $(i = 1, 2)$. Denote by

$$\{Z(t)\}_{t\geq 0} = \{I(t),\, N(t),\, X(t)\}_{t\geq 0}$$

a three-dimensional stochastic process, where the first component $I(t)$ stands the state of environment, the second one $N(t)$ denote the number of failed units at time t and the third one $X(t)$ stands for the elapsed repair time of the unit at time t. The process $\{Z(t)\}_{t\geq 0}$ is obviously Markovian one with state space $E = \{(i, 0),\, (i, n, x) : i, n \in \{1, 2\},\, x \in \mathbb{R}_+\}$. Define the following state probabilities:

(1) $\pi_{(i,0)}(t) = \mathbf{P}\{I(t) = i,\; N(t) = 0\}$ – the probability of a "good" state of both of units, being the system in i-th state of environment at time t;

(2) $\pi_{(i,n)}(t; x)\, dx = \mathbf{P}\{I(t) = i,\; N(t) = n;\; x < X(t) \leq x + dx\}$ – the joint probability that at time t the system is in the i-th state of environment $(i = 1, 2)$, there are n $(n = 1, 2)$ failed units and the repairing unit has elapsed repair time between x and $x + dx$.

By usual approach with the help of Large Number Law for these probabilities the system of forward Kolmogorov partial differential equations jointly with boundary, initial and normalizing conditions has been derived (for analogous system operating in stable environment see [11, 12]. Since the Markov process $\{Z(t)\}_{t\geq 0}$ is a Harris one with positive atom in the states $(i, 0)$, it has steady-state probabilities,

$$\pi_0 = \lim_{t\to\infty} \pi_i(t), \quad \pi_i(x) = \lim_{t\to\infty} \pi_i(t;\, x) \;\; (i = 1, 2).$$

Equations for the stationary regime have been got from the the Kolmogorov equations by taking derivatives with respect to time variable as zero and they have the form, where $\bar{i} = 3 - i$:

$$\begin{aligned}
\Big[\lambda_i + \alpha_i\Big]\pi_{(i,0)} &= \int_0^\infty \pi_{(i,1)}(u)\beta(u)du + \lambda_{\bar{i}}\pi_{(\bar{i},0)},\\
\left[\frac{d}{dx} + \alpha_i + \lambda_i + \beta_i(x)\right]\pi_{(i,1)}(x) &= \lambda_{\bar{i}}\pi_{(\bar{i},1)}(x),\\
\left[\frac{d}{dx} + \lambda_i + \beta_i(x)\right]\pi_{(i,2)}(x) &= \alpha_i\pi_{(i,1)}(x) + \lambda_{\bar{i}}\pi_{(\bar{i},2)}(x),
\end{aligned} \tag{1}$$

with the boundary conditions for $i = 1, 2$

$$\pi_{(i,1)}(0) = \alpha_i\pi_{(i,0)} + \int_0^\infty \pi_{(i,2)}(u)\beta(u)du, \quad \pi_{(i,2)}(0) = 0, \tag{2}$$

together with the normalizing condition

$$\sum_{i=1}^{2}\left[\pi_{(i,0)} + \sum_{n=1}^{2}\int_0^\infty \pi_{(i,n)}(x)\, dx\right] = 1.$$

In order to explain these equations one should consider the process marked transition graph, represented at the Figure 1.

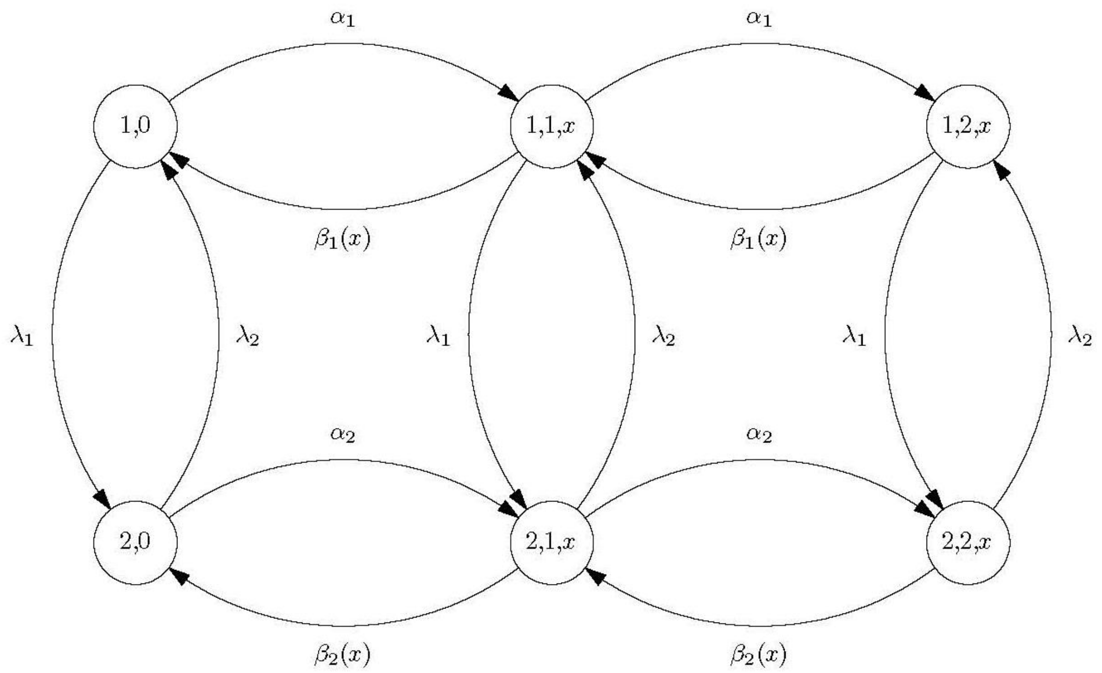

Figure 1. The process transition graph for $< M_2|GI|1(ME) >$ system.

The process passes from one environment state to another and back with intensities λ_1, λ_2. From another side in each environment state i it leaves the states $(i, 1, x)$ and $(i, 2, x)$ with intensity $\beta_i(x)$, otherwise it stays in these states. The process occurs in the boundary state $(i, 1, 0)$ in the case if the operating in the state 0 unit fails, or if the repaired in the state $(i, 2, x)$ unit restores.

This system of equation does not allow analytical solution and examples of its numerical solutions in the section 5 will be considered.

4 Cold redundancy $< GI_2|M|1(ME) >$ system, operating in Markov environment

Consider now analogous system $< GI_2|M|1(ME) >$ with generally distributed life time of the units and exponentially distributed repair time, operating in two-state Markov environment. Consider an analogous to the previous case stochastic process $\{Z(t)\}_{t\geq 0} = \{I(t),\, N(t),\, X(t)\}_{t\geq 0}$ with the same two first components, but the last one denote now elapsed operational time of the working unit. The reader will not be confused by using the same notations for different processes. It has been done for convenience of the results representation with the same notations.

Define the state probabilities:

(1) $\pi_{(i,n)}(t;\, x)\, dx = \mathbf{P}\{I(t) = i,\, N(t) = n, x < X(t) \leq x + dx\}$ – the joint probability that at time t the system is in i-th environment state $(i = 1, 2)$, there are $n\,(n = 0, 1)$ failed units and the operational has elapsed working time between x and $x + dx$;

(2) $\pi_{(i,2)}(t) = \mathbf{P}\{I(t) = i,\, N(t) = 2\}$ – the probability of the "bad" state (complete failure state) of the system at time t being in i-th environment state.

Analogously to the previous case for these probabilities the system of forward Kolmogorov partial differential equations jointly with boundary, initial and normalizing conditions has been derived (for analogous system operating in stable environment see [11, 12]). Since the Markov process $\{Z(t)\}_{t\geq 0}$ is a Harris one with positive atom in the states $(i,2)$, it has steady-state probabilities, equations for which have the form

$$\begin{aligned} \left[\frac{d}{dx}+\alpha_i(x)+\lambda_i\right]\pi_{(i,0)}(x) &= \beta_i\pi_{(i,1)}(x)+\lambda_{\bar{i}}\pi_{(\bar{i},0)}(x), \\ \left[\frac{d}{dx}+\alpha_i(x)+\beta_i+\lambda_i\right]\pi_{(i,1)}(x) &= \lambda_{\bar{i}}\pi_{(\bar{i},1)}(x), \\ (\beta_i+\lambda_i)\pi_{i,2} &= \int_0^\infty \pi_{i,1}(u)\alpha_i(u)du+\lambda_{\bar{i}}\pi_{(\bar{i},2)}. \end{aligned} \tag{3}$$

with the boundary and normalizing conditions

$$\int_0^\infty \pi_{(i,0)}(u)\alpha_i(u)du+\beta_i\pi_{(i,2)} = \pi_{(i,1)}(0), \tag{4}$$

$$\sum_{i=1}^{2}\left[\sum_{n=0}^{1}\int_0^\infty \pi_{(i,n)}(x)\,dx+\pi_{(i,2)}\right] = 1. \tag{5}$$

To explain these equations one can use the analogous to previous case reasonings based on the marked transition graph, represented at the Figure 2.

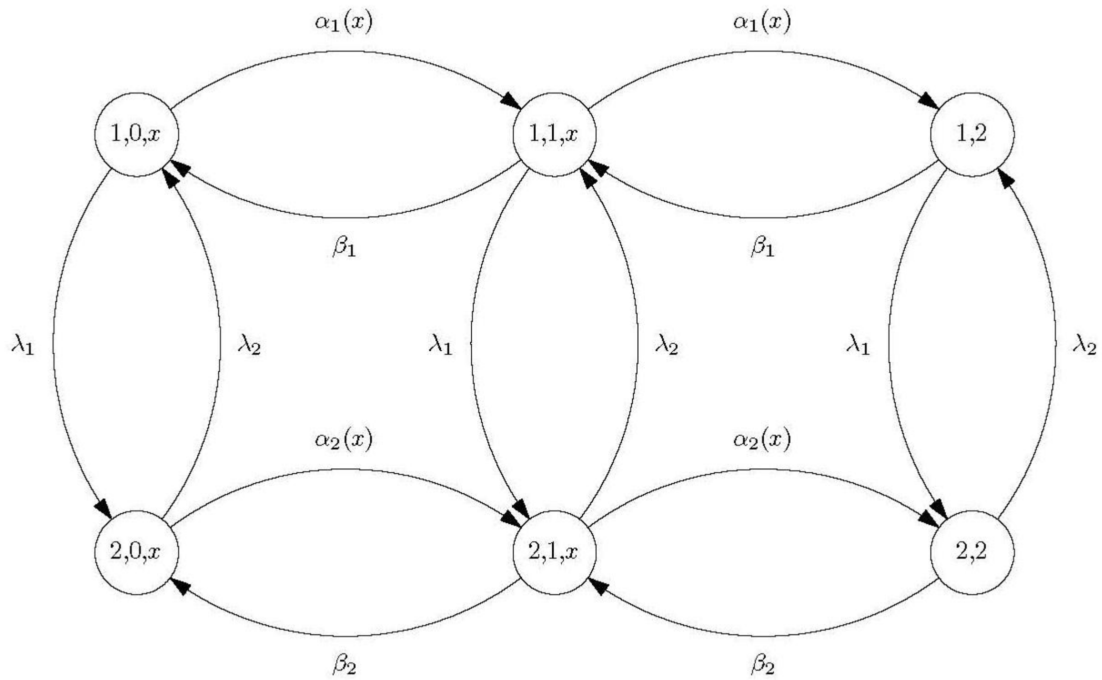

Figure 2. The process transition graph for $< GI_2|M|1(ME) >$ system.

In order to study of environment influence to the reliability characteristics some additional numerical experiment based on the solution of the equations (1) and (3) has been done. Below the results of these investigations will be shown.

5 Numerical investigation of the models

In numerical analysis the Γ-distribution with PDF

$$f(x) = \frac{(x\mu)^{k-1}\,\mathrm{e}^{-x\mu}\mu}{\Gamma(k)}$$

will be used as an example of general distribution. This choice is motivated by the reason that this distribution allows to model the situations with variations grater and smaller than one. As a reliability characteristic an availability coefficient

$$K = \pi_{(1,0)} + \pi_{(2,0)} + \pi_{(1,1)} + \pi_{(2,1)}$$

will be considered. It will be studied depending on mean relative repair rate

$$\rho = \frac{\lambda_1^{-1}\rho_1 + \lambda_2^{-1}\rho_2}{\lambda_1^{-1} + \lambda_2^{-1}}, \quad \text{where} \quad \rho_i = \frac{\bar{a}_i}{\bar{b}_i} \quad \text{and}$$

$\bar{a}_i$, $\bar{b}_i$ $(i = 1, 2)$ have been defined before.

Availability coefficient K also depends on environment relative variability index

$$u = \frac{\lambda_1^{-1}u_1 + \lambda_2^{-1}u_2}{\lambda_1^{-1} + \lambda_2^{-1}}, \quad \text{where} \quad u_i = \lambda_i\,\bar{a}_i,$$

and pure environment variability index (ratio of environment change states intensities)

$$z = \frac{\lambda_2}{\lambda_1}.$$

We consider 3 variants of environment variability: slow ($u = 0.01$), moderate, when mean time between changing states of environment is equivalent mean life time ($u = 1$) and quick ($u = 100$). Also we consider 3 variants of values of parameter z: $0.01, 1$ and 100.

In all numerical experiments the elements are more reliable in first state of environment than in the second one and their mean repair times coincides,

$$\frac{\bar{a}_1}{\bar{a}_2} = 100 \quad \text{and} \quad \frac{\bar{b}_1}{\bar{b}_2} = 1.$$

In all numerical experiments parameters k, μ of Γ-distribution has been chosen in such manner that these ratios are fulfilled.

In order to model different types of the life and repair time distributions the variation V of Γ-distribution has been varied at seven levels: 0.1, 0.4, 0.8, 1, 1.2, 1.6 and 1.9.

5.1 Model $< M_2|\Gamma|1 >$

For the case when Γ-distribution is used as a repair distribution the parameters k, μ of Γ-distribution are determined based on variation V and mean repair time $\bar{b}_j$,

$$k = \frac{1}{V^2}, \quad \mu_j = \frac{1}{\bar{b}_j V^2} \quad (j = 1, 2).$$

Remark 5.1. *For the calculation it is necessary to take into account that for* $x >> 1$ *the expression* $\beta_j(x) = \frac{b_j(x)}{1-B_j(x)}$ *became non-stable, and in this case asymptotic representation* $\beta_j(x) \approx \mu_j - \frac{k-1}{x}$ *has been used for* $x > \frac{k}{\mu_j} + 6\frac{\sqrt{k}}{\mu_j}$.

Moreover, in the case of $V < 1$, *for* Γ*-distribution* $\lim_{x\to 0} \beta_j(x) = \infty$, *therefore in this case for* $x = 0$ *it is used the expression* $\beta_j(0) = b_j\left(\frac{k}{100\mu_j}\right)$.

The results of numerical experiments as the Figure 3 are presented, where the continuous line corresponds to the case of exponential distribution with variation equal one $V = 1$, the point-wise curve corresponds to the case with variation less than one $V < 1$, and the dash-wise line corresponds to the case with variation grater than one $V > 1$.

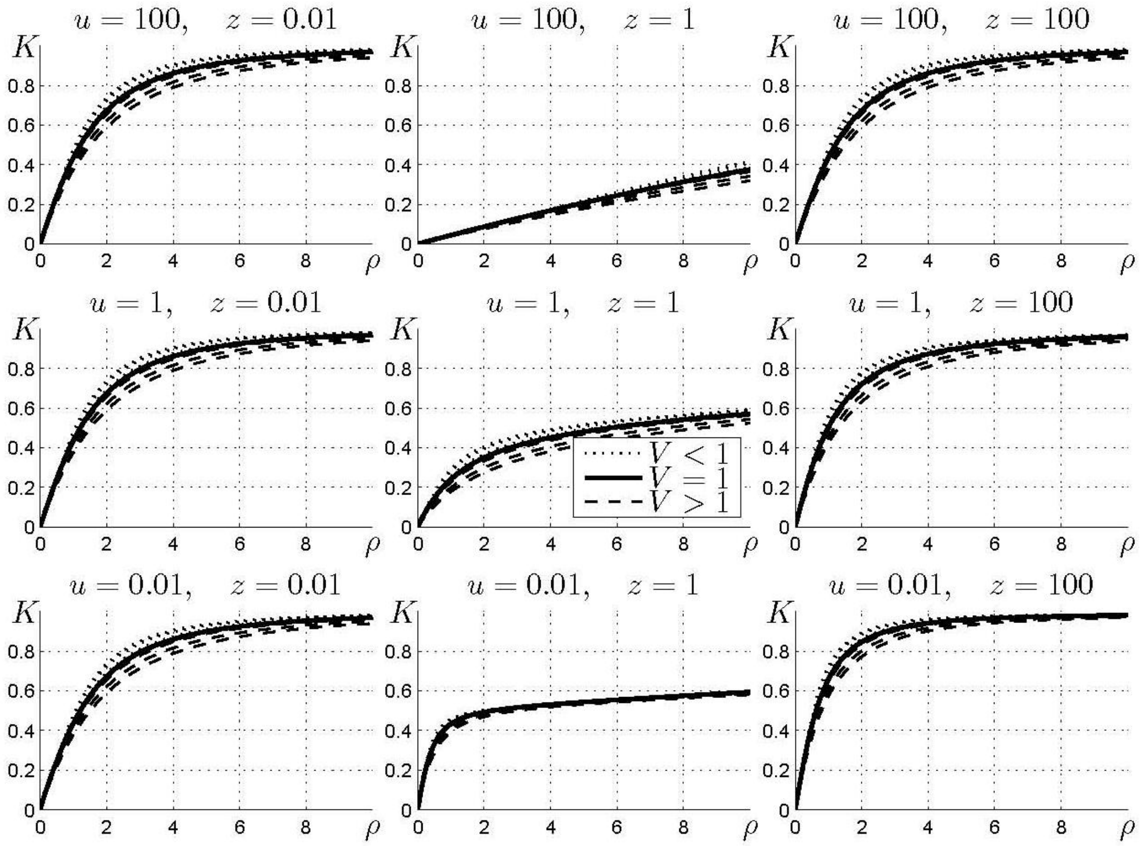

Figure 3. Numerical results for the model $< M_2|\Gamma|1 >$.

The graphs show that the availability coefficient K only a weak sensitive to the shape of the repair time distribution and the sensitivity vanishes with increasing of the relative repair rate coefficient ρ. However, different types of lines demonstrate dependence of the availability coefficient K on variation V: big variation decreases availability while small variation increases it. From another site the influence of the environment relative variability index u to the system availability K is not enough significant, while the pure environment variability index z influences on its convergence rate to one.

5.2 Model $< \Gamma_2|M|1 >$

Consider now the model, when the Γ-distribution is used as elements life time distribution while the repair time distribution is the exponential one. In this case the parameters k, μ of the Γ-distribution are determined based on variation coefficient V and mean life time $\bar{a}_j$ by the following way

$$k = \frac{1}{V^2}, \quad \mu_j = \frac{1}{\bar{a}_j V^2} \quad (j = 1, 2).$$

The graphs of the availability coefficient K versus relative repair rate coefficient ρ for different cases are shown at the Figure 4, where the same type of curves for different values of variations are used.

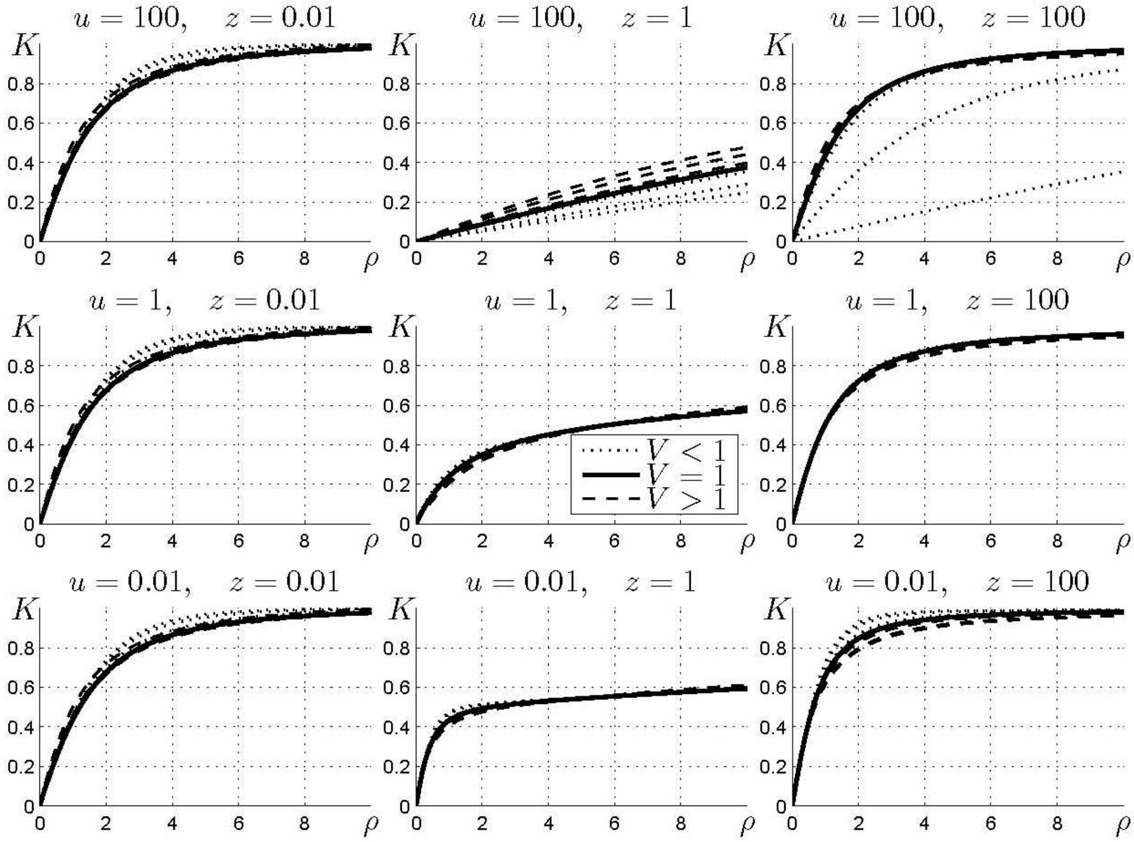

Figure 4. Numerical results for the model $< \Gamma_2|M|1 >$.

Also as before the graphs demonstrate the weak dependence of the system availability coefficient K on the shape of units life time distributions that is vanish under fast restoration (when the relative repair rate coefficient ρ grows to infinity) and the different type of its dependence on the relative and pure environment variability indices u and z. The difference consists only in the variation influence to the availability coefficient K in this case: big variation increases availability while small variation decreases it.

6 Conclusion and gratitude

The sensitivity of availability coefficient for double redundant removable system operating in Markov environment to the shapes of its life and repair time distribution has been considered. It is shown that the sensitivity vanishes with increasing of the elements relative repair rate. From another site the influence of the environment variability to the availability coefficient K is not enough significant and depends on both: relative and pure environment variability indices.

The calculations also show the influence of the elements life and repair time variation to the system availability: the system availability increases when elements life time variation increases and it decreases with increasing the element repair time variation.

The investigation of another cases of redundancy (hot redundant systems, several redundant elements) and another kind (non Markovian) of environment randomness are the problems for further investigations.

The paper has been presented at the MMR-2015 Conference in Tokyo and submitted to ORSJ journal. One of referees gave several remarks and recommend the paper for publication after some modification. Another one “did not see” that the system is modelled by three-dimensional Markov process and reject the paper as too simple. We are grateful the first referee for his remarks that has been used in the present version of the paper.

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