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# RISK ANALYSIS: THEORY \& APPLICATIONS 

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The use of the Weibull distribution to model lifetimes of incandescent lamps was originally suggested by Leff (1990). Following this suggestion, Agrawal and Menon have offered and investigated, in a series of papers, an improved model constructed from physical considerations and laws of mathematical statistics. In the present paper we offer supplementary thoughts concerning the AgrawalMenon model and its several modifications. In addition, we discuss the use of Pinelis's l'Hospital-type calculus rules in the analysis of ageing properties of lifetime distributions.

## Letter from the Editor-in-Chief

I got many letters about the name of our e-journal that indicate that the real scope of the journal is much wider than just reliability. It is difficult not to agree with them: actually we cover a number of topics related to probabilistic modeling and statistical analysis, cost-effectiveness analysis and various of applications of operations research and management sciences that have a nature of Risk Analysis Theory.

Indeed, we cover not only traditional topics of reliability theory (like System Reliability Analysis, Optimum Reliability Allocation, Maintainability, Accelerated and Life Testing, Reliability Data Analysis, Quality Control, Quality Assurance) but also a number of other problems like Counter-terrorism defense models, Security, Safety, Survivability (Vulnerability), Operational Effectiveness, Queuing Models, Optimal Spare Supply, Optimal Inventory Control, Monte Carlo Simulation, etc.

After some discussion, we decided to change the name of the journal to Risk Analysis: Theory and application, to reflect a real face of the journal.

Doing so, we take into consideration the fact that Boris Gnedenko, after who the site is named, was actually a scientist of an extremely wide profile. He started ( as a pupil of one of the greatest mathematician of the $20^{\text {th }}$ century Andrei Kolmogov) with pure probabilistic and statistical problems but after his spectrum became wider: he was involved in many applied problems. He was one of the first Russian mathematicians developing the Queuing Theory, then he founded Russian school in Reliability, he was the author of the first papers on statistical quality control. His last years he dedicated to arising actuarial mathematics in Moscow State University.

Taking into account all these facts and factual state of the journal, we decided to change the name of the journal starting No. 1 of 2008.

We hope that widening the spectrum of publications will help our professional community to exchange ideas and make better contacts with each others.

## Igor Ushakov

# CONFLICTS RESOLUTION AS A GAME WITH PRIORITIES: MULTIDIMENSIONAL CARDINAL PAYOFFS, PART 1 

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#### Abstract

There are two ways to consider increasing the effectiveness of the theory of games in applications. The first is to derive priorities for the payoffs using a cardinal absolute relative scale instead of an ordinal or interval scale to do equilibrium analysis. Our approach using cardinal payoffs is illustrated with one example in an application to OPEC strategies that the author published in the International Journal of Game Theory. Ross Cressman in his book, Evolutionary Dynamics and Extensive Form Games, says that it is inconceivable that current decisions do not depend in an intricate manner on choices made in previous encounters. Such intricate choices can be included in the extensive form of a game, by not explicitly in its normal form. We show how such complex influences can be used in non-cooperative situations using priorities to prescribe best outcomes for the Palestinian-Israeli conflict and for the best strategy for the U.S. in Iraq.


## 1. Introduction

Analysis of equilibria in game theory is based on ordering pairs of strategies according to preference and assigning ordinal numbers accordingly. Often the ordinals are assigned intensities to indicate the degree of preference of one strategy, an intangible, over another, hoping to approximate to a cardinal expression of preference. However assigning such numbers is a fairly arbitrary and intuitive process that leaves one asking, is there a more scientific way to derive numbers to strategies that accords them a more accurate representation of an individual's preferences? How?

There is little doubt that our values and judgments help us determine the relative importance of the numbers we obtain through measurement and the more expert we are, presumably the closer we are to interpret the intensities of numbers in a valid way in so far as they represent the dominance of influences in the real world. The question then is whether any human being including an expert in any field including an individual who is untutored about numbers and arithmetic, has (or can have) the ability to evaluate the relative importance of the intensities of cardinal numbers. If we had such a cardinal representation of payoffs in game theory, how would the analysis of equilibria be theoretically different?

It is known to cognitive psychologists that making comparisons is an intrinsic biological talent that we have. In addition, this talent is used by all people no matter how educated or talented they are. Comparisons can be applied by an expert to derive relative numbers to represent their idea of relative importance or priority. To make sense of these priorities one must have corresponding feelings whose intensity more or less corresponds to the value of the numbers.

The Harvard psychologist Arthur Blumenthal tells us in his book The Process of Cognition, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1977, that there are two types of judgment: "Comparative judgment, which is the identification of some relation between two stimuli both present to the observer, and absolute judgment, which involves the relation between a single stimulus and some information held in short term memory about some
former comparison stimuli or about some previously experienced measurement scale with which the observer rates the single stimulus."

In his book The Number Sense, How the Mind Creates Mathematics, the mathematician and cognitive neuropsychologist Stanislas Dehaene (Oxford University Press 1997, p.73) writes "Introspection suggests that we can mentally represent the meaning of numbers 1 through 9 with actual acuity. Indeed, these symbols seem equivalent to us. They all seem equally easy to work with, and we feel that we can add or compare any two digits in a small and fixed amount of time like a computer. In summary, the invention of numerical symbols should have freed us from the fuzziness of the quantitative representation of numbers."

Professor Michael Maschler the renowned game theorist and one of the editors of the International Journal of Game Theory recently wrote me: "People simply do not possess a utility function, or make mistakes when reporting their priorities. If you ask enough questions they will even state priorities that are not transitive. Thus, you cannot even determine a useful ordinal utility function. I do not know how to alleviate this difficulty. Therefore, at present, non-cooperative game theory can at best shed some insight on the real-life situation but usually it is not capable in suggesting definite recommendations (except for simple cases)".

Conflict resolution today has to be the most important subject for those who think about it and more urgently by those who deal with it as a practical matter on a daily basis [1]. The subject has been a main occupation in this author's career, from his days of working in disarmament in Washington, to teaching and writing $[2,3]$ about the theory of games as a way to deal with conflicts, to many studies involving real political conflicts. Examples are Vietnam [2], terrorism in the Olympics [4], the conflict in Northern Ireland [5], the Middle East conflict, involving intensive meetings and discussions in Cairo, Egypt [6], in the early 1970s, and analysis of the conflict in South Africa in the 1980s [7], spurred by a conference in Pretoria in 1986 [8] resulting in a detailed analysis about the resolution of the conflict, commissioned by a government institute concerned with strategic studies. We need a practical quantitative approach that enables one to synthesize payoffs on different criteria. It delves in greater depth into the fine structures of strategies according to their merits and weaknesses when confronted with those of the opponent than does a game theoretic approach. It makes it possible for the parties to recognize and account for the strengths and weaknesses (political, military, social and so on) of their strategies against those of the opposition. The parties can work together through their representatives (perhaps often through the UN and in the presence of other parties to mitigate exaggerations and excessive claims) or do the analysis on their own with their own judgments partly imputed to what they think the opposition desires. In the absence of one party the judgments are surmised by the analyst from publicly declared positions and subjected to sensitivity analysis in case of uncertainties. In this manner one can evaluate the strategies of each party according to its merits against the strategies of the opponent(s) to improve the parties' understanding of the conflict in which they are involved. This type of analysis involves multi-criteria decisions with intangible payoffs derived from paired comparisons of the relative merits of the strategies against each of the opponent's strategies and then synthesizing the outcome across all merits and weaknesses, analyzed in short, medium and long range time frames.

Traditionally, conflicts have been analyzed quantitatively, using the normal form of a game, with payoffs of different strategies all played at the same time that often need to be measured in different ways [9]. We have a choice to make among the numbers we use to represent these payoffs [10]. Commonly, because of the complexity entailed in different kinds of measurement, ordinal numbers are used to indicate only that one payoff is larger than another [11]. But we can use stronger numbers, cardinal numbers, whose relative magnitudes are meaningful, particularly those used to measure priorities among the things that are traded off. These numbers can be used to measure different things and are then synthesized into a single overall outcome. Theoretical considerations and calculations which appear impossible with ordinal numbers become possible with cardinal numbers. The preliminary steps to understanding the nature of a particular conflict are: 1. Identify the parties to the conflict. 2.

Identify the objectives, needs, and desires of each of the parties. 3. Identify possible outcomes of the conflict and possible "solutions." 4. Clarify the assumptions about the way in which each party views its objectives and, in particular, its view of the relative importance of these objectives and the varying, possibly multidimensional, payoffs for these objectives. 5. Set out these assumptions so each party can view the outcomes and the way in which a given outcome might meet their objectives. This seemingly simple set of steps presents some difficulties, since the perceptions of different parties may differ sharply. "One of the very interesting problems in the study of "perception" and one which is still largely unsolved is of the conditions under which "perceptions" of different individuals converge under the impact of symbolic communication and the conditions under which they diverge" [12].

In practice, as part of the negotiating tactic, parties often do not like to sit facing each other and cooperate to resolve the conflict. This is often a consequence of the fear that they might reveal something that gives the other side an advantage. This, of course, need not always be an issue for representatives of nations in conflict, as their decisions can be vetoed by their leaders back home. More important, mediators and arbitrators are frequently used as buffers who can convey a most agreeable attitude in explaining a tough line or laying out the opening position of each party. Most non-cooperative conflicts are helped by the presence of a third party or organization called in to assist. The mediators' concern with balancing and creating a fair result should outweigh their strict concern for impartiality. The mediators must be careful that their impartiality does not lead them to play into the hands of the stronger party and, similarly, an analyst who desires to model a current conflict and who seeks information from concerned parties runs the risk of allowing his or her formulation to be biased by his understanding of the situation. Even apparently unbiased observers tend to have a slanted view.

Another problem that arises with regard to actual measurement on different scales is how to make comparable assessments made by different players. The relative values obtained for one individual may not be commensurate with the relative values obtained by another individual and the solution then would be to embed the two individuals in a larger framework to determine the commensurability of their relative values. This problem of embedding is a special case of considering both individuals in a single framework with feedback that makes it possible to combine their separate beliefs and influences to obtain a single best outcome in which the question of commensurability is now no longer a pressing issue. Within that framework it is possible to assess the relative importance of the individuals according to various criteria of influence, and also by considering the priorities of their interactions, and the relative importance of their value systems. The outcome is then the one that is best in taking into account their separate points of view.

Thus a major problem in analyzing conflicts in quantitative terms is how to deal with the measurement of intangible factors that arise in a conflict. In the past, people have talked around intangibles and have mostly decided not to include them, as dealing with intangibles can be highly subjective. MacKay [13] writes that pursuing the cardinal approaches is like chasing what cannot be caught. But the situation has changed since he wrote that because of the development of theories for deriving relative measures for intangibles such as the Analytic Hierarchy Process. Without measures for the intangibles, there is likely to be a lack of agreement on trading off values among the parties in the conflict. While each party can reduce the tradeoffs to a single best outcome according to its system of priorities, it remains difficult to trade off values among several parties because of their differing values and objectives. In that case one must find an abstract way to define an index for tradeoffs among the parties that would be hard to reject on grounds of equality and fairness. In this paper, we will propose a way to do that.

Using ordinal payoffs in conflict resolution runs the risk of being arbitrary, because it attempts to summarize in a fell swoop many differing payoffs on different dimensions into a single ordinal number for each party and, in general, that is impossible to do, whether each payoff is expressed with cardinal (ratio scale) numbers
or with ordinal numbers on the different dimensions. This is very similar to the problem of multi-criteria decisionmaking that deals with combining measurements of both tangible and intangible criteria, and we propose an extension of how we deal with it in decision-making theory to conflict resolution.

Decision problems have elements that belong to different domains of knowledge: economic, social, physical, political, environmental, and technological. In each of these areas influences are studied through analysis, by breaking a problem down into components and determining the effects and priorities of the factors involved. The question is how to synthesize this information into a holistic overall outcome that combines all the influences from the different domains. Whereas analysis solves a problem in well defined ways, synthesis leads to compromises across the different fields by prioritizing the importance of their factors relative to each other with respect to higher values and goals that deal with them all at once. Compromise is made according to the particular values and preferences of the people involved. Analysis is needed to study the parts; synthesis is needed to bring together what is known about the parts into a whole. How to model this mathematically has been of considerable interest to some applied mathematicians, economists and operations researchers around the world in recent years.

In this paper we show without too much detail, how to derive payoffs in the form of priorities by developing a hierarchic structure in which the goals and values of the players are represented and use judgments, hopefully provided by the players or surmised from their known positions and writing about them. We then use the Nash equilibrium approach to determine best outcomes in the case of OPEC versus the United States.

## 2. The Quantitative Approach to Conflict Resolution

Decision making and conflict resolution are intimately related. The first deals with best choices by reconciling the multiple values of a single individual and the second by finding an agreed upon outcome to reconcile the values of many. In both cases one seeks the best outcome. The best-known quantitative approach for reconciling the different values of the parties in order to produce a fair resolution to a conflict is the theory of games, which is an abstract approach in search of equilibria for conflicts studied in terms of opposing strategies of several players [14].

The payoffs are usually represented by a single number for each party. These numbers are given by the parties (or assigned by the analyst) as rough estimates for opposing strategies matched in pairs and laid out in a matrix, mostly using ordinal numbers, particularly when dollars or other types of cardinal numbers are unavailable or not easily known. The purpose is to find a strategy that is overall optimal for all the parties, that is finding a cell in the matrix such that if either party moves to a different cell by changing strategy, the opponent can also change strategy to make the first party's payoff less [15]. The strategy oriented normal form of a game - a single matrix of pairs of numbers- over the more complex extensive form gives the mathematician an easy notation for the study of equilibria problems, because it bypasses the question of how strategies are put together, i.e. how the game is actually played. The concept of Nash equilibrium falls in this class of equilibria for non-cooperative games. John Nash made significant contributions to both non-cooperative game theory and to bargaining theory. He proved the existence of a strategic equilibrium for non-cooperative games - the Nash equilibrium - and proposed the "Nash program", in which he proposed dealing with cooperative games via their reduction to non-cooperative form. In his papers on bargaining theory, he founded axiomatic bargaining theory, and proved the existence of the Nash bargaining solution that was the first execution of the Nash program.

There have not been any other effective quantitative ways to analyze conflicts outside the ordinal equilibrium concepts of game theory [16]. When diverse multi-criteria measurements are available, they are assigned to the payoffs by the parties who have their own value systems. When the parties cooperate they can
conceivably align their values (one of yours is worth two of mine). Is there any other way to obtain a best solution? One needs a credible way to combine the payoffs using cardinal measurements into a single overall outcome for each party.

For emphasis, we note again that if there are payoffs that result from a complexity of combinations of different components on different scales of measurement (as a payoff may be composed from factors that have different scales of measurement such as a war that involves money, the lives of people, cultural and social and political influences), it would generally not even be possible to combine them if one were to use ordinals, a question not addressed by game theory, which assumes that a wholesale hypothetical ordinal number can be assigned as the payoff. The idea of equilibrium would undoubtedly involve greater refinement if it were possible to use cardinal instead of ordinal numbers. The question then is what is gained from using cardinal numbers that is stronger than simply using equilibrium solutions. Were one to use cardinal instead of ordinal payoffs, can one obtain a better concept of solution other than the usual game theoretic one with ordinal payoffs? How would the solution be derived and its stability tested in that case?

The Analytic Hierarchy Process (AHP), and its generalization to dependence and feedback the Analytic Network Process (ANP), deal with measuring intangibles in conflicts in a cognitive rather than abstract fashion by deriving priorities of influence for the parties and for the effectiveness of their strategies from actual numerical measurements and from absolute judgments expressed numerically in a pairwise comparison process provided by the parties themselves or by experts knowledgeable about the conflict. The focus is on obtaining accurate judgments that reflect the relative intensity of dominance from which the priorities on which the analysis is based are derived. The analysis is done separately in terms of benefits, opportunities, costs, and risks and then combining them into a single best outcome. Sensitivity analysis is then used to test the stability of the outcome in ranges of values of the priorities derived for different ranges of values of the pairwise judgments. A significant concern is how to incorporate the judgments of different people without the requirement of consensus and how to also include different weights of importance (priorities) for the different judges who provide the judgments. Theorems have been proved to determine methods of synthesis to apply under such circumstances. Several conflicts have been studied in this way and the results communicated to the parties. The best known of these is the analysis of the conflict in South Africa in the 1980s that showed one of the best actions would be to release Nelson Mandela and to remove apartheid, both of which were done in the resolution that followed. Others instances where this approach was applied were terrorism in the Olympics, the conflict in Northern Ireland, the Middle East conflict, the compromise reached between Egypt and Israel in the late 1970s, and the ongoing U.S.- Iraq, China-Taiwan, and U.S.-North Korea conflicts.

In the next sections (3 and 4) we elaborate respectively on Game Theory with its ordinal approach to equilibrium and the AHP with its cardinal multicriteria approach. In section 5, we apply the AHP as a way of deriving cardinal priorities. We then parallel the discussions in these two sections, combining them in section 5 by using a game-theoretic equilibrium approach with the OPEC problem. In section 6 we show how priorities derived with AHP can be used to alert parties in the conflict in Northern Ireland as to what the highest priority strategies they have to pursue resolution of that conflict.

## 3. The Theory of Games; A Normative Theory of Conflict Resolution

The major normative, what-should-be theory that deals with a formalization of the resolution of conflicts is the theory of games. It offers solutions that are thought to be mathematically best in some sense. It is concerned with games of strategy, a well-known rational way to deal with only certain kinds of conflict. Not all conflicts can be formalized as games of strategy and resolved normatively. Its approach requires that strategies be identified in order to think about how to resolve conflicts.

Game theory studies conflict and cooperation by considering the number of players, their strategies and payoffs [17, 18]. Games have been classified as cooperative and non-cooperative and analyzed according to the degree of information available to the players. A game is played with pure and with randomized strategies. The players seek to maximize the expected value of their payoffs. For non-cooperative games the Von Neumann minimax theorem for two-person zero-sum games proves that every finite zero-sum two-person game has a solution in mixed strategies. In 1950 John F. Nash extended this theorem to the existence of a solution of an nperson constant sum game in mixed strategies as a Nash Equilibrium solution. The Prisoner's Dilemma and Chicken are two non-cooperative games that do not yield satisfactory equilibrium solutions, and thus more than the existing concepts of equilibrium is still needed to obtain a good solution for them.

For cooperative games, von Neumann and Morgenstern introduced the idea of a characteristic function of a game and of the worth achievable by a coalition of some of the players independently of the remaining players [9]. A solution is called a stable set with which is associated a core. The core may not always exist. But when it does, it can have a nucleolus, all of which contain the idea of solution to the cooperative game. Many alternative solution concepts have been proposed to deal with coalitions. The Shapely value is another approach to solving a cooperative game. This value sometimes belongs to the core of the game. How to calculate an equilibrium solution can involve nonlinear techniques that may be approximate.

Payoff and expected payoff are central concepts in game theory. But payoff is measured according to what and whose values? How are the values obtained, and are they unique or are there other measures of payoff and do they all yield the same solution? Is it possible to resolve conflicts by other theoretical means that do not parallel the game theoretic approach with multi-dimensional measurements?

An intriguing problem in game theory is the assumption that it is possible to estimate payoffs for strategies in a game before the strategies of one player have been matched against those of the opponent in actual competition. Except for the simplest and most transparent situations it is impossible to spell out all the moves and tactics of a real-life strategy to really get a good idea of how well it would fare in competition. Some broad qualities of a strategy may be known, but exact prescriptions of its effectiveness may encounter such unanticipated problems in practice that it may be difficult to get a "good" estimate of its worth when compared with other strategies.

## 4. A Descriptive Theory of Conflict Resolution; AHP/ANP

Our mind interacts with the real world in two different quantitative ways of measurement. The first is simple and easier to do and that is to determine which of two elements $A$ and $B$ has a property more than the other and simply indicate for example that $B$ has it more than $A$ [19]. In addition if there are several such elements and one wishes to rank them one may use ordinal numbers of any magnitude to indicate their order. There is the possibility that one may make an error in such estimates and thus the outcome may not be exactly as it is in reality. The second relies on our ability to differentiate between magnitudes when the elements are closer with respect to the property and say with a fair amount of certainty approximately how many times more one element has the
property than the other (the lesser one used as the unit). This is a much more difficult task that has many uncertainties. However, if one were to use the judgment of someone who has long term familiarity with the elements, an "expert", one may wish to take that cardinal route simply to see what kind of outcome it leads to and how reliable it is. That approach is no longer simply a less reliable way of guessing at numbers. It is now a well grounded querie that has been developed in considerable mathematical depth and applied to numerous real life situations, and one might add successfully.

Conflict resolution can be regarded as a multiparty, muticriteria and multiperiod (short medium and long term outcomes) decision-making process that involves use of prioritization in the context of benefits, opportunities, costs and risks. From the field of behavioral economics that imports insights from psychology into economics, one learns that conflict resolution is also an evolutionary process of learning to enrich the structure of factors included in the framework of analysis and the interaction and influence of these factors on the outcome with the passing of time. There are many conflict situations in which the grievance that a party has against another party or parties cannot be described in terms of strategies and in terms of responses to these strategies. A helpless person may have many creative and rational complaints against society but has no meaningful strategy to act on his/her grievances if indeed he/she who may also be crippled and inarticulate can. In other words not every wrong in the world can be formed as a game of strategy. Thus conflicts that can be formalized in terms of opposing strategies are a special case of conflicts in general. It is known that non-cooperative games do not always have an equilibrium solution for all the parties involved and these are the most intractable and pressing kinds of conflict including terrorism as a special case. The question is whether there is a way to formalize conflicts rationally in order that one may consider their solution without recourse to the idea of strategy where there may be no strategy, or when there is one, to analyze it as a particular case of a more general concept? It is easy to give examples of conflicts where no solution is possible. In a hungry society with little food to go around, the hungry would be opposed to the well fed for the threat of their survival. With increasing population and despite creativity and progress it may be that the world would reach a point where not all essential amenities would be potentially available to every one.

Let us look at some concepts developed in the AHP/ANP about conflict resolution and how they were applied in practice that take into consideration: multiple payoffs to each party, cooperation and non-cooperation, and the question of priorities for each party that need not be compatible with the priorities of their opponents, and how they were used in practice. The AHP/ANP evolved out of my experience at the Arms Control and Disarmament Agency (ACDA) in the Department of State during the Kennedy and Johnson years. ACDA negotiated arms agreements with the Soviets in Geneva. I was invited to join ACDA, I think because of work I had done for the military using Operations Research mathematics. I published on it and wrote the first book on mathematical methods of operations research. At ACDA I supervised a team of foremost internationally known scientists, economists and game theorists (coincidentally including four people who later won the Nobel Prize in economics: Debreu, Harsanyi, Selten and Aumann) who advised ACDA on arms tradeoffs, but we had some insurmountable difficulties in making lucid and usable recommendations to our highly intelligent and experienced negotiators who were guided by strong intuition deriving from long practice.

We consider three types of uses of measurement. One is a strict game theoretic context. The other is a game against nature and the third is an equilibrium game with gains and losses. We don't think that it is possible to present the entire details of the conflicts involved but only to give the reader an idea of a new area of possibilities for conflict resolution.

## 5. OPEC

Here we analyze the relative effectiveness of the strategies if engaged against each strategy of the opponent. This yields a vector of the relative strengths of the strategies against each strategy of the opponent's. These vectors form the columns of a matrix. Each row of this "engagement" matrix is weighted by the corresponding "intrinsic" weight of the strategy from the first step to obtain the payoff matrix. The process is repeated to obtain the opponent's payoff matrix.

The next steps in the process are as follows:

1. Construct a hierarchy of objectives and strategies for each actor.
2. Prioritize these objectives.
3. Compute "constant values" of each actor's strategies; for example, the relative effectiveness of each strategy in satisfying the actor's objectives.
4. Compute "current values" of the strategies; for example, the relative strengths of the strategies of one actor against those of the opponent.
5. Compute the payoffs to each actor by multiplying the current value of each strategy by its constant value.

This results in a payoff matrix showing the payoffs to the actors for each pair of their strategies.
6. Search for a "Nash equilibrium solution."

The objective of the method is first to assign payoffs to the strategies of the actors by taking into consideration both their "constant" and their "current" values and then to determine the equilibrium solution (s).

The United States may choose one or a mix of the following strategies [20]:
$\mathrm{U}_{1}$ : Reduce oil imports from OPEC by increasing imports from non-OPEC oil producers, accelerating the development of indigenous resources, and reducing oil consumption.
$\mathrm{U}_{2}$ : Limit petroleum imports by tariffs and quotas.
$\mathrm{U}_{3}$ : Prepare an emergency scheme for dealing with sudden oil shortages, such as establishing strategic petroleum reserves, oil rationing programs, and emergency oil sharing.
$\mathrm{U}_{4}$ : Devalue the dollar against other major currencies.
$\mathrm{U}_{5}$ : Take military action against OPEC.
$\mathrm{U}_{6}$ : Impose embargoes of various kinds of goods and services to OPEC.
$\mathrm{U}_{7}$ : Weaken or break up OPEC by a joint consumer action.
$\mathrm{U}_{8}$ : Help Israel in its confrontations with the Arabs.
$\mathrm{U}_{9}$ : Encourage and support a "just" political settlement of the Arab-Israeli conflict.
$\mathrm{U}_{10}$ : Increase interdependence with OPEC members.
$\mathrm{U}_{11}$ : Increase arms sales to OPEC members.
OPEC members, either individually or collectively, may choose one or a mix of the following strategies:
$\mathrm{O}_{1}$ : Impose an oil embargo.
$\mathrm{O}_{2}$ : Cut back production.
$\mathrm{O}_{3}$ : Base the price of oil on the nearest alternative energy source.
$\mathrm{O}_{4}$ : Link crude oil prices to an index of prices of goods that OPEC members need to import.
$\mathrm{O}_{5}$ : Reduce the price of oil drastically
$\mathrm{O}_{6}$ : Use SDRs (Saudi Dinars) or a basket of majorindicator.
$\mathrm{O}_{7}$ : Increase oil prices gradually.
$\mathrm{O}_{6}$ : Impose sudden oil price hikes.
$\mathrm{O}_{9}$ : Search for an alternative to OPEC.
$\mathrm{O}_{10}$ : Increase interdependence with the oil importers.
$\mathrm{O}_{11}:=$ Do nothing.

Table 1 Payoffs to U.S. and OPEC by Matching their Strategies

|  | $\mathrm{O}_{1}$ | $\mathrm{O}_{2}$ | $\mathrm{O}_{3}$ | $\mathrm{O}_{4}$ | $\mathrm{O}_{5}$ | $\mathrm{O}_{6}$ | $\mathrm{O}_{7}$ | $\mathrm{O}_{8}$ | $\mathrm{O}_{9}$ | $\mathrm{O}_{10}$ |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\mathrm{U}_{1}$ | $6.6,0$ | $12,1.2$ | 14,0 | $10,1.1$ | $5.6,6$ | $9.7,0$ | 13,0 | $8.2,0$ | $19, .35$ | $3.5,3.5$ | $17, .28$ |
| $\mathrm{U}_{2}$ | $.22, .14$ | $.25,2.5$ | $.26, .42$ | $.25, .65$ | $2.8,0$ | $.59, .76$ | $.41, .25$ | $.17, .4$ | $.37, .38$ | $0,3.3$ |  |
| $\mathrm{U}_{3}$ | 3,0 | $2.6,0$ | $.78, .26$ | $0,2.9$ | $.35,0$ | $.49, .46$ | $.34,1.3$ | $.71,0$ | $1.2, .5$ | 0,6 | $.34,0$ |
| $\mathrm{U}_{4}$ | 0,0 | $.14,1.1$ | $0, .65$ | $0,9.3$ | 0,0 | $0,2.3$ | $.47, .44$ | $.43, .34$ | $0, .32$ | $0, .27$ |  |
| $\mathrm{U}_{5}$ | $.02, .51$ | $0,2.3$ | 0,0 | 0,0 | 0,0 | $0, .39$ | 0,0 | 0,0 | 0,0 | $0,1.8$ |  |
| $\mathrm{U}_{6}$ | $.05, .24$ | $0,1.1$ | 0,0 | 0,0 | 0,0 | $0, .54$ | 0,0 | $0, .15$ | 0,0 | 0,0 |  |
| $\mathrm{U}_{7}$ | $6, .05$ | $.79,2$ | $1.1,0$ | $.52, .67$ | $.45,0$ | $.31,0$ | $.75,0$ | $.99, .43$ | $0, .57$ | $0,1.4$ |  |
| $\mathrm{U}_{8}$ | $.02, .51$ | $0,1.1$ | 0,0 | 0,0 | 0,0 | $0, .39$ | 0,0 | $0, .93$ | 0,0 | 0,0 |  |
| $\mathrm{U}_{9}$ | $2.6,0$ | 0,0 | 0,0 | $0,6.3$ | 0,0 | 0,0 | $0,1.6$ | 0,0 | 0,0 | 4.0 |  |
| $\mathrm{U}_{10}$ | $2.3,0$ | $2.7,0$ | $2.7,0$ | $6.9,2.8$ | $3.5,0$ | $7.6,0$ | $4.3,0$ | $6.1,0$ | $3.8,3.2$ | $9.2,7.9$ | $4.6, .12$ |
| $\mathrm{U}_{11}$ | 0,0 | 0,0 | 0,0 | $.31,6.2$ | $.31,0$ | 0,0 | $0,8.5$ | 0,0 | 0,0 | 0,0 |  |

The payoff matrix of the U.S.-OPEC conflict is given in Table 1. The $\left(U_{i}, O_{j}\right)$ entry in the matrix represents the payoff to the United States if it adopts strategy $U_{i}$, and to OPEC if it adopts strategy $\mathrm{O}_{\mathrm{j}}$. These payoffs are obtained by weighting each current value (the relative strength of the strategies of one actor against those of the opponent) by the constant value (the relative effectiveness of each strategy in satisfying the actors' objectives) of the corresponding strategy.

The next step is to find the Nash equilibrium solution of the nonzero sum U.S.-OPEC game. The Nash equilibrium solution is a pair of strategies (one for each player) such that no player is able to improve his payoff by changing his strategy choice while the other player holds his strategy fixed. In our case the solution is $\left(\mathrm{U}_{10}\right.$, $\mathrm{O}_{10}$ ); that is, to increase interdependence between the United States and OPEC members.

Increasing interdependence between the oil consumers and oil producers appears to be a rational strategy. By exercising restraint in price hikes and by investing in the economies of the oil-consuming countries, particularly the United States, OPEC members are encouraging this type of outcome. ( $\mathrm{U}_{10}, \mathrm{O}_{10}$ ) places the United States in a good position against threats by OPEC members regarding production cutbacks and oil price hikes. It also boosts U.S. exports, thereby providing more jobs and improving the U.S. balance of payments. From the OPEC members' viewpoint, interdependence not only ensures an oil market but also provides OPEC countries with U.S. technology, capital, and management skills needed for development.

Looking at the equilibrium solution $\left(\mathrm{U}_{10}, \mathrm{O}_{10}\right)$, which deals with interdependence, we see that the payoff for the United States is higher than that for OPEC $\left(\mathrm{U}_{10}=9.2, \mathrm{O}_{10}=7.9\right)$. This is because the oil producers, with their large oil revenues, should be able to buy capital, technology, and know-how almost anywhere. The United States, however, does not have a wide choice for its oil imports. With interdependence, the U.S. is benefited more than OPEC.

If other consumers were to follow a policy of interdependence, the difference between the two payoffs (of producers and consumers) would be drastically reduced. This is because each oil consumer would be tied to several oil producers, which would reduce the chances of OPEC members switching customers. (A Paretooptimal point has a payoff not worse for each coordinate than that of any other point).

If the "interdependence" strategies of the United States and OPEC $\left(\mathrm{U}_{10}, \mathrm{O}_{10}\right)$ are removed, we would have several Pareto-optimal points at $\left(\mathrm{U}_{11}, \mathrm{O}_{4}\right),\left(\mathrm{U}_{1}, \mathrm{O}_{2}\right),\left(\mathrm{U}_{1}, \mathrm{O}_{1}\right)$, and $\left(\mathrm{U}_{1}, \mathrm{O}_{9}\right)$, Among these points, only $\left(\mathrm{U}_{1}, \mathrm{O}_{2}\right)$ is an equilibrium solution, with the United States reducing its dependence on OPEC by increasing its imports from non-OPEC sources, accelerating the development of indigenous resources, and reducing oil consumption through energy conservation; and OPEC cutting back production in order to prevent a glut in the oil market due to reduced demand for its oil and to keep the Price of oil from falling. Note that in the $\left(\mathrm{U}_{1}, \mathrm{O}_{2}\right)$ equilibrium, OPEC's payoff has decreased more than six times while the U.S. payoff has increased slightly.

If the strategies with $\mathrm{U}_{10}, \mathrm{O}_{10}$ and $\mathrm{U}_{1}, \mathrm{O}_{2}$ are eliminated, there would be no equilibrium solution in single strategies, and the likelihood of conflict of interest between OPEC members and oil consumers, particularly the United States, would be great. That would bring losses to all the actors in the international oil market. We could calculate the expected value of each strategy by summing and normalizing the row (column) payoffs to the U.S. (OPEC).

Two points have been illustrated in this section. The first is how to derive payoffs in the form of priorities on a relative scale of absolute numbers instead of attaching ordinals and the second is to use the idea of Nash equilibrium to determine the best strategies for the players. This is a good way to deal with situations where everything is known in advance by all the parties and they are fairly sure about their strategies and payoffs. In real life the situation is often different.

## 6. Unilateral Approach that Includes Opponents' Concerns

Fudenberg and Levine [21] in their book The Theory of Learning in Games, write that traditional explanation of equilibrium is that it results from analysis and introspection by the players in a situation where the rules of the game, the rationality of the players, and the player's payoff functions are all common knowledge. Both conceptually and empirically, these theories have many problems." They go on to say that they deal with situations in which the players are less than fully rational, who grope for optimality over time and the learning models they use do not lead to any equilibrium beyond the very weak notion of rationalizability. It is along similar lines that the real life applications described below were made (see also Elster [22]).

In our last example about the US-OPEC relations we had Nash equilibria to determine the best outcome by using the normal form of a game. But Nash equilibria do not always exist and often one is faced with trading off not strategies but move within a strategy thus requiring something like the extensive form of a game, a general approach is needed to enable one to resolve a conflict based on tradeoffs among payoffs within and between strategies in an appropriately determined sequential way.

The best way to illustrate this approach is through an example. It illustrates how the use of priorities helps the player to discover how important in the general scheme of things they and their strategies are thus tempering the degree and intensity of credible claims they can make. By knowing that often the parties have resorted to extremes like terrorism to increase their relative power as in Northern Ireland, in the Balkans and in the Middle East.

## The Northern Ireland Conflict

An early application of the AHP was made in 1977 to the conflict in Northern Ireland [23,24]. This early analysis was repeated over the years as the situation changed [24]. These studies created a great deal of interest in official as well as in academic circles. Alexander was invited to present her work on Northern Ireland in Canada and the People's Republic of China, as well as in Northern Ireland. In the United States, she made presentations on these studies at the Pentagon and at West Point, in addition to a number of academic venues.

In 2004, a further study using the AHP looked at the attitudes to the ongoing problems of Northern Ireland shown by the Scotch-Irish (Ulster-Americans), and compared what they wanted to happen with what they thought actually would happen [24]. A presentation based on this paper was given at the Ulster American Heritage Symposium in Northern Ireland, where it aroused considerable interest.

A number of the insights derived from the earlier studies seemed to have an effect on the way in which the respondents in the latest study viewed the problem, particularly in their assessment of the relative power of the participants in the conflict.

The first step in each analysis was to identify the participants to the conflict, those individuals or groups who may have an influence on the outcome. The problem is often described in religious terms, although this is a gross over-simplification. Throughout, there was a conscious effort to use basic descriptors and to avoid terms that used religious denominations, in order to remove any potential prejudice. This approach owes much to the influence of Richard Rose, who defined the Northern Ireland problem in classical political science terms of allegiance (or non-allegiance) to a regime.

To ensure that assumptions made in the analyses were valid, Alexander spent a number of extended periods in Northern Ireland to study the problems at first hand. She met with political, church, and community leaders, who gave generously of their time and who seemed pleased to discuss a wide range of issues. We are grateful to these busy people for their help.

The main participants in the Northern Ireland conflict are:
The British Government (BRITAIN), which controls Northern Ireland.
The Protestant (Unionist) majority community (ALLEGIANTS), which wants Northern Ireland to remain separate from the Republic of Ireland and which would find a substantial measure of minority participation acceptable. (This group does not include those who support violence.)
The Loyalist groups (DEFENSE), whose needs are similar to those of the ALLEGIANTS, but who are prepared to use force to resist the creation of a United Ireland.
The Roman Catholic (Nationalist) minority community (MODERATES), which includes both those who would prefer to join Northern Ireland with the Republic of Ireland and those who would be content to have Northern Ireland remain separate, provided that a structure that provides for substantial minority participation is established. (This group does not include those who support violence.)
The Irish Republican Army (IRA) (which includes not only the Provisional and Official subgroups and their supporters, but also the so-called splinter groups), which considers violence to be an acceptable path to a United Ireland.

The Government of the Republic of Ireland (DUBLIN), which seeks to act on an equal footing with the British Government in determining what should happen in Northern Ireland. It also aspires to a United Ireland.

The list of objectives of each participant is long: the reader is referred to Alexander 2004 and to the earlier studies.

The current political structure in Northern Ireland is that set up by the Belfast Agreement (Good Friday Agreement) of 1998, i.e., an elected Assembly with built-in guarantees of participation by all major groups in the ruling Executive. Cross-border bodies are, in theory, responsible to the elected Assembly in Belfast and the Dail (parliament) in Dublin. (In practice, these cross-border bodies seem to be under the control of London and Dublin.) The British Government is ultimately in control of Northern Ireland, with considerable input from the Irish Government.

Some of the citizens of Northern Ireland would like to be ruled directly by the British Government with a fully integrated Parliament, on the same basis as other regions of the United Kingdom (with the exception of Scotland and Wales, both of which have a considerable measure of autonomy). Others want Northern Ireland to be joined with the Republic of Ireland in a unitary state.

A further option, for which there is a measure of support, is for Northern Ireland to become a separate state, independent of both Britain and the Republic of Ireland. This could be, for example, a state within the (British) Commonwealth or it could be a state within the European Union.

Thus, the possible Political Outcomes* [24] are:
(a) the Good Friday Agreement: AGREEMENT
(b) an Integrated Parliament: INT-PARLIAMENT
(c) a separate independent state: INDEPENDENCE
(d) union of Northern Ireland and the Republic of Ireland: UNITED-IRELAND

* In the 1977 and subsequent studies, two outcomes of an elected Assembly, with or without a Council of Ireland, were listed. In the 2004 study, the Good Friday Agreement replaced these outcomes since the Agreement included an elected Assembly.

In the two studies of the Northern Ireland conflict carried out in 1977 [23], Alexander and Saaty showed that the most likely outcome would be some form of legislative independence for Northern Ireland, followed closely by a local Assembly. This suggested that a strong local Assembly, with a considerable measure of autonomy, would provide a workable solution. Since this outcome might well satisfy the MODERATES, but would certainly not satisfy the IRA and would probably not satisfy DUBLIN, it was reasonable to ask if a change in the relative power of these two participants relative to the remaining participants would affect the outcome and, if so, how much of a change would be necessary. By using what is known as the backward process through the hierarchy and varying the power of the participants, they were able to find thresholds of power to indicate by how much the power of both the IRA and DUBLIN would have to be increased relative to the other participants to change the outcome.

It now appears that these increases in power may have been achieved, at least in the eyes of the respondents in the 2004 study. In this latest study, the AGREEMENT outcome came out first, but barely ahead of the UNITED IRELAND outcome, using the probabilities obtained from the AHP analysis. When asked to rank their personal preferences, however, the group ranked the AGREEMENT first and the UNITED IRELAND outcome last. The respondents considered it desirable for the people of Ulster (Northern Ireland) to remain separate from the Republic of Ireland. They understood and sympathized with the desire of their kinsmen not to be absorbed into an all-Ireland state. Nevertheless, they saw the power of both the IRA and DUBLIN as having been so enhanced over recent years that their influence was now considerable. They saw the forces arrayed against the Ulster majority as being so powerful that the probability that a United Ireland will occur is now almost as great as the probability of successful implementation of the Good Friday Agreement.

The use of the AHP to analyze this problem enables one to shed light on a complex problem. The respondents in the 2004 study had no previous experience with a study of this nature, but were able to make straightforward comparisons and thus enable the analyst to calculate the required probabilities. The sharp dichotomy between what they saw as desirable and what emerged from their answers in the AHP part of the study seems to show that their personal preferences did not influence their judgments on the comparison weights.

What of the future? The situation described is volatile and the end result may not be the triumph of violence suggested by the latest study. Ongoing analyses of this problem may provide further understanding.

## 7. Conclusion

So far we have dealt with conflicts in terms of equilibria to resolve a conflict or in terms of priorities of strategies within hierarchic structures to enable players to assess their relative power and what they can and cannot achieve against their opponents. Particularly in the application to the conflict in Northern Ireland, it was possible to inform one of the parties about what it could not accomplish because of its low priority of influence relative to the other parties and the options it had to increase its effectiveness.

In the next paper we explore a different way of conflict resolution by giving examples that use network structures with dependence and feedback to derive different kinds of payoffs involving benefits, opportunities, costs and risk and then combine them into an overall payoff used to determine the best strategy to follow or to tradeoff different moves in a strategy to benefit the parties according to balance between their own value systems rather than according to an overall abstract strategic equilibrium.

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# FURTHER ANALYSIS OF CONFIDENCE INTERVALS FOR LARGE CLIENT/SERVER COMPUTER NETWORKS 

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#### Abstract

In the recent paper [Abramov, RTA, 2 (2007), pp. 34-42], confidence intervals have been derived for symmetric large client/server computer networks with client servers, which are subject to breakdowns. The present paper mainly discusses the case of asymmetric network and provides another representation of confidence intervals.


Key words: Closed networks, Performance analysis, Normalized queue-length process, Confidence intervals.
2000 Mathematical Subject Classification: 60K30, 60 K 25

## 1. Introduction

Consider a large closed queuing network containing a server station (infinite-server queuing system) and $k$ single-server client stations. The total number of customers (units) is $N$, where $N$ is assumed to be a large parameter. The departure process from client stations is assumed to be autonomous. For the definition of queuing systems with autonomous service mechanism in the simplest case of single arrivals and departures see [1], where there are references to other papers related to that subject.

The arrival process from the server to the $i$-th client station is denoted $A_{i, N}(t)$. The service time of each unit in the server station is exponentially distributed with parameter $\lambda$. Therefore, the rate of arrival to client stations depends on the number of units in the server station. If there is $N_{t}$ units in the server station in time $t$, then the rate of departure of units from the server in time $t$ is $\lambda N_{t}$. There are $k$ client stations in total, and each client station is a subject to breakdown. The lifetime of each client station is a continuous random variable independent of lifetimes of other client stations. The probability distribution of $i$-th client station is $G_{i}(x)$. The client stations are not necessarily identical, and a unit transmitted from the server station chooses each one with corresponding probability $p_{i}$. This probability $p_{i}$ depends on configuration of the system in a given time moment, that is on the number of available (not failure) client servers and there indexes. In general such kind of dependence is very complicated. However, in the case of the network with only two client stations this dependence is simple. This simplest case is just discussed in the present paper.

The departure instants from $j$-th client station $(j=1,2, \ldots, k)$ are $\xi_{j, N, 1}, \xi_{j, N, 1}+\xi_{j, N, 2}, \xi_{j, N, 1}+\xi_{j, N, 2}+\xi_{j, N, 3}, \ldots$ where each the sequence $\xi_{j, N, 1}, \xi_{j, N, 2}, \ldots$ forms a strictly stationary and ergodic sequence of random variables ( $N$ is the series parameter). The corresponding point process associated with departures from the client station $j$ is denoted

$$
S_{j, N}(t)=\sum_{i=1}^{\infty} \mathrm{I}\left\{\sum_{l=1}^{i} \xi_{j, N, l} \leq t\right\},
$$

and satisfies the condition

$$
\mathrm{P}\left\{\lim _{t \rightarrow \infty} \frac{S_{j, N}(t)}{t}=\mu_{j} N\right\}=1 .
$$

The relations between parameters $\lambda, p_{j}, \mu_{j}(j=1,2, \ldots, k)$ and $k$ are assumed to be
(1.1) $\frac{\lambda p_{j}}{\mu_{j}}<1$, for all $j=1,2, \ldots, k$,
and
(1.2) $\frac{\lambda}{\mu_{j}}>1$, at least for one of $j$ 's, $j=1,2, \ldots, k$.

In the sequel, asymmetric networks will be discussed for $k=2$, and the relations (1.1) and (1.2) will be assumed for this value of parameter $k$.

In the case of symmetric network, where $\mu_{j}=\mu$ for all $j$, and $p_{j}=p_{j}(l)=1 / l$, where $l$ is the number of available client stations (in a given time moment), conditions (1.1) and (1.2) correspondingly are as follows:

$$
\frac{\lambda}{k \mu}<1,
$$

and

$$
\begin{equation*}
\frac{\lambda}{\mu}>1 \tag{1.4}
\end{equation*}
$$

In this case, condition (1.4) means that after one or other breakdown the entire client stations become bottleneck, and there is a value $l_{0}=\max \left\{l: \frac{\lambda}{l \mu}>1\right\}$.

The queue-length in the $j$-th client station is defined as
$Q_{j, N}(t)=A_{j, N}(t)-\int_{0}^{t} \mathrm{I}\left\{Q_{j, N}(s-)>0\right\} d S_{j, N}(s)$.
In the case of $N$ large, the behavior of the queue-length process is as follows. When all of $k$ client stations are available in time $t$, most of units are concentrated at the server station, and normalized queue-lengths $q_{j, N}(t)=Q_{j, N}(t) / N$ vanish as $N$ increases indefinitely. When after one or another failure the client stations are
overloaded in time $t$, then $q_{j, N}(t)$ converge in probability, as $N$ increases indefinitely, to some positive value. Then the queue-lengths in client stations increase more and more as $t$ increases. Then the system is assumed to be at risk if the total number of units in queues in client stations increases the value $\alpha N$.

Confidence intervals for symmetric large client/server computer networks have been studied in Abramov [1]. The motivation of this problem, review of the related literature and technical details are given in [1]. The present paper mainly discusses confidence intervals for large asymmetric client/server computer networks and provides new representations for confidence intervals.

In the case of symmetric network, a confidence interval is characterized by parameter $\gamma \leq \alpha$. More specifically, for given level of probability $P$, say $P=0.95$, there is value $\gamma$ characterizing the guaranteed level of normalized cumulated queue, and a (random) confidence interval is associated with this value of $\gamma$. In other words, along with parameter $\alpha$ characterizing the system at risk we have another parameter $\gamma$, which is closely associated with $\alpha$ and with probability $P$. In the particular case $k=2$, the explicit representation for $\gamma$ has been established in [1].

In the case of asymmetric network such deterministic parameter cannot longer characterize confidence intervals. To see it, consider a network the parameters of which are: $\lambda=1, p_{1}=p_{2}=1 / 2, \mu_{1}=4 / 3, \mu_{2}=2 / 3$. Then, condition (1.1) is fulfilled, and $\frac{\lambda p_{1}}{\mu_{1}}=\frac{3}{8}<1, \frac{\lambda p_{2}}{\mu_{2}}=\frac{3}{4}<1$. Condition (1.2) is fulfilled as well, and $\frac{\lambda}{\mu_{1}}=\frac{3}{4}<1, \frac{\lambda}{\mu_{2}}=\frac{3}{2}>1$. In this case one can expect the situation when the second server breakdowns first, and the cumulative normalized queue-length process will converge to zero as $N \rightarrow \infty$ for all $t$, and there is no observable parameter. Therefore, the network can breakdown unexpectedly without any information on its state. In other example, where the parameters of network are: $\lambda=1, p_{1}=p_{2}=1 / 2, \mu_{1}=3 / 4, \mu_{2}=2 / 3$, we have the following situation. Condition (1.1) is fulfilled with $\frac{\lambda p_{1}}{\mu_{1}}=\frac{2}{3}<1, \frac{\lambda p_{2}}{\mu_{2}}=\frac{3}{4}<1$. Condition (1.2) is fulfilled with $\frac{\lambda}{\mu_{1}}=\frac{4}{3}>1$ and $\frac{\lambda}{\mu_{2}}=\frac{3}{2}>1$. Therefore in the case when the first client station breakdowns first, the limiting cumulative normalized queue-length will increase with the rate, different from that would be in the case when the second client station breakdowns first. Therefore, by following up the limiting cumulative normalized queue-length one cannot uniquely characterize a confidence interval as it has been done in the case of symmetric networks. For this reason we need in another representation for confidence intervals.

The rest of the paper is organized as follows. In Section 2 we recall main equations for limiting (as $N \rightarrow \infty$ ) cumulated normalized queue-length process from earlier paper [1], and derive slightly more general representation than in [1]. We then derive an explicit value for a confidence interval, which are closely related to the result, obtained in [1]. In Section 3, the results are derived for asymmetric networks in the case $k=2$. We conclude the paper in Section 4.

## 2. The case of symmetric network

Limiting as $N \rightarrow \infty$ cumulated normalized queue-length process is denoted $q(t)$. Let $l_{0}=\max \left\{l: \frac{\lambda}{l \mu}>1\right\}$, let $\tau_{1}, \tau_{2}, \ldots, \tau_{k}$ be the moments of breakdown of client stations, $0 \leq \tau_{1} \leq \tau_{2} \leq \ldots \leq \tau_{k}$. Then $q(t)=0$ for all $t \leq \tau_{k-l_{0}}$, and in any arbitrary time interval $\left[\tau_{i}, \tau_{i+1}\right), i=k-l_{0}, k-l_{0}+1, \ldots, k-1$, we have the equation

$$
\begin{aligned}
q(t)= & q\left(\tau_{i}\right)+\left[1-q\left(\tau_{i}\right)\right]\left\{\left(\left[1-q\left(\tau_{i}\right)\right] \lambda-\mu(k-i)\right)\left(t-\tau_{i}\right)\right. \\
& \left.-\left[1-q\left(\tau_{i}\right)\right] \lambda \int_{\tau_{i}}^{t} r\left(s-\tau_{i}\right) d s\right\},
\end{aligned}
$$

where
$r(t)=\left(1-\frac{\mu(k-i)}{\left[1-q\left(\tau_{i}\right)\right] \lambda}\right)\left(1-e^{-\left[1-q\left(\tau_{i}\right)\right] t}\right)$.

In the last endpoint $\tau_{k}$ we set $q(t)=1$. In the case of $k=2$ the confidence time interval is the sum of two intervals. The first interval is [0, $\tau_{1}$ ). The second one is [ $\tau_{1}, \theta$ ), where the endpoint $\theta$ is defined as follows. In [ $\tau_{1}, \tau_{2}$ ) for $q(t)$ we have the equation:
$q(t)=(\lambda-\mu)\left(t-\tau_{1}\right)-\lambda \int_{\tau_{1}}^{t} r\left(s-\tau_{1}\right) d s$,
where
$r(s)=\left(1-\frac{\mu}{\lambda}\right)\left(1-e^{-\lambda t}\right)$.
To find $\theta$ we have the following equations:
$\mathrm{P}\{q(t)=0\}=[1-G(t)]^{2}$,
$\mathrm{P}\{q(t) \leq \gamma<1\}=[1-G(t)]\left[1-G\left(t-t_{\gamma}\right)\right]$,
where $G(x)$ denotes lifetime distribution of each of identical client stations and $t_{\gamma}$ is such the value of $t$ under which
$(\lambda-\mu) t-\lambda \int_{0}^{t} r(s) d s=\gamma$.

The value of $t_{\gamma}$ can be found from the relation

$$
\frac{\int_{0}^{\infty}[1-G(t)]\left[1-G\left(t-t_{\gamma}\right)\right] d t}{\int_{0}^{\infty}\left[1-G\left(t-t_{\gamma}\right)\right]^{2} d t}=P .
$$

If the corresponding value of $\gamma$ is not greater than $\alpha$, then the value $\theta$ of the interval $\left[\tau_{1}, \theta\right)$ should be taken $\theta=\tau_{1}+t_{\gamma}$. Otherwise, if $\gamma>\alpha$, the value $\theta$ should be taken $\theta=\tau_{1}+t_{\alpha}$.

The above result has been obtained in [1]. Let us now extend this result for a more general situation of an arbitrary number of client stations $k \geq 2$ under the special setting assuming that $l_{0}=1$. This means that $q(t)=0$ in the random interval $\left[0, \tau_{k-1}\right)$, and $q(t)>0$ in the interval $\left(\tau_{k-1}, \tau_{k}\right)$. In this case we have the following relationships:

$$
\begin{aligned}
& \mathrm{P}\{q(t)=0\}=\sum_{i=2}^{k}\binom{k}{i}[1-G(t)]^{i}[G(t)]^{k-i}, \\
& \mathrm{P}\{q(t) \leq \gamma<1\}=[1-G(t)] \sum_{i=1}^{k-1}\binom{k-1}{i}\left[1-G\left(t-t_{\gamma}\right)\right]^{i}\left[G\left(t-t_{\gamma}\right)\right]^{k-i},
\end{aligned}
$$

where the value of $t_{\gamma}$ can be found from the relation

$$
\frac{\int_{0}^{\infty}[1-G(t)] \sum_{i=1}^{k-1}\binom{k-1}{i}\left[1-G\left(t-t_{\gamma}\right)\right]^{i}\left[G\left(t-t_{\gamma}\right)\right]^{k-i} d t}{\int_{0}^{\infty} \sum_{i=2}^{k}\binom{k}{i}\left[1-G\left(t-t_{\gamma}\right)\right]^{i}\left[G\left(t-t_{\gamma}\right)\right]^{k-i} d t}=P .
$$

Again, if the corresponding value of $\gamma$ is not greater than $\alpha$, then the value $\theta$ of the interval $\left[\tau_{k-1}, \theta\right)$ should be taken $\theta=\tau_{k-1}+t_{\gamma}$. Otherwise, if $\gamma>\alpha$, the value $\theta$ should be taken $\theta=\tau_{k-1}+t_{\alpha}$.

The above construction gives us a random confidence interval $[0, \theta)$ corresponding to the level of probability not smaller than $P$. We now find a deterministic confidence interval corresponding to the level of probability not smaller than $P$. That deterministic confidence interval will be a guaranteed interval, and the probability that the system will be available is not smaller than $P$.

We have
$\mathrm{P}\{\theta>t\}=\mathrm{P}\left\{\tau_{1}+t_{\gamma}>t\right\}=\mathrm{P}\left\{\tau_{1}>t-t_{\gamma}\right\}=\left[1-G\left(t-t_{\gamma}\right)\right]^{2}=P$.

Therefore, the desired deterministic interval is $\left[0, z+t_{\gamma}\right]$, where the value $z$ is given from the condition $[1-G(z)]^{2}=P$.

Then the construction of deterministic interval in the case $k=2$ is as follows.

- According to the aforementioned relations we find the value of interval $t_{\gamma}$. If the corresponding value of $\gamma$ is not greater than $\alpha$, then we accept this interval and set $\mathrm{T}:=t_{\gamma}$.
- Otherwise, we set $\mathrm{T}:=t_{\alpha}$, where the value $t_{\alpha}$ is determined from the relation $(\lambda-\mu) t-\lambda \int_{0}^{t} r(s) d s=\alpha$, and $r(s)=\left(1-\frac{\mu}{\lambda}\right)\left(1-e^{-\lambda t}\right)$.
- We find the value $t$ from the relation $[1-G(t)]^{2}=P$.
- The confidence interval is then taken as $[0, t+\mathrm{T}]$.


## 3. The case of asymmetric network

The case of asymmetric network is similar to that of symmetric network. It is based on the formula for the total probability. Specifically, in the case $k=2$ we are to study the cases as (1) the first client station breakdowns first and (2) the second client station breakdowns first. Using the notation, $G_{i}(x)=\mathrm{P}\left\{\chi_{i} \leq x\right\}$, we have $\mathrm{P}\left\{\chi_{1} \leq \chi_{2}\right\}=\int_{0}^{\infty}\left[1-G_{2}(x)\right] d G_{1}(x)$. Next, we have the following two values $\gamma_{1}$ and $\gamma_{2}$, such that
$\mathrm{P}\left\{q(t) \leq \gamma_{1}<1 \mid \chi_{1} \leq \chi_{2}\right\}=\left[1-G_{2}(t)\right]\left[1-G_{1}\left(t-t_{\gamma_{1}}\right)\right]$,
$\mathrm{P}\left\{q(t) \leq \gamma_{2}<1 \mid \chi_{2} \leq \chi_{1}\right\}=\left[1-G_{1}(t)\right]\left[1-G_{2}\left(t-t_{\gamma_{2}}\right)\right]$,
and the corresponding values of $t_{\gamma_{1}}$ and $t_{\gamma_{2}}$ are found from the relationships

$$
\frac{\int_{0}^{\infty}\left[1-G_{2}(t)\right]\left[1-G_{1}\left(t-t_{\gamma_{1}}\right)\right] d t}{\int_{0}^{\infty}\left[1-G_{1}\left(t-t_{\gamma_{1}}\right)\right]\left[1-G_{2}\left(t-t_{\gamma_{1}}\right)\right] d t}=P,
$$

$$
\frac{\int_{0}^{\infty}\left[1-G_{1}(t)\right]\left[1-G_{2}\left(t-t_{\gamma_{2}}\right)\right] d t}{\int_{0}^{\infty}\left[1-G_{1}\left(t-t_{\gamma_{2}}\right)\right]\left[1-G_{2}\left(t-t_{\gamma_{2}}\right)\right] d t}=P .
$$

where in each case, if $\gamma_{1}$ or $\gamma_{2}$ is greater than $\alpha$, then the corresponding value is replaced by $\alpha$, and then the corresponding value of $t_{\gamma_{1}}$ or $t_{\gamma_{2}}$ is to be replaced by $t_{\alpha}$ as well.

Similarly to the case of symmetric network in this case we have the following.

- We find the value of intervals $t_{\gamma_{1}}$ and $t_{\gamma_{2}}$. If the corresponding value of $\gamma_{1}$ or $\gamma_{2}$ is not greater than $\alpha$, then we accept this interval and set $\mathrm{T}_{1}:=t_{\gamma_{1}}$ or $\mathrm{T}_{2}:=t_{\gamma_{2}}$
- Otherwise, we set $\mathrm{T}_{1}:=t_{\alpha}$ or $\mathrm{T}_{2}:=t_{\alpha}$, where the value $t_{\alpha}$ is determined from the relation $(\lambda-\mu) t-\lambda \int_{0}^{t} r(s) d s=\alpha$, and $r(s)=\left(1-\frac{\mu}{\lambda}\right)\left(1-e^{-\lambda t}\right)$.
- Using the formula for the total expectation we find $\mathrm{T}=\mathrm{T}_{1} \int_{0}^{\infty}\left[1-G_{2}(x)\right] d G_{1}(x)+\mathrm{T}_{2} \int_{0}^{\infty} G_{2}(x) d G_{1}(x)$.
- We find value $t$ from the relation $\left[1-G_{1}(t)\right]\left[1-G_{2}(t)\right]=P$.
- The confidence interval is then taken as $[0, t+\mathrm{T}]$.


## 4. Concluding remark

In the present paper we established confidence intervals for large closed client/server computer networks with two client stations. Unlike the earlier result established in [1] for symmetric network, the confidence intervals are deterministic. The advantage of the result of [1] is that one can judge about the quality of system from the information on the system state. However, approach of [1] is not longer available for asymmetric systems. The advantage of the results of the present paper is that they provide confidence intervals for both symmetric and asymmetric networks that give us the entire lifetime of data system in the network with probability not smaller than $P$.

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# AGING AND LONGEVITY CONTROL OF BIOLOGICAL SYSTEMS VIA DRUGS - A RELIABILITY MODEL 

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#### Abstract

The treatments in bio-systems correspond to respective repairs known in reliability. Some treatments may make the biological objects younger; others may make them older, or not deteriorate their current age. Such kind of "maintenance" has some analogous failure/repair models in reliability. We use it to incorporate some results of reliability and bio modeling for the quantitative studies of the aging and resistance of bio-systems to environmental stress factors. We call "calendar age" the age of a bio-object which does not use treatments, or uses it without age improvement, or deterioration. All bio-objects, which are using treatments of same strength and direction of effect, have "virtual age". We explain here what the virtual age is, and how is it related to age correcting factors. We illustrate our common results about the virtual ages on the example of the GompertzMakenham law of mortality, and discuss the relations of the longevity, mechanism of aging and age affecting control. As a consequence, a concept of age determination is proposed. Numeric and graphical examples are provided.


## 1. Introduction

This study has been induced by the number of presentations and articles presented at the First FrenchRussian Conference on Longevity, Aging and Degradation Models in Reliability, Public Health, Medicine and Biology (LAD'2004), hosted by the Saint Petersburg State Polytechnic University, Russia in June 5-11, 2004 [1], [5], [6]. It has been a demonstration that probability and statistical methods cover an enormous ground of research and can successfully handle most of situations involving uncertainty in any area of human activity. One of these is the study of biological systems, particular case of which is every kind of live organisms.

### 1.1 Bio-Systems as Devices

Bio-systems can be considered as a specifically organized devices constructed to perform some preset functions, according to their genetic programs. These functions are performed in the presence of a great number of random factors (environmental conditions). Following Koltover [5], we may schematically consider any operation bio-system as a multi-dimensional time-dependent random vector $\mathbf{Y}(t)=\left(Y_{1}(t), Y_{2}(t), \ldots\right)$ each component of which corresponds to a relevant functional parameter of the device. There exists a relevant subset $\boldsymbol{S}$ of points in the space where $\boldsymbol{Y}(t)$ belongs which corresponds to the feasible (admissible) values (limits) of the functional parameters. If $\boldsymbol{Y}(t) \in \boldsymbol{S}$, then the device (bio-system) is defined as having normal operation at the time $t$. Whenever $\boldsymbol{Y}(t)$ passes beyond the limits of $\boldsymbol{S}$, then the device gets a failure. Sometimes $\boldsymbol{S}$ may also depend on time, or may be a random set. It is assumed $\boldsymbol{Y}(0) \in \boldsymbol{S}$. Life time of the device as a whole is defined by the random variable

$$
\begin{equation*}
\tau=\max \{t: t \geq 0, \mathrm{Y}(t) \in S\} . \tag{1}
\end{equation*}
$$

It represents the time of non-failure operation of the bio-system. The life time distribution is presented by the function

$$
\begin{equation*}
F(t)=P(\tau \leq t) \tag{2}
\end{equation*}
$$

### 1.2 Reliability of Bio-Systems

Reliability of the bio-system is the probability of non-failure operation within the interval of time $(0, t)$, i.e.

$$
\begin{equation*}
R(t)=P(\tau>t) \tag{3}
\end{equation*}
$$

For bio-systems it is known as survival function, and denoted by $S(t)$
Consequently, all the statistical procedures used in reliability theory can be used in evaluation of the reliability function $R(t)$ from many independent copies, $N$, of independently operating bio-systems as the ratio $N$ $(t) / N$. Here $N(t)$ is the number of those bio-systems which are alive (do normally operate) at the instant $t$. Also, the mortality rate function

$$
\begin{equation*}
\lambda(t)=-\frac{d}{d t} \ln R(t)=-\frac{R^{\prime}(t)}{R(t)}=\frac{f(t)}{1-F(t)} \tag{4}
\end{equation*}
$$

appears as an analogue to the failure rate function, used in technical reliability. The $f(t)$ here is probability density function of the life time distribution. The time $t$ just survived by a bio-system is called its calendar age. Therefore, the same mathematical theory of reliability is essentially applicable to the mathematics of mortality. Our article is a step in this direction.

## 2. Main Models and Results

The specification of components of the random vector $\boldsymbol{Y}(t)$ for bio-systems, the use of random modeling and analysis helps to understand how the improvement of its reliability can be attained, and how to keep a better control on the survival of such systems. There are lots of analogies as well a number of specific differences in modeling and studies of reliability of technical devices and for the bio-systems. For instance, bio-systems are obviously subject of wearing and aging. Bio-systems have a proven life-span (something like a maximal value of the life time $\tau$ beyond which no copy of the bio-system can pass). Life span for people is, for instance, 120 years. Life spans have also most of the functional components of the bio-systems. Life span for people's brain is 250 years. For technical devices the exponential, the Weibull, the Gamma, and even the Norman distributions frequently fit for modeling the life times. For the bio-systems, despite of their complexity, there exists some "universal kinetics of the growth of mortality with the age", expressed by the Gompertz-Makenham law of mortality

$$
\begin{equation*}
\lambda(t)=\beta+\alpha e^{\gamma t} . \tag{5}
\end{equation*}
$$

Here the parameters $\alpha, \beta$, and $\gamma>0$ are independent on time. The Gompertz - Makenham mortality law has been confirmed for people and for other mammals, flies, mollusks [5] with specific values of its parameters. From [5] we find that for people parameter $\beta \neq 0$ if the age is less than 35 years, and $\beta=0$ if the age is greater than 35 . We treat this parameter $\beta$ as a collaterals mortality rate (e.g. accidentals, casualties), and guess that its numerical value may vary for different countries and species. For our numeric and graphic examples later we take the value of $\beta=$ .0025. Values of parameters $\alpha \approx 42.827 \pm 8.85$ years, and $\gamma \approx .094 \pm .0014$ years $^{-1}$ are assumed (evaluated) for ages above 35 and less than 95 , according to [5].

We consider here the effects of drug use that may slow down, or accelerate the aging for people in a proportional fashion as it is modeled and used for technical items in [2] and [3]. Then we study graphically the
effects on mortality rates and the life span on people as functions of the age reducing, or age accelerating effects depending on the dosage of medication or treatment. Numerical examples are using values seeming reasonable for the people.

## 3. The drug use effects

The most convenient description should be given in terms of the mortality rate function $\lambda(t)$ and the related to it hazard function $\Lambda(t)=\int_{0}^{t} \lambda(u) d u$. There is a convenient relationship between the probability characteristics of the original lifetime $X$ of an individual and these functions.

Let the initial life time, $X$, be a continuous random variable (r.v.) with c.d.f.
$F(x)=P(X \leq x)$, and have a p.d.f. $f(x)=\frac{d}{d x} F(x)$. Then its hazard function is

$$
\begin{equation*}
\Lambda(t)=-\ln [1-F(t)] \text { for } t \geq 0 \tag{6}
\end{equation*}
$$

and its mortality rate function is

$$
\begin{equation*}
\lambda(t)=\frac{d}{d t} \Lambda(t)=\frac{f(t)}{1-F(t)} \tag{7}
\end{equation*}
$$

In reliability works is shown that the temporary failures which do not affect the failure rate after recovery (known as minimal repairs), have Poisson distribution with mean $\Lambda(u+v)-\Lambda(u)$ for their total number within any time interval $[u, u+v), u, v>0$.

Drug use activities may improve the performance of the individuals and give them a "new life". The specifics of the drugs, its intake amount of labor, recovery time, or money invested in the health care may have significant impact on the health improvement, which directly affects the longevity of life. If assume that health improvement prolongs the life of such individuals by certain percentage $\delta$, we call it an age- reducing factor. Fig. 1 a) and the model (8) below explain how it happens with actions made on the individual mortality rate.


Fig. 1 a) Individual mortality rate under age-reducing factor


Fig. 1 b) Individual mortality rate under age-accelerating factor

Fig. 1. Changes in the individual mortality rates in time with an age affecting factor
We consider also drug abuses (analogous to the reliability maintenance under imperfect repairs) that may shorten the life. Such actions affect the future performance of the individual, and are related to an age-accelerating factor $\delta$, which is equivalent to reducing the overall life of such individuals by certain percentage $\delta$. Fig. 1 b ) and the model and the theorem below explain how it happens changes in the individual mortality rate.

Let $X_{i}$ denote time intervals between successive epochs of drugs intake that affect the individual. Assume that $\delta_{i}$ denote the lack of perfection the life system of an individual may get as a result of the $i$-th action. The values

$$
\begin{equation*}
T_{0}, T_{i}=T_{i-1}+X_{i} \delta_{i}, \quad I=1,2,3, \ldots \tag{8}
\end{equation*}
$$

are understood as virtual ages of the individual right after the $i$-th action. When $\delta_{i}=1$, then no improvement or deterioration of the virtual age of the individual occurs at the $i$-th epoch of action. When $\delta_{i}<1$ (or if $\delta_{i}>1$ ), then an improvement (or a deterioration) of the virtual age of the individual occurs at that epoch.

The model described here is also known in Reliability as Kijima's model II [4].
We consider this model with the assumption that $\delta_{i}=\delta \neq 0$, and call this $\delta$ an age-correcting factor. If $\delta<$ 1, we call it age-reducing factor, and if $\delta>1$, we call age- accelerating factor.

Assume instantaneous effects, continuous non-decreasing virtual hazard function $\Lambda^{*}(t)$ as a function of the time parameter $t$, and having right derivative $\lambda^{*}(t)=\left.\frac{d}{d t} \Lambda^{*}(t)\right|_{t+0}$. The subscript here indicates that the value of $\Lambda^{*}(t)$ is considered immediately after an occasional age-reducing action is completed.

Consider the sequence $T_{0}<T_{1}<T_{2}<\ldots<T_{n}<\ldots$ of times representing the virtual product age after the $n$-th coincident action. Assume that X has a c.d.f. $F(x)$ with $F(0)=0$ and $F(x)<1$ for all $x>0$. Denote the survival function by $\bar{F}(t)=1-F(t)=P\{X>t\}$.
We derive the following expressions.
The $n$-th step transition probability function is

$$
\begin{equation*}
P\left\{T_{n+1}>t \mid T_{1}, \cdots, T_{n}\right\}=P\left\{X>\frac{t}{\delta} \left\lvert\, X>\frac{T_{n}}{\delta}\right.\right\}=\frac{\bar{F}\left(\max \left[\frac{T_{n}}{\delta}, \frac{t}{\delta}\right]\right)}{F\left(\frac{T_{n}}{\delta}\right)}, \tag{9}
\end{equation*}
$$

for $n=1,2, \cdots$. The initial distribution is

$$
\begin{equation*}
P\left\{T_{1}>t\right\}=P\left\{X_{1}>\frac{t}{\delta}\right\}=\bar{F}\left(\frac{t}{\delta}\right) . \tag{10}
\end{equation*}
$$

From (9) and (10), by induction, we get that for any non-negative measurable function $g\left(t_{1}, \cdots, t_{n}\right)$ and for any $n$ it is true that:
$\boldsymbol{E}\left[g\left(T_{1}, \cdots, T_{n}\right)\right]=$

$$
\begin{equation*}
\int_{0}^{\infty}\left(\bar{F}\left(\frac{t_{1}}{\delta}\right)\right)^{-1} \int_{t_{1}}^{\infty}\left(\bar{F}\left(\frac{t_{2}}{\delta}\right)\right)^{-1} \int_{t_{2}}^{\infty}\left(\bar{F}\left(\frac{t_{3}}{\delta}\right)\right)^{-1} \cdots \int_{t_{n-2}}^{\infty}\left(\bar{F}\left(\frac{t_{n-1}}{\delta}\right)\right)^{-1} \int_{t_{n-1}}^{\infty} g\left(t_{1}, \cdots, t_{n}\right) d F\left(\frac{t_{n}}{\delta}\right) \cdots d F\left(\frac{t_{1}}{\delta}\right) . \tag{11}
\end{equation*}
$$

Let $\left\{N_{t}^{v}, t \geq 0\right\}$ be the counting process corresponding to the point process $\left\{T_{n}\right\}_{n=0}^{\infty}$ defined by

$$
N_{t}^{v}=\sum_{n=0}^{\infty} I_{[0, t)}\left(T_{n}\right),
$$

where $I_{B}(\cdot)$ is the indicator function of the set $B$.

Theorem 1: The random process $\left\{N_{t}^{v}, t \geq 0\right\}$ is a non-homogeneous Poisson process with a leading function

$$
\begin{equation*}
\Lambda^{v}(t)=-\log \left[1-F\left(\frac{t}{\delta}\right)\right]=\Lambda\left(\frac{t}{\delta}\right) \tag{12}
\end{equation*}
$$

where $\Lambda(t)=-\ln [1-F(t)]$ is the leading function of the NPP associated to the life time $X$ of this individual.
The proof can be found in [3].
Equation (12) shows that the transformation between the calendar and the virtual time scales is $t \rightarrow \frac{t^{v}}{\delta}$, i.e., if the virtual age of the individual is $t^{\nu}$ its corresponding calendar age $t$ is $t^{\nu} / \delta$. Therefore, we may expect that when the calendar age of an individual acting under age correcting factor $\delta$ is $t$, then its virtual (we would say, actual) age is $\delta t$.

Denote by $T$ the r.v. representing the virtual lifetime of the individual. The c.d.f. of $T$ is $F_{T}(t)=1-e^{-\Lambda^{\nu}(t)}$. Equation (12) also shows that $P(T \leq t)=F_{T}(t)=F_{X}\left(\frac{t}{\delta}\right)=P\left(X \leq \frac{t}{\delta}\right)$. Therefore, $P(T \leq t)=P(\delta X \leq t)$, and this means that the virtual lifetime $\boldsymbol{T}$ and the multiplied by $\delta$ calendar lifetime $X$ are equal in distribution, i.e.

$$
\begin{equation*}
T=^{d} \delta X \tag{13}
\end{equation*}
$$

When the individual is at calendar age $x$ its virtual age measured at the calendar age scale is $\delta x$. At calendar age $x$ an individual maintaining himself under age-correcting medication of factor $\delta$, lives as a new individual at age $\delta x$. Thus

$$
\lambda^{*}(x) d x=\lambda(\delta x) d x
$$

i.e., the probability to have a failure of the individual from the original population within the interval $[x, x+d x)$ is the same as the probability to have a failure from the population of individuals, maintained by age-affecting actions of factor $\delta$, within the interval $[x, x+\delta d x)$.

The relation

$$
\Lambda^{*}(x)=\int_{0}^{x} \lambda^{*}(u) d u, \text { and } \Lambda(x)=\int_{0}^{x} \lambda(u) d u,
$$

leads to

$$
\begin{equation*}
\Lambda^{*}(x)=\frac{1}{\delta} \Lambda(\delta x) \tag{14}
\end{equation*}
$$

Theorem 2: The virtual failure rate $\lambda^{*}(x)$ at calendar age $x$, and the original failure rate are related by the equality

$$
\begin{equation*}
\lambda^{*}(x)=\lambda(\delta x), x \geq 0, \delta \neq 0 \tag{15}
\end{equation*}
$$

The virtual hazard rate $\Lambda^{*}(x)$ and the original hazard rate $\Lambda(x)$ are related by equation (14).
An age-reducing factor $\delta$ slows down the aging process of the individual by $100(1-\delta) \%$.
Example 1 The Gompertz-Makenham life-time distribution with an age-affecting factor.
Consider the Gompertz-Makenham life-time distribution, $X \in \operatorname{Gompertz}(\lambda(t, \beta, \alpha, \gamma))$, which is proven to fit the cells and most mammal's life [5]. From relationships (10) - (12) we get

$$
\begin{equation*}
F(t)=P\left\{T_{1} \leq t\right\}=1-e^{-\int_{0}^{t}\left(\beta+\alpha e^{r u}\right) d u}=1-e^{-\beta t-(\alpha / \gamma)\left(1-e^{r t}\right)}, \tag{16}
\end{equation*}
$$

and

$$
f(t)=\left(\beta+\alpha e^{\gamma t}\right) e^{-\beta t-(\alpha / \gamma)\left(1-e^{-t}\right)}
$$

with $\lambda(t)$ given by equation (10), and

$$
\Lambda(\mathrm{t})=\beta t+\frac{\alpha}{\gamma}\left(1-e^{\gamma t}\right), \mathrm{t} \geq 0
$$

Here $\beta$ is a constant rate parameter, $\alpha$ is a secondary time-scaling parameter, and $\gamma$ is an aging rate parameter.

If the individual has the Gompertz-Makenham life-time distribution with parameters $\beta$, $\alpha$, and $\gamma$, and is maintained under age-affecting factor $\delta>0$, then its virtual failure rate and virtual hazard rates are given by the equations (14), (15), namely

$$
\begin{equation*}
\lambda^{*}(t)=\beta+\alpha e^{\gamma \delta t}, \mathrm{t}>0, \delta \neq 0 \tag{17}
\end{equation*}
$$

and

$$
\begin{equation*}
\Lambda^{*}(t)=\beta t+\frac{\alpha}{\delta \gamma}\left(1-e^{\gamma \delta t}\right), \mathrm{t} \geq 0 \tag{18}
\end{equation*}
$$

Figure 2 illustrates the behavior of the two functions $\lambda^{*}(t)$ and $\Lambda^{*}(t)$ under various values of the ageaffecting parameter $\delta$. For values of the parameters $\alpha, \beta$, and $\gamma$ are taken the numbers: $\beta=.0025$ when $t \leq 35$, and $\beta=$ 0 for $t \geq 0$. For both cases $\alpha=42.827$, and $\gamma=.094$ as proven to be valid for the human beings with an age between 35 years and 94 years, according to [5]. And respective graphs also are for the ages between 35 and 120 years. For ages between 0 and 35 we assume $\beta=.0025$


Fig. 2 a. Mortality rates $\lambda^{*}(t)$ for various $\delta$.


Fig. 2.b.1. Age below 35 years


Fig. 2.b.2. Age above 35 years

Fig. 2.b. Integrated mortality rate (hazard rate, integrated risk) $\Lambda^{*}(t)$ various $\delta$.
Fig. 2. Mortality Rates and Integrated Risks in the calendar ages

## 4. Comparative ages between different groups

Now we illustrate one possible use of the results obtained in Section 3 by considering again the GompertzMakenham life-time distribution. We will think in terms of the human beings as belonging to various groups (or populations) determined by the values of the age-correcting factor $\delta$. As we noticed before, even being at a same calendar age $t$ the individuals from different populations would have different virtual (or as we say, actual) age. Since the only convenient time scale is the calendar age, it makes sense to speak about comparable or equivalent ages between the individuals from different groups. Using the relationships (6), (16) and (18) we find the life-time distribution function for each population determined by the value $\delta$ of its age-correcting factor:

$$
\begin{equation*}
F_{\delta}(t)=1-e^{-\beta_{t}-\frac{\alpha}{\delta \gamma}\left(1-e^{\delta_{\gamma}}\right)} . \tag{19}
\end{equation*}
$$

Equation (19) represents the probability that an arbitrary selected individual from the population with agecorrecting factor $\delta$ will not survive the calendar age $t$. The function

$$
\begin{equation*}
S_{\delta}(t)=1-F_{\delta}(t)=e^{-\beta t-\frac{\alpha}{\delta \gamma}\left(1-e^{\delta t}\right)} \tag{20}
\end{equation*}
$$

is known as the survival function for the individuals in this population. Its meaning is the same as for reliability expressed by equations (1)-(3). We abandon the notation $R(t)$ and leave it for cases of technical issues.

Definition: We say the age $T_{\delta_{1}}$ of the individual from the population with age-correcting factor $\delta_{1}$ is equivalent to the age $T_{\delta_{2}}$ of the individual from the population with age-correcting factor $\delta_{2}$, if it is fulfilled

$$
\begin{equation*}
S_{\delta_{1}}\left(T_{\delta_{1}}\right)=S_{\delta_{2}}\left(T_{\delta_{2}}\right) \tag{21}
\end{equation*}
$$

In the sense of this Definition, every age of one of the two populations has equivalent comparable age to the other population. Ages are equivalent when the probabilities to survive these respective values $T_{\delta_{1}}$ and $T_{\delta_{2}}$ are equal. When we pick $\delta=1$, we see to what calendar age of the normal human population and individual from the
population with age-correcting factor $\delta$ will be equivalent. Since this is the only available age information, when we know $\delta$ we may see what the true age of an individual from the respective population.

However, we notice that equations (21) may be quantified, since these are probabilities. If we select any probability level $p \in(0,1)$, we will get all the equivalent ages $T_{\delta}(\mathrm{p})$ at this level for all the populations just as solutions of the equations

$$
S_{\delta}\left(T_{\delta}\right)=p
$$



Fig. 3. The survival functions and equivalent ages for the individuals from the population with age-correcting factor $\delta$.

Fig. 3 illustrates the equivalency between the ages of several $\delta$-populations at the survival probability levels $p=.9$ and $p=.7$. The continuous black line traces the "normal population" where $\delta=1$. The meaning of the numbers is following: At level .9 the age of the normal population is, say, 33. The age-accelerated population with $\delta=1.5$ will look like this 33 years old people already being at age 28. Similarly, looking on the graph for the equivalent ages at probability level .7 , we notice that the end of life for average individual from the normal population is about 95 years, while for these from the age-accelerated population with $\delta=1.5$ it will be about 64 .

We evaluated the equivalent ages for several probability levels, and the results are shown on next table. In bold black digits are shown the respective ages for the normal population at the corresponding level of survival. In the same line of the level are shown the ages in the other populations, at which an average individual would look (have the age) like this in the normal population. One may see some unreal numbers which we also need to comment. For instance, at level .25 individuals in the normal population have properties as 75 years old, while same properties would be in possession by the average individuals at 82 years age if they reduce their ages regularly by a factor $\delta=.9$ (an improvement by $10 \%$ compare to the normal). In the same line we see that the same properties would have the 114 year old people in the population who got $40 \%$ improvements regularly. Finally, we see that as a 75 year old normal individual would have been an individual at age 197 if it was possible to reduce the age regularly by $70 \%$. Numbers in the last column are somewhat unreal, because they represent a mystic dream for such high level of age reduction.

Table 4.1. Equivalent ages under various age correcting factors at different levels of survival.

| $\mathbf{P} \mathbf{\delta}$ | 1.5 | 1.25 | 1.1 | $\mathbf{1}$ | .9 | .6 | .3 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| .9 | 28 | 30 | 31 | $\mathbf{3 3}$ | 34 | 37 | 39 |
| .7 | 42 | 45 | 50 | $\mathbf{5 8}$ | 61 | 80 | 112 |
| .5 | 48 | 55 | 61 | $\mathbf{6 6}$ | 72 | 98 | 158 |
| .25 | 53 | 62 | 69 | $\mathbf{7 5}$ | 82 | 114 | 197 |
| .1 | 57 | 66 | 75 | $\mathbf{8 1}$ | 89 | 125 | 220 |



Fig. 4. Equivalent ages for various values of age correcting factor and various values of the age-correcting factor.

On Fig. 4 we show the same numbers in the table as a graph of the equivalent ages that can be approximate to get equivalent ages at any survival level. This graph could be used as a chart for a number of useful comparisons. The black middle line shows the calendar age of the normal population. If one chooses an age, let say, 50 within his population, and draws a vertical line at the heights of the points of intersection with the other graphs one may find at what calendar age the individuals from other populations are, which is equivalent to his. Taking an individual from the normal population at any age in the same way it can be shown to what age he would have been if he was using age corrections as do the individuals from the other populations.

We believe that there is a fresh idea about how to treat age related questions, how to compare ages among different populations, and even, how to work out an approach towards the accelerated life testing based on the above considerations. And this discussion we postpone for another article.

## 5. Life supporting (insurance) costs associated with an individual during some assigned time

Certainly, there are some costs associated with any improvement or deterioration. Denote by $C_{r}(u, \delta)$ the cost of an age-affecting action of factor $\delta$ at calendar age $u$ of the individual. A natural assumption is that $C_{r}(u, \delta)$ satisfies the inequalities

$$
C_{m}(u) \leq C_{r}(u, \delta) \leq C_{c}(u),
$$

where $C_{m}(u)$ is the cost for the minimal support of the individual at calendar age $u$, and $C_{c}(u)$ is the cost of the complete (radical) loss of the individual at age $u$. In the sequel we develop a radical modeling for comparing of the expected expenses associated with the life support of individuals under regular life-supporting "maintenance", and these with age-affecting factors. As a matter of fact, the comparison of the virtual ages of individuals at the same calendar age $T$ indicates that those who have used an age affecting action of factor $\delta$ differ from what this calendar age is. These are based on comparing the values of the survival functions given by equations (20), Table 4.1 and Fig. 4 shown in the previous section. Struggle to keeping a constant age-correcting factor we call maintenance policy for the life style. Hence, the life supporting costs depend on the insurance agreement and the maintenance policy.

We assume that an individual is covered by insurance for some calendar time of duration $T$, according to a certain policy agreement.

Consider cost modeling from insurer's point of view. The insurer covers all, or a portion of the expenses associated with the needs and supports of the individual starting at age $t_{0}$, until the expiration of the insurance coverage, or until the death of this individual. In occasion of a death at age $u$ the insurer pays the amount $C(u, \delta)$, if the case is within the time of the action of the insurance agreement.

### 5.1. Expected insurance costs for a policy with limited validity

Consider an insurance policy, which starts at age $t_{0}$ of an individual, and will last time of duration $W$, i.e. the insurance policy is valid during the calendar age $\left[t_{0}, t_{0}+W\right]$ of that individual. On the other hand this individual has been maintained, and will maintain his life under an age-affecting factor $\delta$. Assume, that the original life time $X$ of the population at this site has failure rate function $\lambda(t)$. During the coverage no partial claims are possible.

The effect of the initial age $t_{0}$ of the individual at the start of the insurance coverage is also a parameter of interest, which may affect the expected insurer's expenses. The collected premiums usually are proportional to the elapsed time, $u$, and may depend on the initial age $t_{0}$ when the policy starts, and the supposed age-affecting factor $\delta$. Therefore, the collected premium on the interval $\left[t_{0}, t_{0}+\mathrm{u}\right.$ ) equals $C\left(t_{0}, \delta\right) u$. If the failure is beyond the assigned insurance period, the insurer incurs no expenses. If the death occurs at a moment on the interval $\left[t_{0}, t_{0}+\mathrm{u}\right)$, the insurer refunds the insured by the amount $C(u, \delta)$, thus his expenses are determined according to

$$
C(u)=C(u, \delta)-C\left(t_{0}, \delta\right) u .
$$

Particular forms of these functions are assumed. Most common seems

$$
\begin{equation*}
C(u, \delta)=C=\text { const }, \text { and } C\left(t_{0}, \delta\right)=c_{0}+c_{1} t_{0}+c_{2}(\delta-1), \tag{19}
\end{equation*}
$$

so that if the individual maintains the regular way of life he pays no addition or gets no discount expressed by the third component.

Lemma 1 The expected insurance cost associated with an age correcting factor $\delta$ for an individual insured at age $t_{0}$ with coverage of duration $W$ is given by

$$
\begin{equation*}
C_{W}\left(t_{0}, \delta\right)=\int_{0}^{W}\left\{\mathrm{C}\left(\mathrm{t}_{0}+\mathrm{u}, \delta\right)-\mathrm{C}\left(\mathrm{t}_{0}, \delta\right) \mathrm{u}\right\} \lambda\left(\delta\left(\mathrm{t}_{0}+\mathrm{u}\right)\right) \mathrm{e}^{-\left[\frac{1}{\delta} \Lambda\left(\delta\left(\mathrm{t}_{0}+u\right)\right)-\Lambda\left(\delta \mathrm{t}_{0}\right)\right]} d u \tag{20}
\end{equation*}
$$

where $\lambda(t)$ and $\Lambda(t)$ are the original failure rate and hazard rate functions associated with the life time $X$ of an individual in the population.

Example 2 The Gompertz-Makenham life-time distribution with an age-affecting factor
Assume, that the contract prices $C(u, \delta)$ and $C\left(t_{0}, \delta\right)$ for an individual insured at age $t_{0}$, are given by the equations (19). Then as a Corollary of Lemma 1 and the previous discussion we get:

Corollary 1. Under the conditions of Lemma 1, and (17)-(19), the expected incurred insurance costs are given by the expression
$\mathrm{V} C_{W}\left(t_{0}, \delta\right)=$

$$
C\left[e^{-\beta t_{0}-(\alpha / \gamma)\left(1-e^{\gamma \delta_{0}}\right)}-e^{-\beta\left(t_{0}+W\right)-(\alpha / \gamma)\left(1-e^{\gamma \delta\left(c_{0}+W\right)}\right)}\right]-\left[c_{0}+c_{1} t_{0}+c_{2}(\delta-1)\right] \int_{t_{0}}^{t_{0}+W} u\left(\beta+\alpha e^{\gamma \delta u}\right) e^{-\beta u-\frac{\alpha}{\delta \gamma}\left(1-e^{\gamma \delta u}\right)} d u
$$

We work on graphical illustrations of the dependence of $C_{W}\left(t_{0}, \delta\right)$ on the individual's age $t_{0}$ at the time of enrolment into the insurance, for the Gompertz-Makenham lifetime distribution with parameters as for human beings, and for different values of the coverage period $W$ in years.

The effects of increase or decrease of the insurance premiums and costs should be justified. The dependence of the costs and premiums on the calendar age $t_{0}$ of individuals and its interaction with the age correcting factors might be reviled.

## 6. Conclusions

Age-affecting actions on live individuals may have feasible models, and the life supporting (insurance) cost for some natural policies can be analyzed by making use of approaches similar to those in reliability maintenance and warranty cost analysis.

It is shown that the failure rate function and hazard function provide more convenient tools to agedependent life modeling than the approaches based on the direct use of probability distributions of the individual's life times.

Numerical and graphical examples illustrate the use of the proposed models with Gompertz-Makenham mortality rates and respectively distributed life times with an age- correcting factor for comparing ages between individuals.

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# RELIABILITY AND RISK ASSESSMENT OF SYSTEMS OF PROTECTION AND BLOCKING WITH FAST RESTORATION 

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#### Abstract

There is examined a system with fast restoration which should be operational beginning from some moments of time. If beginning from these moments of time the system is defective during the time more than the assigned random time interval it is considered failed. Such system includes the models of systems with the protection and blocking and systems with the discrete periodic functions. The estimations of indices of failure-free performance and maintainability of these systems and the estimation of indices of risk and losses, connected with the failure (accident) of the system with protection are obtained. This material was presented in the Mathematical Methods in Reliability 2007 Conference in Glasgow, UK.


## 1. Introduction and Motivation

August 2003. The largest in the history of the USA a de-energizing of eastern regions of the USA and Canada for several days has left extensive territories and huge quantity of the population without the electric energy. Losses from this blackout were incredible. What did cause this catastrophic failure? The system of protection and blockings decided that power supply became too high and not to harm power plants they were cut off. In just three minutes the system of protection and blockings produced a cascade avalanche-like cut-off of 21 power plants.

September 2003. The "power outage that affected all of Italy-except the island of Sardinia-for 9 hours and part of Switzerland near Geneva for 3 hours on 28 September 2003. It was the largest blackout in the series of blackouts in 2003, affecting a total of 56 million people" (http://en.wikipedia.org/wiki/2003_Italy_blackout).

May 2005. The blackout in Moscow and in the adjacent regions of Russia.

November 2006. "Two high voltage power lines in Germany failed. This triggered a cascade of cuts as automatic safety devices cut millions of customers in order to prevent a total blackout of the continent. Parts of Germany, Belgium, France (including parts of Paris), Spain, and Italy were affected. High speed railways were also impacted. Power was restored within two hours. Later reports said that Austria and Croatia were also affected" (http://en.wikinews.org/wiki/Europe_suffers_widespread_power_cuts).

These events show the importance of the reliable and correct functioning of systems with protection and blockings. The same events tightly connect the concepts of reliability and engineering risk (risk, appearing as a result of human activity), under which follow to Henley and Kumamoto [10] we understand consequences (on resources, on environment, victims and so on) arising as results of unreliable work (failures) of technical systems and/or intentional terrorists activity.

There are examined systems, for which it is required, that the system would be serviceable only beginning from specific moments of time. If, beginning from specific moments of time, the system is malfunction during the time
not less than $\eta, \mathrm{P}\{\eta<\mathrm{x}\}=\mathrm{H}(\mathrm{x})$, the system is considered failed. Such systems include the models of systems with discretely periodical functions and the models of systems with the protection and blockings.

The system with discrete periodic function is considered failed, if it is faulty during the time not less than $\eta$ after a demand for the function's service arrival. At other moments of time the system can be faulty, but this does not affect the reliability of the system with discrete periodic function.

The system with protection and blockings (SPB) contains a certain object, which periodically falls into a before accident situation (BAS), and a unit of protection and blockings (UPB) that should prevent an accident. Such an object can be, for example, a system of power plants together with power lines and end users. Power stations can fail; short circuits or breaks are possible on power lines and at end users. Many of such events represent a BAS which signals should be sent to UPB. Then during a "short" interval of time UPB should make a reconfiguration of system in purpose to disconnect or the failed plant, or/and failed power lines and the end user, and to redistribute the power supply that was delivered by the failed plant (if any) between other plants and/or power lines. Otherwise there can be an accident, for example not authorized redistribution of power supply which indeed can lead to cascade switching-off of plants.

For given above examples with de-energizing the BAS events could be failures of some power plants (a deficiency of electric power), or some short circuits on the power lines (as it was in the USA), or a short circuit on transformer substations (as it was in Russia), or a result of an external intentional terrorist activity directed toward the destruction of systems of electric power's production and delivery. Power plants are united into the power grid. Some power plants can temporarily stop the electric energy generation. But the electric energy generation by remained power plants is sufficient to ensure the needs of all basic users.

UPB receiving a BAS signal should prevent the accident. For our examples it means that during the assigned time interval, which is less then $\eta$, some actions must be executed in order or to block the failed power plants, temporarily excluding them from the power grid, or to block some sections of the power transmission line, where the short circuit or the break occurred, or to block users who had a short circuit that led to the big power consumption and to reconfigure the layout of the power delivery. UPB being also a technical system itself is subjected to failures and is continuously monitoring so that failures arising in it are eliminating.

A failure of SPB (the accident) occurs when BAS signal arrives at the failed UPB who therefore cannot prevent accident. In this case the UPB failure leads to the SPB failure (accident). But the event "UPB failed but had time to be restored before arrival of the BAS signal" will not affect on the SPB reliability. So, not every UPB failure leads to the SPB failure. There can be "dangerous" and "safe" failures of UPB that should be considered in the criterion of the SPB failure (accident).

In this article we will use terms, relating to the systems with protection and blockings. For evaluating the reliability of SPB we will use ideas of the reliability assessment of systems with fast restoration presented in our article [1].

## 2. Model Description

In the general case UPB is a space-distributed repairable system, and contains $n$ elements and $k$ repair units (RU). UPB come to the models of section 2 of [1]. Each element of system can be only in the operational or the failed state. Each operational element can be located in the loaded or unloaded regime. Let $F_{i}(x)$ and $f_{i}(x)$ are
accordingly the distribution function (DF) and the distribution density (DD) of the time of failure-free operation of the $i$-th element in the system, $i=\overline{1, n}$, and $m_{i}$ is the mean value of this time, $m_{i}<\infty$.

We will examine only the systems of the $1^{\text {st }}$ and the $3^{\mathrm{d}}$ types [1], working in the steady-state operation. The systems of the $2^{\text {nd }}$ type [1], that don't have the steady-state operation section of work, are not examined within the framework of this article. But taking into account [1] and the ideas given below in the sections 4 and 5 it is also possible to carry out the estimation of the reliability of the systems of the $2^{\text {nd }}$ type.

The failed elements are restored. Different interruptions of restoration are permitted, but DF of summary recovery time of the $i$-th element by the $j$-th RU is equal to $G_{i j}(x)$, independent of the number and the duration of the interruptions Genis [3]. Class $D$ of restoration disciplines [1] includes, in particular, the discipline FIFO $d_{1}$ with the straight order of maintenance, where the priority for the restoration have the elements, failed the first, discipline LIFO $d_{2}$ with the reverse order of maintenance, where the priority for the restoration have the elements, failed the last, discipline with time sharing $d_{3}$, where all failed elements are restored with the same speed, and discipline $d_{4}$, where the priority for the restoration have the elements with the shortest residual recovery time. The indices of reliability of the same systems for various restoration disciplines are essentially different. Therefore the reliability assessment for various restoration disciplines allows choosing the most effective discipline.

It is set the criterion of SPB failure that can include and a condition of time reservation.

UPB works in conditions of fast restoration (FR). Practically it means that the average time of restoration of a system's element is essentially less than the average time between any two failures of elements in the system [3].

The problem consists in estimating of SPB indices of non-failure operation and maintainability in conditions of fast restoration of UPB.

## 3. Mathematical Formulation of Problem

Behavior of UPB is described by the alternating process, in which the intervals where all elements are operational are changed by intervals, when in UPB there are any failures of elements, which possibly are not leading to the failure or the malfunction of UPB [1]. Let us call the last intervals as intervals of the malfunction (IM). Let us call the interval of malfunction, which begun in the interval ( $\mathrm{z}, \mathrm{z}+\mathrm{dz}$ ), as $\operatorname{IM} z$.

The state of the elements of system at the moment z is assigned by the vector $\vec{v}(z)=\left\{v_{1}(z), \ldots, v_{n}(z)\right\}$, where each component can take the values of $\{0,1, \ldots, n\}$. Number 0 corresponds to failed elements; numbers from 1 to $n$ correspond to operational elements. Vector $\vec{v}(z)$ helps to estimate the reliability of concrete systems.

Let E is the set of the states of the system, $\{\vec{v}(z)\}=E=E_{+} \cup E_{-}$, where $E_{+}$is the area of the operational, and $E_{-}$ is the area of the defective states of the system. The system is considered as defective at the moment $z$ if $\vec{v}(z) \in E_{-}$ and failed if its malfunction lasts time not smaller then $\eta, P\{\eta<x\}=H(x)$ [1].

Let $\vec{b}$ is a certain state vector of elements of the system directly before IM, and $\vec{b}^{N}$ is the state vector of the elements of the system on the same IM immediately after the moment of passing the state vector of system from the region $E_{+}$into the area $E_{-}$. Let $\pi$ is the way leading from $\vec{b} \in E_{+}$into the state $\vec{b}^{N} \in E_{-}$on the IM. Then $\pi$ is
the sequence of the state vectors of elements, beginning from the vector $\vec{b}$, which directly precede the beginning of the IM, and ending with the vector $\vec{b}^{N}$, which corresponds to the first onset of malfunction of the system on this IM; the passage from one state vector to the following occurs only due to that, that exactly one element of system fails or ends to be restored [1].

The path length is equal to the number of state vectors, being contained on this path, not counting the initial state $\vec{b}$. Let us call the way monotonic if on it there are no restorations of elements. Let us call the monotonic way minimal for $\vec{b}$ if its length $l(\vec{b})$ is equal to the minimum of path lengths, leading from $\vec{b}$ into $E_{-}[1]$.

All introduced notations help to understand the obtained results and are used to prove them.
Let us determine the concept of fast restoration. Let $G(x)=\min G_{i j}(x), G *(x)=\max G_{i j}(x)$, where the minimum and the maximum are taken according to the numbers $j$ of RU, accessible to $i$-th element, and on $i=\overline{1, n}$ (here $G(x)$ and $G *(x)$ are DF of the correspondingly greatest and shortest recovery time of elements); s is the minimum number of elements, failure of which can cause the malfunction of the system; $\bar{\Gamma}()=1-\Gamma()$ for whichever DF $\Gamma()$; $m_{R}^{(j)}=j \int_{0}^{\infty} x^{j-1} \bar{G}(x) d x, \quad m_{R}=m_{R}^{(1)}, \quad m_{R^{*}}(\eta)=\int_{0}^{\infty} \int_{0}^{\infty} \bar{G} *(x+u) d x d H(u) ;$
$€$ and $\underline{\lambda}$ are the maximum and the minimum failure rates of elements in the operational system [1].
Let us say, that in the system is satisfied the condition of FR if $\underline{\lambda}>0$ and

$$
\begin{equation*}
\alpha=\left[€^{\mathrm{s}} m_{R}^{(s)} /\left(m_{R}\right)^{s-1}\right] \rightarrow 0 \tag{3.1}
\end{equation*}
$$

and in this case for all $\operatorname{DF} F_{i}(x), \quad i=\overline{1, n}$, there exist limited DDs.

In practice it is necessary to evaluate the reliability of a concrete system with fixed $\mathrm{DF} F_{k}(x)$ and $G_{i j}(x)$. Therefore it is possible to count without the damage for the generality, that $\mathrm{DF} F_{k}(x)$ are fixed, and $\mathrm{DF} G_{i j}(x)$ are the element of a certain infinite sequence in the diagram of series. More precise we will assume, that are satisfied the following conditions introduced by A.D. Solovyev in [7]:

1) $\mathrm{DF} F_{k}(x), k=\overline{1, n}$, are fixed and have a limited and continuous in zero DD ;
2) $\mathrm{DF} G_{i j}(x)$ have the form

$$
G_{i j}(x)=G_{i j}^{(0)}(x / \xi),
$$

where $G_{i j}^{(0)}(x)$ are fixed, and

$$
\begin{equation*}
\xi \rightarrow 0 ; \tag{3.2}
\end{equation*}
$$

in this case $G(x)=G^{(0)}(x / \xi), G_{*}(x)=G_{*}^{(0)}(x / \xi) ;$
3) There is a finite moment $a^{(s)}$, where

$$
a^{(j)}=j \int_{0}^{\infty} x^{j-1} \bar{G}^{(0)}(x) d x, a=a^{(1)} .
$$

Let us note that the conditions (3.1) and (3.2) are equivalent under assumptions 1) - 3). Actually, under assumption 1) the value $\hat{\lambda}$ is limited and under assumptions 2 ) and 3) when $\xi \rightarrow 0$

$$
\left(m_{R}^{(s)} /\left(m_{R}\right)^{s-1}\right)=\xi\left(a^{(s)} / a^{s-1}\right) \rightarrow 0 .
$$

Conversely, if we assume $\xi=\left(m_{R}^{(s)} /\left(m_{R}\right)^{s-1}\right) /\left(a^{(s)} / a^{s-1}\right)$, than from condition (3.1) and 3) it follows $\xi \rightarrow 0$.

Under the condition of fast restoration almost always the failure of system occurs along the monotonic path [1], if only the probability of this failure is not zero. However, the sufficient condition of that, that the probability of the failure of system on the monotonic path is different from zero, is the condition

$$
\begin{equation*}
a_{*}(\eta)=\int_{0}^{\infty} \int_{0}^{\infty} \bar{G} *(x+u) d x d H(u)>0 \tag{3.3}
\end{equation*}
$$

Next index unites the conditions for fast restoration (3.1) and (3.3)

$$
\begin{equation*}
\varphi_{1}=\left[\hat{\lambda}^{s} m_{R}^{(s)} / \underline{\lambda}^{s-1}\left[m_{R^{*}}(\eta]^{s-1}\right] \rightarrow 0, \underline{\lambda}>0\right. \tag{3.4}
\end{equation*}
$$

The condition

$$
\begin{equation*}
\varphi_{2}=€_{m_{R}} \rightarrow 0 \tag{3.5}
\end{equation*}
$$

ensures the convergence of DF of time to the first failure for the system of the $1^{\text {st }}$ and $3^{\mathrm{d}}$ type to the exponential function, and for the $2^{\text {nd }}$ type to $\exp \left\{-\int_{0}^{x} \beta(u) d u\right\}$, that is shown in [1].

In practically important cases $m_{R}^{(s)} \leq C\left(m_{R}\right)^{s}$, where C - some constant. In these cases at small $s(s \approx 2 \div 4)$, closely related among themselves $G(x)$ and $G *(x)$, that is reached due to unification of procedure of restoration, and a small time reserve ( $m_{R} \approx m_{R}(\eta)$ ) condition (3.4) is possible to replace by condition (3.1) or condition (3.5).

In section 8 [1] it is shown, that under the conditions of FR the estimation of the indices of the reliability of complex system can be brought to the estimations of the indices of reliability of its series-connected in the sense of the reliability schemes of the form $p$ out of $m$, calculated under the assumption, that these schemes operate autonomously. The scheme $p$ out of $m$ has $m$ of elements. Its malfunction occurs with the failure of not less than $p$ elements out of $m, p \leq m$, and its failure begins then, when the malfunction of the scheme lasts not less than $\eta, P\{\eta<x\}=H(x)$. Therefore within the framework of this article we will count that UPB is the scheme $p$ out of $m$ 。

In our estimations we will count, that all RU are identical and therefore $G_{i j}(x)=G_{i}(x)$. Let $m_{R i}^{(j)}(\eta / a)=j \int_{0}^{\infty} \int_{0}^{\infty} x^{j-1} \bar{G}_{i}(x+u / a) d x d H(u), \quad m_{R i}(\eta)=m_{R i}^{(1)}(\eta), \quad m_{R i}^{(j)}(u)=j \int_{0}^{\infty} x^{j-1} \bar{G}_{i}(x+u) d x, \quad \varphi_{i}(u)=m_{R i}^{(1)}(u)$, $m_{R i}^{(j)}=m_{R i}^{(j)}(0), m_{R i}=m_{R i}^{(1)}$. Let in the steady-state operation section of work with $d \in D$ and $k \operatorname{RU} \beta_{p}(d, k)$ is the
estimation of the failure rate of the scheme $p$ out of $m$ with $k$ RU and the restoration discipline $d \in D$ taking into account only monotonic ways of failure, $\tau^{\prime \prime}(d, k)$ is the random system recovery time after failure, $T_{R}(d, k)$ is the average value of this time, $K_{A}(d, k)$ is the availability function of the system.

## 4. Estimation of the indices of failure-free performance

Let $\tau_{i}(t)$ is the interval from the moment $t$ to the first failure of system after moment $t$. The details of the determination of time to the first failure of the system $\tau_{1}(t)$ and time between $(j-1)$-th and $j$-th failures of the system $\tau_{j}(t), j \geq 2$, are given in [1].

Let $F_{*}(x)$ and $m_{*}$ are the distribution function of time between the adjacent BAS signals and mean time between them, moreover $F_{*}(x)$ is an absolutely continuous distribution function with the limited distribution density. In the steady-state operating conditions of the system the DF of residual time before the appearance of a BAS signal is

$$
E(x)=\int_{0}^{x} \overline{F_{*}}(x) d x / m_{*} .
$$

The object, which sends BAS signals, is considered as one of the elements of the system containing $(n+1)$ elements. We will investigate two cases:

1) the condition for the fast restoration is satisfied also relatively to the time between the appearances of the BAS signals (time between the adjacent BAS signals is considered as the object operating time between failures; the restoration of UPB leads to the restoration of SPB);
2) this condition is not satisfied, but in UPB the restoration is fast (restoration of UPB leads to the restoration of SPB);

In both cases is valid theorem 5.3 and estimation (5.8) from [1], and in steady-state operation $\mathrm{DF} \tau_{i}(t), i \geq 1$, converge to exponential. It remains to estimate the parameter of these distributions.

In the first case the SPB model come to the model of section 2 of [1], but containing ( $n+1$ ) elements. In this model is fixed the element, failing the last before the failure of the system (if the time reserve is absent and $\eta \equiv 0$ ) or before the beginning of the malfunction of the system (when $\eta \neq 0$ ). This element is object. On IM this element is not restored, and the time reserve of system is equal to $\eta$. In this case all estimations (6.2) and (6.4) - (6.7) from [1] are carried out, if in them in all expressions, besides $d H(u)$, to replace " u " by " $\mathrm{v}+\mathrm{u}$ " and $\int_{0}^{\infty} \ldots d H(u)$ by $\int_{0}^{\infty} \int_{0}^{\infty} \ldots \overline{F_{*}}(v) d v d H(u) / m_{*}$ or $\int_{0}^{\infty} \int_{0}^{\infty} \ldots d v d H(u) / m_{*}$. There " $v$ " represents the residual time to the arrival of BAS signal and " $u$ " is the time reserve, during which must be executed actions on averting the accident. Replacement $\int_{0}^{\infty} \ldots d H(u)$ by $\int_{0}^{\infty} \int_{0}^{\infty} \ldots d v d H(u) / m_{*}$ is carried out on the basis of the condition for fast restoration that is satisfied also relatively to the time between the appearances of BAS signals, when $\overline{F_{*}}(v) d v / m_{*} \approx d v / m_{*}$.

In the same case with $n=1$ (structural reserve in UPB it is absent) because of the fast restoration and relative to the time between the appearances of the BAS signals $\left(\hat{\lambda} m_{R}^{(2)} / m_{R}\right) \rightarrow 0$, DF $\tau_{i}(t), i \geq 1$, converge to exponential and with $\mathrm{k}=1$ in accordance with the criterion of the failure of the system

$$
\begin{equation*}
\beta_{1}(d, 1) \approx \frac{1}{m_{1} m_{*}} \int_{0}^{\infty} \int_{0}^{\infty} \overline{G_{1}}(u+v) d v d H(u) \approx \frac{m_{r 1}(\eta)}{m_{1} m_{*}} \tag{4.1}
\end{equation*}
$$

Result (4.1) was obtained by Turbin with co-authors [5] for $\eta \equiv 0, d=d_{1}$ under more stronger assumptions. In particular, there was required the absolute continuity of d.f. $G_{1}(x)$.

In the second case the system with protection and blockings is reduced to the model of section 2 [1], in which in accordance with the criterion of the failure of the system this failure begins when malfunction of UPB lasts not less than the time $(\gamma+\eta), \quad P\{\eta<x\}=H(x)$, and in the steady-state operating conditions of the system $P\{\gamma<x\}=E(x)$. Therefore and in this case with the presence of structural reserve in SPB are carried out all previous estimations for the indices of failure-free performance, if in them in all expressions, besides $d H(u)$, to replace " $u$ " by " $v+u$ " and $\int_{0}^{\infty} \ldots d H(u)$ by $\int_{0}^{\infty} \int_{0}^{\infty} \ldots \overline{F_{*}}(v) d v d H(u) / m_{*}$.

Thus it is proven
Theorem 4.1. For examined cases of systems with protection and blockings there are carried out the estimations $\beta_{p}(d, k)$, undertaken for the scheme $p$ from $n$, if in these estimations in all expressions, besides $d H(u)$, to replace " $u$ " by " $v+u$ " and $\int_{0}^{\infty} \ldots d H(u)$ by $\int_{0}^{\infty} \int_{0}^{\infty} \overline{F_{*}}(v) d v d H(u) / m_{*}$. In the case if the fast restoration is satisfied also relatively to the time between the appearances of signals of before accident situations it is allowed to substitute $\int_{0}^{\infty} \ldots d H(u)$ by $\int_{0}^{\infty} \int_{0}^{\infty} \ldots d v d H(u) / m_{*}$.

Corollary 4.1. If UPB represents $n>1$ parallel-connected in the sense of the reliability elements used in the loaded regime and $k=1$ than for the system with protection and blockings under the conditions for fast restoration that are satisfied also relatively to the time between the appearances of BAS signals

$$
\begin{align*}
& \beta_{n}\left(d_{1}, 1\right) \approx \frac{n-1}{m_{1} \ldots m_{n} m_{*}} \sum_{j=1}^{n} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} x^{n-2} \overline{G_{j}}(x+v+u) d x d v d H(u)= \\
&=\sum_{j=1}^{n} m_{R j}^{(n)}(\eta) /\left(m_{1} \ldots m_{n} m_{*} n\right) ;  \tag{4.2}\\
& \beta_{n}\left(d_{2}, 1\right) \approx \frac{(n-1)!}{m_{1} \ldots m_{n} m_{*}} \sum_{j=1}^{n} \int_{0}^{\infty} \int_{0}^{\infty} \overline{G_{j}}(v+u) \prod_{k \neq j} m_{R k} d v d H(u)= \\
& \quad=(n-1)!\sum_{j=1}^{n} m_{R j}(\eta) \prod_{k \neq j} m_{R k} /\left(m_{1} \ldots m_{n} m_{*}\right) ; \tag{4.3}
\end{align*}
$$

$$
\begin{gather*}
\beta_{n}\left(d_{3}, 1\right) \approx \frac{(n-1)!}{m_{1} \ldots m_{n} m_{*}} \sum_{j=1}^{n} \int_{0}^{\infty} \int_{0}^{\infty} \overline{G_{j}}\left(\frac{v+u}{n}\right) \prod_{k \neq j} \varphi_{k}\left(\frac{v+u}{n}\right) d v d H(u)= \\
=\frac{n!}{m_{1} \ldots m_{n} m_{*}} \int_{0}^{\infty}\left(\prod_{j=1}^{n} \int_{0}^{\infty} \bar{G}_{j}\left(y+\frac{u}{n}\right) d y\right) d H(u) ;  \tag{4.4}\\
\beta_{n}\left(d_{4}, 1\right) \approx \sum_{j=1}^{n} \int_{0}^{\infty} \int_{0}^{\infty} \overline{G_{j}}(v+u) \prod_{k \neq j} \varphi_{k}(v+u) d v d H(u)= \\
=\frac{1}{m_{1} \ldots m_{n} m_{*}} \int_{0}^{\infty}\left(\prod_{j=1}^{n} \int_{0}^{\infty} \bar{G}_{j}(y+u) d y\right) d H(u) . \tag{4.5}
\end{gather*}
$$

For the proof of Corollary 4.1 the probabilities of system failures along all minimal monotonic paths are summarized. At that for estimation of indices of failure-free performance for various disciplines of restoration the results of Corollary 6.1 of [1] were used.

All necessary proofs in the article are given in the Appendix.
For the system with the discrete periodic function the probability $Q_{D}(d, k)$ of the failure of system to the requirement on the fulfillment of the function is determined from the formula

$$
\begin{equation*}
Q_{D}(d, k) \approx m_{*} \beta(d, k) . \tag{4.6}
\end{equation*}
$$

## 5. Indices of maintainability and availability function

It is examined a system, described in paragraph 2 , with the condition $\eta=h=$ const. This condition is typical. Let us define recovery time of SPB as time, passed from the moment of the failure of the system to the moment of the restoration of SPB capacity for work. For evaluating the indices of the maintainability and availability function we will need the concept of $x$-failure of SPB [1]. Let us say, what the system $x$-failed, if its failure lasts not less than the time $x$. Through $\beta_{x}(d, r)$ we will designate the intensity of SPB $x$-failures. Let us recall, that in the SPB model the BAS signals enter the last before the UPB failure.

Since with $\eta=h=$ const $H(u)=0$ with $u<h$ and $H(u)=1$ with $u \geq h$ than taking into account theorem 4.1 we will obtain

Corollary 5.1. For the system with protection and blockings and the time reserve, equal to the constant, $\eta=h=$ const , the estimation $\beta_{x}(d, r)$ is obtained

- or from the estimations $\beta_{p}(d, r)$ for the scheme $p$ out of $m$ from [1], if in them in all expressions, besides $d H(u)$, to replace the arguments " $u$ " by " $h+x+v$ ", and replace $\int_{0}^{\infty} \ldots d H(u)$ by $\int_{0}^{\infty} \ldots \bar{F}(v) d v / m_{*}$; in the case of
the fast restoration and relatively to the time between the appearances of the signals of before accident situations it is allowed to substitute $\int_{0}^{\infty} \ldots d H(u)$ by $\int_{0}^{\infty} \ldots d v$;
- or from the estimations $\beta_{p}(d, r)$ of section 4 of this work, if in them in all expressions, besides $d H(u)$, to replace the arguments " u " by " $\mathrm{h}+\mathrm{x}$ " and to remove the external integral on $d H(u)$.

Corollary 5.2. If UPB is a system $n$ out of $n$ with the loaded reserve (parallel in the sense of reliability connection of elements), $k=1$ and the condition of fast restoration is satisfied and relatively to the time between the appearances of BAS signals, than
for $d=d_{1}$

$$
\begin{gather*}
P\left\{\tau^{\prime \prime}\left(d_{1}, 1\right) \geq x\right\} \approx \sum_{i=1}^{n} m_{R i}^{(n)}(h+x) / \sum_{j=1}^{n} m_{R i}^{(n)}(h),  \tag{5.1}\\
T_{R}\left(d_{1}, 1\right) \approx \sum_{i=1}^{n} m_{R i}^{(n+1)}(h) /\left((n+1) \sum_{j=1}^{n} m_{R j}^{(n)}(h)\right),  \tag{5.2}\\
K_{A}\left(d_{1}, 1\right) \approx 1-\sum_{i=1}^{n} m_{r i}^{(n+1)}(h) /\left(n(n+1) m_{*} \prod_{j=1}^{n} m_{j}\right) ; \tag{5.3}
\end{gather*}
$$

for $d=d_{2}$

$$
\begin{gather*}
P\left\{\tau^{\prime \prime}\left(d_{2}, 1\right\} \geq x\right\} \approx\left[\sum_{j=1}^{n} m_{R j}(h+x) / m_{R j}\right] /\left[\sum_{i=1}^{n} m_{R i}(h) / m_{R i}\right],  \tag{5.4}\\
T_{R}\left(d_{2}, 1\right) \approx\left[\sum_{j=1}^{n} m_{R j}^{(2)}(h) / m_{R j}\right] /\left[2 \sum_{i=1}^{n} m_{R i}(h) / m_{R i}\right],  \tag{5.5}\\
K_{A}\left(d_{2}, 1\right) \approx 1-(n-1)!\sum_{j=1}^{n} m_{R j}^{(2)}(h) \prod_{i \neq j} m_{R i} /\left(2 m_{*} \prod_{k=1}^{n} m_{k}\right) ; \tag{5.6}
\end{gather*}
$$

for $d=d_{3}$

$$
\begin{align*}
& P\left\{\tau^{\prime}\left(d_{3}, 1\right) \geq x\right\} \approx \prod_{i=1}^{n} m_{R i}\left(\frac{h+x}{n}\right) / m_{R i}\left(\frac{h}{n}\right),  \tag{5.7}\\
& T_{R}\left(d_{3}, 1\right) \approx \int_{0}^{\infty} \prod_{i=1}^{n}\left[m_{R i}\left(\frac{h+x}{n}\right) / m_{R i}\left(\frac{h}{n}\right)\right] d x, \tag{5.8}
\end{align*}
$$

$$
\begin{equation*}
K_{A}\left(d_{3}, 1\right) \approx 1-n!\int_{0}^{\infty} \prod_{i=1}^{n}\left[m_{R i}\left(\frac{h+x}{n}\right) /\left(m_{*} m_{i}\right)\right] d x \tag{5.9}
\end{equation*}
$$

for $d=d_{4}$

$$
\begin{gather*}
P\left\{\tau^{\prime \prime}\left(d_{4}, 1\right) \geq x\right\} \approx \prod_{i=1}^{n} m_{R i}(h+x) / m_{R i}(h),  \tag{5.10}\\
T_{R}\left(d_{4}, 1\right) \approx \int_{0}^{\infty} \prod_{i=1}^{n}\left[m_{R i}(h+x) / m_{R i}(h)\right] d x,  \tag{5.11}\\
K_{A}\left(d_{4}, 1\right) \approx 1-\int_{0}^{\infty} \prod_{i=1}^{n}\left[m_{R i}(h+x) /\left(m_{*} m_{i}\right)\right] d x . \tag{5.12}
\end{gather*}
$$

For the proof of Corollary 5.2 were used estimations of indices of maintainability and availability function for various disciplines of restoration that are given in Corollary 6.2 of [1].

For $k=1$ and $n=1$ (UPB consists of one element) and with any discipline $d \in D$

$$
\begin{gathered}
P\left\{\tau^{\prime \prime}(d, 1) \geq x\right\} \approx m_{R}(h+x) / m_{R}(h), \\
T_{R}(d, 1) \approx m_{R}^{(2)}(h) /\left(2 m_{R}(h),\right. \\
K_{A}(d, 1) \approx 1-m_{R}^{(2)}(h) /\left(2 m m_{*}\right) .
\end{gathered}
$$

In much the same way it is possible to obtain the estimations of the indices of maintainability and availability function also for other types of UPB.

Let us note that it is possible to remove the requirement of absolute continuity of d.f . $F_{*}(x)$. Thus, the period between two adjacent entries of BAS signals (or of requirements for the fulfillment of a function in the system with the discretely carried out functions) can be constant. In this case it is possible to obtain the same estimations of the indices of reliability, using an apparatus, connected with the rare events in the regenerating process [6].

## 6. Estimation of risk and losses

It is natural to ask the question how to estimate the risk and losses connected with the failure of the system of protection and blockings? One of the possible approaches is the following.

When SPB fails the local systems of protection and blockings as a rule soften the losses from the accident. Let there are possible N different ways of the development of the accident when SPB fails. The probability of the $i$-th way of the development of the accident is $p_{i}, \sum p_{i}=1$ on $i=\overline{1, N}$, and the losses on this way are $L_{i}$. Then risk and losses from the SPB failure [9] can be estimated as

$$
\begin{equation*}
R_{1}=\sum_{i=1}^{N} p_{i} L_{i} \tag{6.1}
\end{equation*}
$$

In practice [9] the value $L_{i}$ is substituted by the loss function or the function of the usefulness $U\left(L_{i}\right)$ (when some losses are disregarded, and some losses are exaggerated), and the probabilities $p_{i}$ are substituted by the subjective probabilities $f\left(p_{i}\right)$ (when some small probabilities are disregarded). In this case risk and losses from the SPB failure can be estimated as

$$
\begin{equation*}
R_{2}=\sum_{i=1}^{N} f\left(p_{i}\right) U\left(L_{i}\right) \tag{6.2}
\end{equation*}
$$

With this approach the estimation of risk and losses can be conducted only for the concrete system in the stages of its design and operation.

## Appendix

The following three lemmas will make it possible to simplify calculation formulas.
Let $M_{J}^{(l)}$ is the set of all permutations from the collection of the numbers $J=\left(j_{1}, \ldots, j_{l}\right)$ and $i^{(l)}=\left(i_{1}, \ldots, i_{l}\right)$ is a certain permutation from the set $M_{J}^{(l)}$.

Lemma A.1. For $i \geq 1$ and any fixed $x \geq 0$ and $u \geq 0$ the next inequality is correct

$$
\begin{equation*}
\sum_{i^{(l)} \in M_{J}^{(j)}} \int_{x<y_{1}<\ldots<y_{l}} \ldots \prod_{k=1}^{l} \bar{G}_{i_{k}}\left(y_{k}+u\right) d y_{1} \ldots d y_{l}=\prod_{k=1}^{l} \int_{x}^{\infty} \bar{G}_{i_{k}}(y+u) d y . \tag{A.1}
\end{equation*}
$$

Proof of lemma A.1. When $l=1$ (A.1) is obviously. Let (A.1) is true for $l=w$. Then we will show that (A.1) is true for $l=w+1$.

$$
\begin{aligned}
& \sum_{i^{(w+1)} \in M y_{j}^{(w+1)}} \int_{x<y_{1}<\ldots<y_{w+1}} \ldots \prod_{k=1}^{w+1} \bar{G}_{i_{k}}\left(y_{k}+u\right) d y_{1} \ldots d y_{w+1}=\sum_{i^{(w)} \in M_{j}^{(w)}} \int_{x<z_{1}<\ldots<z_{w}} \ldots \prod_{k=1}^{w} \bar{G}_{i_{k}}\left(z_{k}+u\right) d z_{1} \ldots d z_{w} * \\
& *\left[\int_{x}^{z_{1}} \bar{G}_{i_{w+1}}(y+u) d y+\int_{z_{1}}^{z_{2}} \bar{G}_{i_{w+1}}(y+u) d y+\ldots+\int_{z_{w}}^{\infty} \bar{G}_{i_{w+1}}(y+u) d y\right]=\prod_{k=1}^{w+1} \int_{x}^{\infty} \bar{G}_{i_{k}}(y+u) d y .
\end{aligned}
$$

Lemma A. 1 is proved.

Lemma A.2. For any integers $k \geq 1$ and $h=$ const

$$
\begin{equation*}
\int_{0}^{\infty} m_{R i}^{(k-1)}(h+x) d x=m_{R i}^{(k)}(h) / k . \tag{A.2}
\end{equation*}
$$

Proof of lemma A.2. After using the replacement of variable and a change in the order of integration, we will obtain:

$$
\begin{gathered}
\int_{0}^{\infty} m_{R i}^{(k-1)}(h+x) d x=(k-1) \int_{0}^{\infty} \int_{0}^{\infty} u^{k-2} \vec{G}_{i}(u+h+x) d u d x=(k-1) \int_{0}^{\infty} \int_{x}^{\infty}(u-x)^{k-2} \bar{G}_{i}(u+h) d u d x= \\
=(k-1) \int_{0}^{\infty} \bar{G}_{i}(u+h) \int_{0}^{u}(u-x)^{k-2} d x d u=\int_{0}^{\infty} \bar{G}_{i}(u+h) \int_{u}^{0} d_{x}(u-x)^{k-1} d u= \\
=\int_{0}^{\infty} u^{k-1} \bar{G}_{i}(u+h) d u=m_{R i}^{(k)}(h) / k
\end{gathered}
$$

and lemma A. 2 is proved.

Lemma A.3. For any integers $N \geq 1$, any $u<\infty, 0<a<\infty$ and any functions $f_{i}(x)$ such, that $\int_{0}^{\infty} f_{i}(x) d x<\infty$, next identity is carried out

$$
\begin{equation*}
\sum_{i=1}^{N} \int_{0}^{\infty} f_{i}\left(\frac{u+y}{a}\right) d y \prod_{\substack{j=1,1, \frac{u+y}{a} \\ j \neq i}}^{N} f_{j}(v) d v=a \prod_{j=1}^{N} \int_{\frac{u}{a}}^{\infty} f_{j}(y) d y . \tag{A.3}
\end{equation*}
$$

Proof of lemma A.3. Since

$$
\sum_{i=1}^{N} \int_{0}^{\infty} f_{i}\left(\frac{u+y}{a}\right) d y \prod_{\substack{j=1,1 \\ j \neq i}}^{N} \int_{\frac{u+y}{a}}^{\infty} f_{j}(v) d v=a \sum_{i=1}^{N} \int_{\frac{u}{a}}^{\infty} f_{i}(y) d y \prod_{\substack{j=1,1 \\ j \neq i}}^{N} \int_{y}^{\infty} f_{j}(v) d v,
$$

then for the proof of (A.3) it is sufficient to show, that

$$
\begin{equation*}
\sum_{i=1}^{N} \int_{u}^{\infty} f_{i}(y) d y \prod_{\substack{j=1 \\ j \neq i}}^{N} \int_{y}^{\infty} f_{j}(v) d v=\prod_{k=1}^{N} \int_{u}^{\infty} f_{k}(y) d y . \tag{A.4}
\end{equation*}
$$

Since

$$
=-\left.\sum_{i=1}^{N} \int_{u}^{\infty} f_{i}(v)\right|_{v=y} ^{v=\infty} d y \prod_{\substack{j=1, j \neq i}}^{N} \int_{y}^{\infty} f_{j}(v) d v=\sum_{i=1}^{N} \int_{u}^{\infty} f_{i}(y) d y \prod_{\substack{j=1, y \\ j \neq i}}^{N} \int_{y}^{\infty} f_{j}(v) d v,
$$

And lemma (A.3) is proved.
Proof of corollary 4.1. For the scheme $n$ out of $n$ all monotonic ways leave from $\vec{b} \equiv \overrightarrow{1}$, where all $n$ of elements are operational, and they fall into $\vec{b}^{j} \equiv \overrightarrow{0}$, where all $n$ of elements failed [1]. The length of the monotonic way leading from $\vec{b} \equiv \overrightarrow{1}$ into $\vec{b}^{j} \equiv \overrightarrow{0}$ equals $(n+1)$. The BAS signal on the monotonic way always enters the last.

With $d=d_{1}$ let us fix the first failed element, which will be restored by the single RU. With the fixed first element and with the fixed last element (object) remained ( $n-1$ ) elements will give ( $n-1$ )! ways, leading from $\vec{b} \equiv \overrightarrow{1}$ into $\vec{b}^{j} \equiv \overrightarrow{0}$. Therefore taking into account (6.2) from [1] and theorem 4.1 it follows

$$
\begin{align*}
& \beta_{n}\left(d_{1}, 1\right) \approx \frac{(n-1)!}{m_{1} \ldots m_{n} m_{*}} \sum_{j=1}^{n} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0<x_{2}<\ldots<x_{n}} \ldots \bar{G}_{j}\left(x_{n}+v+u\right) d x_{2} \ldots d x_{n} d v d H(u)= \\
& =\frac{(n-1)!}{m_{1} \ldots m_{n} m_{*}} \sum_{j=1}^{n} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \bar{G}_{j}\left(x_{n}+v+u\right) d x_{n} \int_{0}^{x_{n}} d x_{n-1} \ldots \int_{0}^{x_{3}} d x_{2} d v d H(u)= \\
& =\frac{(n-1)}{m_{1} \ldots m_{n} m_{*}} \sum_{j=1}^{n} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} x^{n-2} \bar{G}_{j}(x+v+u) d x d v d H(u) . \tag{A.5}
\end{align*}
$$

Passing in (A.5) from the internal double integral to the iterated and after making the change of variables $(\mathrm{x}+\mathrm{v})=$ $\mathrm{y}, \mathrm{dv}=\mathrm{dy}, 0<x<(x+v)=y$, we will obtain

$$
\begin{aligned}
\beta_{n}\left(d_{1}, 1\right) & \approx \frac{(n-1)}{m_{1} \ldots m_{n} m_{*}} \sum_{j=1}^{n} \int_{0}^{\infty} \int_{0}^{\infty} \bar{G}_{j}(y+u) \int_{0}^{y} x^{n-2} d x d y d H(u)= \\
& =\frac{1}{m_{1} \ldots m_{n} m_{*} n} \sum_{j=1}^{n} n \int_{0}^{\infty} \int_{0}^{\infty} y^{n-1} \bar{G}_{j}(y+u) d y d H(u)
\end{aligned}
$$

and statement (4.2) is proven.
With $d=d_{2}$ let us fix the last failed element of UPB which precedes the BAS signal. Those remaining ( $n-1$ ) elements of UPB will give ( $n-1$ )! ways leading from $\vec{b} \equiv \overrightarrow{1}$ into $\vec{b}^{j} \equiv \overrightarrow{0}$ with the last fixed element. Therefore taking into account (6.2) from [1] and theorems 4.1 it follows

$$
\beta_{n}\left(d_{2}, 1\right) \approx \frac{(n-1)!}{m_{1} \ldots m_{n} m_{*}} \sum_{j_{n}=1}^{n} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0<x_{2}<\ldots x_{n}} \ldots \bar{G}_{j_{1}}\left(x_{2}\right) \bar{G}_{j_{2}}\left(x_{3}-x_{2}\right) \ldots \bar{G}_{j_{n-1}}\left(x_{n}-x_{n-1}\right) \bar{G}_{j_{n}}(v+u) d x_{2} \ldots d x_{n} d v d H(u)=
$$

$$
=\frac{(n-1)!}{m_{1} \ldots m_{n} m_{*}} \sum_{j_{n}=1}^{n} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \bar{G}_{j_{1}}\left(x_{2}\right) d x_{2} \int_{x_{2}}^{\infty} \bar{G}_{j_{2}}\left(x_{3}-x_{2}\right) d x_{3} \ldots \int_{x_{n-1}}^{\infty} \bar{G}_{j_{n-1}}\left(x_{n}-x_{n-1}\right) d x_{n} \bar{G}_{j_{n}}(v+u) d v d H(u) .
$$

After taking into consideration, that $x_{1}=0$, making the change of variables $x_{j}-x_{j-1}=y, d x_{j}=d y$, and changing limits of integration from $x_{j-1}<x_{j}<\infty$ то $0<y<\infty, j=\overline{2, n}$, we will obtain, that

$$
\beta_{n}\left(d_{2}, 1\right) \approx \frac{(n-1)!}{m_{1} \ldots m_{n} m_{*}} \sum_{j=1}^{n} \int_{0}^{\infty} \int_{0}^{\infty} \bar{G}_{j}(v+u) \prod_{i \neq j} m_{R i} d v d H(u)=\frac{(n-1)!}{m_{1} \ldots m_{n} m_{*}} \sum_{j=1}^{n} m_{R j}(\eta) \prod_{i \neq j} m_{R i} .
$$

Statement (4.3) is proven.
With $d=d_{3}$, let us fix the last failed element of UPB ( $i$-th element), which precedes the BAS signal. Therefore taking into account (6.7) from [1] and theorem 4.1 we will obtain

$$
\begin{gather*}
\beta_{n}\left(d_{3}, 1\right) \approx \frac{(n-1)!}{m_{1} \ldots m_{n} m_{*}} \sum_{i=1}^{n} \sum_{j^{(n-1} \in M_{j}^{(n-1)}} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0<y_{1}<\ldots<y_{n-1}} \ldots \bar{G}_{j_{n-1}}\left(y_{1}+\frac{v+u}{n}\right) \ldots \bar{G}_{j_{1}}\left(y_{n-1}+\frac{v+u}{n}\right) d y_{1} \ldots d y_{n-1} * \\
* \bar{G}_{i}\left(\frac{v+u}{n}\right) d v d H(u) . \tag{A.6}
\end{gather*}
$$

Since integral expression, standing under the sign of sums in (A.6), is converged and it is equal to the probability of failure of SPB on one of the monotonic ways, leading from $\vec{b} \equiv \overrightarrow{1}$ into $\vec{b}^{j} \equiv \overrightarrow{0}$ than according [8] it is possible to interchange the positions of summing up and integration. Therefore

$$
\begin{gather*}
\beta_{n}\left(d_{3}, 1\right) \approx \frac{(n-1)!}{m_{1} \ldots m_{n} m_{*}} \sum_{i=1}^{n} \int_{0}^{\infty} \int_{0}^{\infty}\left(\sum_{j^{(n-1)} \in M_{J^{(n-1)}}} \int_{0<y_{1}<\ldots<y_{n-1}} \ldots \bar{G}_{j_{n-1}}\left(y_{1}+\frac{v+u}{n}\right) \ldots \bar{G}_{j_{1}}\left(y_{n-1}+\frac{v+u}{n}\right) d y_{1} \ldots d y_{n-1}\right) * \\
* \bar{G}_{i}\left(\frac{v+u}{n}\right) d v d H(u) . \tag{A.7}
\end{gather*}
$$

After using to the internal iterated integral in (A.7) lemma A.1, then after making the change of variables $[y+(v+u) / n]=y^{\prime}$, and then using lemma A.3, we will obtain

$$
\begin{aligned}
\beta_{n}\left(d_{3}, 1\right) & \approx \frac{(n-1)!}{m_{1} \ldots m_{n} m_{*}} \sum_{i=1}^{n} \iint_{0}^{\infty} \int_{0}^{\infty} \bar{G}_{i}\left(\frac{v+u}{n}\right) \prod_{\substack{j=1 \\
j \neq i}}^{n} \int_{0}^{\infty} \bar{G}_{j}\left(y+\frac{v+u}{n}\right) d y d v d H(u)= \\
& =\frac{(n-1)!}{m_{1} \ldots m_{n} m_{*}} \sum_{j=1}^{n} \int_{0}^{\infty} \int_{0}^{\infty} \bar{G}_{j}\left(\frac{v+u}{n}\right) \prod_{i \neq j} \int_{\frac{v+u}{n}}^{\infty} \bar{G}_{i}(y) d y d v d H(u)=
\end{aligned}
$$

$$
\begin{gathered}
\frac{(n-1)!}{m_{1} \ldots m_{n} m_{*}} \int_{0}^{\infty}\left(\sum_{i=1}^{n} \int_{0}^{\infty} \bar{G}_{i}\left(\frac{v+u}{n}\right) \prod_{\substack{j=1 \\
j \neq i}}^{n} \int_{\frac{v+u}{n}}^{\infty} \bar{G}_{j}(y) d y\right) d v d H(u)= \\
=\frac{n!}{m_{1} \ldots m_{n} m_{*}} \int_{0}^{\infty}\left(\prod_{j=1}^{n} \int_{0}^{\infty} \bar{G}_{j}\left(y+\frac{u}{n}\right) d y\right) d H(u),
\end{gathered}
$$

and we obtained (4.4).
Let us pass to discipline $d_{4}$. By definition of discipline $d_{4} A\left(d_{4}, 1, \pi, u\right)=A\left(d_{1}, l, \pi, u\right)$. Furthermore, exactly so, as was proven lemma 1 in [4], it is possible to show, that with the condition for fast restoration and $k \geq 1$ $A\left(d_{4}, k, \pi, u\right)=A\left(d_{1}, l, \pi, u\right)$, and we obtain statement 4) of corollary 6.1 from [1]. Hence with $d=d_{4}, l=n, k=1$ let us fix the last failed element of UPB and taking into account (6.5) from [1] and theorem 4.1 we will obtain

$$
\begin{gathered}
\beta_{n}\left(d_{4}, 1\right) \approx \frac{1}{m_{1} \ldots m_{n} m_{*}} \sum_{j_{n}=1}^{n} \int_{0}^{\infty} \int_{0}^{\infty}\left(\sum_{j^{(n-1)} \in M_{j}^{(n-1)}} \int_{0<y_{1}<\ldots<y_{n-1}} \ldots \bar{G}_{j_{n-1}}\left(y_{1}+v+u\right) \ldots \bar{G}_{j_{1}}\left(y_{n-1}+v+u\right) d y_{1} \ldots d y_{n-1}\right) * \\
* \bar{G}_{j_{n}}(v+u) d v d H(u) .
\end{gathered}
$$

After using to the internal iterated integral lemma A.1, then after making the change of variables $[y+(v+u)]=y^{\prime}$, and then using lemma A.3, we will obtain

$$
\begin{gathered}
\beta_{n}\left(d_{4}, 1\right) \approx \frac{1}{m_{1} \ldots m_{n} m_{*}} \sum_{j=1}^{n} \int_{0}^{\infty} \int_{0}^{\infty} \bar{G}_{j}(v+u) \prod_{i \neq j} \int_{0}^{\infty} \bar{G}_{i}(y+v+u) d y d v d H(u)= \\
=\frac{1}{m_{1} \ldots m_{n} m_{*}} \sum_{j=1}^{n} \int_{0}^{\infty} \int_{0}^{\infty} \bar{G}_{j}(v+u) \prod_{i \neq j} \int_{v+u}^{\infty} \bar{G}_{i}(y) d y d v d H(u)= \\
=\frac{1}{m_{1} \ldots m_{n} m_{*}} \int_{0}^{\infty}\left(\prod_{j=1}^{n} \int_{0}^{\infty} \bar{G}_{j}(y+u) d y\right) d H(u),
\end{gathered}
$$

and we obtained (4.5). Corollary 4.1 is proven.
Proof of corollary 5.2. Let us take an advantage of corollary 6.2 of [1] and the second recommendation of corollary 5.1.

With $r=1, d=d_{1}$, and $\eta=h=$ const from (4.2)

$$
\beta_{n}\left(d_{1}, 1\right) \approx \frac{1}{m_{1} \ldots m_{n} m_{*} n} \sum_{j=1}^{n} n \int_{0}^{\infty} y^{n-1} \bar{G}_{j}(y+h) d y=\frac{1}{m_{1} \ldots m_{n} m_{*} n} \sum_{j=1}^{n} m_{R j}^{(n)}(h),
$$

and

$$
\beta_{x n}\left(d_{1}, 1\right) \approx \frac{1}{m_{1} \ldots m_{n} m_{*} n} \sum_{j=1}^{n} n \int_{0}^{\infty} y^{n-1} \bar{G}_{j}(y+h+x) d y=\frac{1}{m_{1} \ldots m_{n} m_{*} n} \sum_{j=1}^{n} m_{R j}^{(n)}(h+x) .
$$

Hence from (6.9) [1] it follows (5.1).
On the lemma A. 2

$$
\int_{0}^{\infty} m_{R j}^{(n)}(h+x) d x=m_{R j}^{(n+1)}(h) /(n+1)
$$

and in accordance with (6.10) and (6.11) from [1] we obtain (5.2) and (5.3).
With $r=1, d=d_{2}$, and $\eta=h=$ const from (4.3) we will obtain

$$
\begin{gathered}
\beta_{n}\left(d_{2}, 1\right) \approx \frac{(n-1)!}{m_{1} \ldots m_{n} m_{*}} \sum_{j=1}^{n} m_{R j}(h) \prod_{i \neq j} m_{R i}, \\
\beta_{x n}\left(d_{2}, 1\right) \approx \frac{(n-1)!}{m_{1} \ldots m_{n} m_{*}} \sum_{j=1}^{n} m_{R j}(h+x) \prod_{i \neq j} m_{R i} .
\end{gathered}
$$

We obtained (5.4).
On lemma A. 2

$$
\int_{0}^{\infty} m_{R j}(h+x) d x=m_{R j}^{(2)}(h) / 2 .
$$

And we obtained (5.5) and (5.6).
With $r=1, d=d_{3}$, and $\eta=h=$ const from (4.4) we will obtain

$$
\begin{gathered}
\beta_{n}\left(d_{3}, 1\right) \approx \frac{n!}{m_{1} \ldots m_{n} m_{*}} \prod_{j=1}^{n} m_{R j}\left(\frac{h}{n}\right), \\
\beta_{x n}\left(d_{3}, 1\right) \approx \frac{n!}{m_{1} \ldots m_{n} m_{*}} \prod_{j=1}^{n} m_{R j}\left(\frac{h+x}{n}\right) .
\end{gathered}
$$

Hence taking into account corollary 6.2 from [1] we will obtain (5.7), (5.8) and (5.9).

With $r=1, d=d_{4}$, and $\eta=h=$ const from (4.4) we will obtain

$$
\begin{gathered}
\beta_{n}\left(d_{4}, 1\right) \approx \frac{1}{m_{1} \ldots m_{n} m_{*}} \prod_{j=1}^{n} m_{R j}(h), \\
\beta_{x n}\left(d_{4}, 1\right) \approx \frac{1}{m_{1} \ldots m_{n} m_{*}} \prod_{j=1}^{n} m_{R j}(h+x),
\end{gathered}
$$

which proves (5.10), (5.11), and (5.12).
Corollary 5.2 is proven.

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# AST ALGORITHMS OF ASYMPTOTIC ANALYSIS OF NETWORKS WITH UNRELIABE EDGES 

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#### Abstract

A problem of a reliability in networks with unreliable elements naturally origin in technical applications [1]. But a direct calculation of the reliability demands a number of operations which increases geometrically dependently on a number of edges. So it is necessary to use approximate methods and particularly asymptotic one. In [2] a reliability asymptotic is calculated in analogous asymptotic suggestions on the network edges. Main parameters in these asymptotic are a shortest way length and a maximal flow in a network. In this paper different partial classes of networks are considered and effective algorithms of their parameters calculations are suggested. These networks are networks originated by dynamic systems, networks with integer-valued lengths of edges, superposition of networks and bridge schemes.


## 1. Preliminaries

Define the graph $\Gamma$ with the finite nodes set $U$ and the set $W$ of edges $w=(u, v)$. The graph $\Gamma$ may contain cycles or not, its edges may be oriented or not. Denote by $R(u)$ the set of all ways $R$ of the graph $\Gamma$, which connect the nodes $u_{0}, u$, and assume that $\mathcal{R}(u) \neq \varnothing, u \in U$. Suppose that $\Gamma(u)$ is the sub-graph of the graph $\Gamma$, which consists of the ways $R \in \mathcal{R}(u)$. Consider the sets

$$
\mathcal{A}(u)=\left\{A \subset U: u_{0} \in A, u \notin A\right\}, L=L(A)=\left\{\left(u, u^{\prime}\right): u \in A, u^{\prime} \notin A\right\}
$$

And the set $\mathcal{L}(u)=\{L(A), A \in \mathcal{A}(u)\}$ of all sections of the sub-graph $\Gamma(u)$.
Characterize each edge $w \in W$ of the graph $\Gamma$ by the logic number $\alpha(w)=I$ (the edge $w$ works), where $I(B)$ is the indicator function of the event $B$. Denote

$$
\beta(u)=\underset{R \in \mathbb{R}(u)}{\vee} \wedge_{w \in R} \alpha(w)
$$

the characteristic of the nodes $u_{0}, u$ connectivity in the graph $\Gamma$. Suppose that $\alpha(w), w \in W$, are independent random variables, $P(\alpha(w)=1)=p_{w}(h), q_{w}(h)=1-p_{w}(h)$, where $h$ is small parameter: $h \rightarrow 0$. In [2] the following statements are proved.

Theorem 1. Suppose that $p_{w}(h) \sim \exp \left(-h^{-c(w)}\right), h \rightarrow 0$, where $c(w)>0, w \in W$. Then

$$
\begin{equation*}
-\ln P(\beta(u)=1) \sim h^{-D(u)}, \quad D(u)=\min _{R \in \mathcal{R}(u)} \max _{w \in R} c(w) . \tag{1}
\end{equation*}
$$

Theorem 2. Suppose that $q_{w}(h) \sim \exp \left(-h^{-c_{1}(w)}\right), h \rightarrow 0$, where $c_{1}(w)>0, w \in W$. Then

$$
\begin{equation*}
-\ln P(\beta(u)=0) \sim h^{-D_{1}(u)}, \quad D_{1}(u)=\max _{R \in \mathcal{R}(u)} \min _{w \in R} c_{1}(w) \tag{2}
\end{equation*}
$$

Theorem 3. Suppose that $p_{w}(h) \sim h^{g(w)}, h \rightarrow 0$, where $g(w)>0, w \in W$. Then

$$
\begin{equation*}
-\ln P(\beta(u)=1) \sim T(u) \ln h, \quad T(u)=\min _{R \in \mathcal{R}(u)} \sum_{w \in R} g(w) . \tag{3}
\end{equation*}
$$

Theorem 4. Suppose that $q_{w}(h) \sim h^{g(w)}, h \rightarrow 0$, where $g(w)>0, w \in W$. Then

$$
\begin{equation*}
-\ln P(\beta(u)=0) \sim T_{1}(u) \ln h, \quad T_{1}(u)=\min _{L \in L(u)} \sum_{w \in L} g(w) . \tag{4}
\end{equation*}
$$

Statement 1. Suppose that all $c(w)\left(\right.$ all $\left.c_{1}(w)\right), w \in W$, are different. Then there is the single edge $w(u)$ (there is the single edge $\left.w_{1}(u)\right)$, so that $c(w(u))=D(u)\left(c_{1}\left(w_{1}(u)\right)=D_{1}(u)\right)$. It is called the critical edge.

## 2. Graphs generated by dynamic systems

Suppose that the set $U$ consists of non-intersected subsets $U_{0}, U_{1}, \ldots, U_{m}$, and the set $U_{0}$ contains the single vertex $u_{0}$, which is called initial. All edges of the oriented graph $\Gamma$ are represented as $\left(u_{i}, u_{j}\right), 1 \leq i<j \leq m$, $u_{i} \in U_{i}, u_{j} \in U_{j}$, and each vertexis accessible from the initial vertex $u_{0}$. Described graphs are generated by dynamic systems with a delay. In this section we calculate $D(u), D_{1}(u), T(u)$ and find critical edges $w(u), w_{1}(u)$ for a fixed $u_{0}$.

A main idea of this section is an application of the Floyd algorithm [3], when a solution is calculated for all $u \in U$. To construct fast algorithms it is natural to constrict a class of considered graphs. An idea of such a constriction is illustrated in [4] but for a fixed $u$.

Suppose that $D\left(u_{0}\right)=D_{1}\left(u_{0}\right)=T\left(u_{0}\right)=0$, for all $u \in U_{1}$ put

$$
D(u)=D_{1}(u)=T(u)=c(u), w(u)=w_{1}(u)=\left(u_{0}, u\right) .
$$

For $u \in U$ define $S(u)=\{v:(v, u) \in W\},|S(u)|$ a number of elements in the finite set $S(u)$. Assume that for all $u \in U_{1}, \ldots, U_{k}$ the meanings $D(u), D_{1}(u), T(u), w(u), w_{1}(u)$ are defined. Take $u \in U_{k+1}$ and in an accordance with the formulas (1), (2) put

$$
\begin{gather*}
D(u)=\min _{v \in S(u)} \max (c(v, u), D(v)), \quad D_{1}(u)=\max _{v \in S(u)} \min \left(c(v, u), D_{1}(v)\right),  \tag{5}\\
T(u)=\min _{v \in S(u)}(c(v, u)+T(v)), k \geq 1 . \tag{6}
\end{gather*}
$$

To calculate each element from the set $D(u), D_{1}(u), T(u), u \in U$ it is necessary $2|S(u)|-1$ arithmetic operations and this number can not be decreased. So the algorithm (5), (6) is optimal. And if for fixed $u \in U$ $D(u), D_{1}(u), T(u)$ are calculated by the algorithm (5), then we find $D(v), D_{1}(v), T(v)$ for all nodes $v$ from which the vertex $u$ is accessible.

To define critical edges it is necessary to complement the formulas (5) by

$$
\begin{gather*}
w(u)=w_{1}(u)=\left(u_{0}, u\right), \text { if } u_{0} \in S(u), \\
w(u)=\left\{\begin{array}{c}
w(v), \text { if } D(u)=\max (D(v), c(v, u))>c(v, u), \\
(v, u), \text { if } D(u)=\max (D(v), c(v, u))>D(v),
\end{array}\right.  \tag{7}\\
w_{1}(u)=\left\{\begin{array}{c}
w_{1}(v), \text { if } D_{1}(u)=\max \left(D_{1}(v), c(v, u)\right)<c(v, u), \\
(v, u), \text { if } D_{1}(u)=\max \left(D_{1}(v), c(v, u)\right)<D(v) .
\end{array}\right. \tag{8}
\end{gather*}
$$

## 3. Graphs with integer-valued lengths of edges

In this section we consider a calculation of $T(u)$ for all $u \in U$ in graphs with integer-valued lengths of edges. Suppose that $g(w), w \in W$, are natural numbers, $g(w) \leq \bar{g}<\infty$ and define

$$
\begin{equation*}
G_{\Gamma}=\sum_{w \in W} g(w) . \tag{9}
\end{equation*}
$$

Divide each edge of the graph $\Gamma$ into edges with unit lengths by an introduction of intermediary nodes. As a result obtain the graph $\Gamma^{1}$ with the nodes set $U^{1}, U \subseteq U^{1}$ and with the edges set $W^{1}$. Denote $N\left(u^{1}\right)$ the minimal number of the graph edges in ways, which connect the nodes $u_{0}, u^{1}$. It is easy to obtain that

$$
\begin{equation*}
N(u)=G(u), u \in U \tag{10}
\end{equation*}
$$

Consider now an algorithm of $N\left(u^{1}\right), u^{1} \in U^{1}$ calculation.
Suppose that all nodes of the graph. $\Gamma^{1}$ are not marked. Mark the vertex $u_{0}$, and put $U_{0}^{1}=\left\{u_{0}\right\}$. Then construct a recurrent procedure of non-intersected sets $U_{k}^{1}, k \geq 0$, definition. Suppose that the sets $U_{k}^{1}$, $V_{k}^{1}=\bigcup_{0 \leq i \leq k} U_{i}^{1}$ are known and all nodes of the set $V_{k}^{1}$ are marked and all other nodes are not marked. Define the set. $U_{k+1}^{1}$ as a set of all unmarked nodes from $U^{1}$, which are connected directly with some vertexfrom the set $U_{k}^{1}$. By a definition the set $U_{k+1}^{1}$ satisfies the formula

$$
U_{k+1}^{1}=\left\{u^{1}: N\left(u^{1}\right)=k+1\right\} .
$$

Mark all nodes of the set $U_{k+1}^{1}$ and define the set $V_{k+1}^{1}=V_{k}^{1} \cup U_{k+1}^{1}$.
Estimate a number of operations which are necessary to calculate $U_{k+1}^{1}$ if each vertexof the graph $\Gamma$ is connected directly with no more $l$ nodes. Then a number of operations to define $U_{k+1}^{1}$ does not exceed $l\left|U_{k}^{1}\right|$. Define $M$ by the formula

$$
V_{0}^{1} \subset V_{1}^{1} \subset \ldots \subset V_{M}^{1}=V_{M+1}^{1}=\ldots
$$

then to construct the sequence $U_{1}^{1}, \ldots, U_{M}^{1}$ it is necessary no more $l G_{\Gamma}$ operations where $l G_{\Gamma} \leq l^{2} \bar{g}|U|$. Compare these results with the results of Deikstra [4], in a case when $c(w)$ is not integer-valued. To calculate $D(u), u \in U$ in a general case it is necessary no more $K_{1}|U|^{2}$ operations and for a dendriform graph - no more $K_{2}|U| \ln |U|$ operations, where $K_{1}, K_{2}<\infty$.

## 4. Superposition of graphs

Fix in the graph $\Gamma$ some vertex $v_{0}$. Assume that $\Gamma^{\prime}$ is non-oriented graph with the nodes set $U^{\prime}=\left\{1^{\prime}, \ldots, m^{\prime}\right\}$, $U \cap U^{\prime}=\varnothing$ and with the edges set $W^{\prime}\left(i^{\prime}, j^{\prime}\right),\left(i^{\prime}, i\right) \notin W^{\prime}$. Distinguish in the graph $\Gamma^{\prime}$ initial and final nodes $u_{0}{ }^{\prime}, v_{0}^{\prime}$ and in the set $U$ - two nodes $\bar{u}, \bar{v}$ so that $\bar{w}=(\bar{u}, \bar{v}) \in W$. Denote by $\mathcal{R}^{\prime}$ the set of all ways $R^{\prime}$ of the graph $\Gamma^{\prime}$ from $u_{0}{ }^{\prime}$ to $v_{0}{ }^{\prime}$.

Define the superposition $\bar{\Gamma}=\Gamma \stackrel{\bar{w}}{\otimes} \Gamma^{\prime}$ of the graphs $\Gamma, \Gamma^{\prime}$ with a replacement of the edge $(\bar{u}, \bar{v})$ from the graph $\Gamma$ by the graph $\Gamma^{\prime}$ and with an aliasing of the nodes $\bar{u}$ with $u_{0}{ }^{\prime}$ and of the nodes $\bar{v}$ with $v_{0}{ }^{\prime}$ correspondingly. Denote by $\bar{U}$ the nodes set, by $\bar{W}$ - the edges set and by $\overline{\mathcal{R}}$ - the set of ways from the vertex $u_{0}$ to the vertex $v_{0}$ in the graph $\bar{\Gamma}$. Put $\mathcal{R}$ the set of ways from $u_{0}$ to $v_{0}$ in the graph $\Gamma, \mathcal{R}^{\prime}$ - the set of ways from $u_{0}^{\prime}$ to $v_{0}^{\prime}$ in the graph $\Gamma^{\prime}$. Analogously define $\overline{\mathcal{L}}, \mathcal{L}, \mathcal{L}^{\prime}$ the sets of sections in the graphs $\bar{\Gamma}, \Gamma, \Gamma^{\prime}$ with pairs of initial and final nodes $\left(\overline{u_{0}}, \overline{v_{0}}\right),\left(u_{0}, v_{0}\right),\left(u_{0}{ }^{\prime}, v_{0}{ }^{\prime}\right)$ correspondingly. Define

$$
\beta=\underset{R \in \mathcal{R}}{\vee} \hat{w \in R}^{\wedge} \alpha(w), \bar{\beta}=\frac{\vee}{\bar{R} \in \overline{\mathcal{R}}}, \wedge_{w \in \bar{R}}^{\wedge} \alpha(w)
$$

characteristics of a connectivity between the nodes $u_{0}, v_{0}$ in the graphs $\Gamma, \bar{\Gamma}$ correspondingly. Then from the theorems 1-4 it is possible to obtain asymptotic formulas for the superposition $\bar{\Gamma}$.
Theorem 5. Suppose that $p_{w}(h) \sim \exp \left(-h^{-c(w)}\right), h \rightarrow 0$, where $c(w)>0, w \in \bar{W}$. Then
$-\ln P(\bar{\beta}=1) \sim h^{-\bar{D}}, \bar{D}=\min _{R \in \mathcal{R}} \max _{w \in R} \bar{c}(w)$,

$$
\bar{c}(w)=c(w), w \neq \bar{w}, \quad \bar{c}(\bar{w})=\min _{R^{\prime} \in \mathcal{R}^{\prime}} \max _{w \in R^{\prime}} c(w) .
$$

Theorem 6. Suppose that $q_{w}(h) \sim \exp \left(-h^{-c_{1}(w)}\right), h \rightarrow 0$, where $c_{1}(w)>0, w \in \bar{W}$. Then

$$
\begin{gathered}
-\ln P(\bar{\beta}=0) \sim h^{-\bar{D}_{1}}, \quad \bar{D}_{1}=\min _{L \in \mathcal{L}} \max _{w \in R} \bar{c}_{1}(w), \\
\bar{c}_{1}(w)=c_{1}(w), w \neq \bar{w}, \quad \bar{c}_{1}(\bar{w})=\min _{L^{\prime} \in L^{\prime}} \max _{w \in L^{\prime}} c_{1}(w)
\end{gathered}
$$

Theorem 7. Suppose that $p_{w}(h) \sim h^{g(w)}, h \rightarrow 0$, where $g(w)>0, w \in \bar{W}$. Then

$$
\begin{gathered}
\ln P(\bar{\beta}=1) \sim \bar{T} \ln h, \quad \bar{T}=\min _{R \in \mathcal{R}} \sum_{w \in R} \bar{g}(w), \\
\bar{g}(w)=g(w), w \neq \bar{w}, \quad \bar{g}(\bar{w})=\min _{R^{\prime} \in \mathcal{R}^{\prime}} \sum_{w \in R^{\prime}} g(w) .
\end{gathered}
$$

Theorem 8. Suppose that $q_{w}(h) \sim h^{g(w)}, h \rightarrow 0$, where $g(w)>0, w \in \bar{W}$. Then

$$
\begin{gathered}
\ln P(\bar{\beta}=0) \sim \bar{T}_{1} \ln h, \quad \bar{T}_{1}=\min _{L \in \mathcal{L}} \sum_{w \in L} \bar{g}_{1}(w), \\
\bar{g}_{1}(w)=g_{1}(w), w \neq \bar{w}, \quad \bar{g}_{1}(\bar{w})=\min _{L^{\prime} \in L^{\prime}} \sum_{w \in L^{\prime}} g_{1}(w) .
\end{gathered}
$$

It is obvious that the formulas from these theorems allow calculating asymptotic of a reliability for superposition of networks with unreliable elements rationally. These formulas may be used to calculate a reliability in recursively defined networks which are widely used in the solid state physics and in the nanotechnology.

## 5. Asymptotic analysis of bridge scheme

The simplest superposition of graphs is parallel-sequential graphs. But there are graphs widely used in the reliability theory, which are not parallel - sequential. One of them is a bridge scheme.

Consider the non-oriented graph $\Gamma$ with the nodes set $U=\left\{u_{i}, i=0, \ldots, 3\right\}$ and with the edges set $W=\left\{w_{j}, j=1, . ., 5\right\}$, where

$$
w_{1}=\left(u_{0}, u_{1}\right), w_{2}=\left(u_{0}, u_{2}\right), w_{3}=\left(u_{1}, u_{3}\right), w_{4}=\left(u_{2}, u_{3}\right), w_{5}=\left(u_{1}, u_{2}\right) .
$$

The vertex $u_{0}$ is initial and the vertex $u_{3}$ is final. The edge $w_{5}$ is a bridge element in the graph $\Gamma$. The graph $\Gamma$ is called the bridge scheme in the reliability theory. Define the $\Gamma_{1}$ by a deleting of the edge $w_{5}$ from the graph $\Gamma$. Introduce the graph $\Gamma_{2}$ by an aliasing of the nodes $u_{1}, u_{2}$ in the graph $\Gamma_{1}$.


Fig. 1. Graph $\Gamma$.


Fig. 2. Graph $\Gamma_{1}$.


Fig. 3. Graph $\Gamma_{2}$.
Suppose that the edges $w_{1}, \ldots, w_{5}$ work independently and define positive numbers $c\left(w_{i}\right)=c_{i}, 1 \leq i \leq 5$,

$$
C_{1}=\min \left(\max \left(c_{1}, c_{3}\right), \max \left(c_{2}, c_{4}\right)\right), C_{2}=\max \left(\min \left(c_{1}, c_{2}\right), \min \left(c_{3}, c_{4}\right)\right), C_{2} \leq C_{1} .
$$

If random logical variables $\beta, \beta_{1}, \beta_{2}$ characterize the nodes $u_{0}, u_{3}$ connectivity in the graphs $\Gamma, \Gamma_{1}, \Gamma_{2}$, correspondingly, then from the complete probability formula we have:

$$
\begin{equation*}
P(\beta=1)=p_{w_{5}}(h) P\left(\beta_{2}=1\right)+\left(1-p_{w_{5}}(h)\right) P\left(\beta_{1}=1\right), P\left(\beta_{1}=1\right) \leq P\left(\beta_{2}=1\right) \tag{11}
\end{equation*}
$$

From the theorem 1 and the equalities (11) obtain ) the statement which characterizes a role of the bridge element.

Theorem 9. If $p_{w}(h) \sim \exp \left(-h^{-c(w)}\right), h \rightarrow 0$, where $c(w)>0, w \in W$, then

$$
\begin{equation*}
-\ln P(\beta=1) \sim h^{-D}, \quad D=\min \left(C_{1}, \max \left(C_{2}, c_{5}\right)\right) \tag{12}
\end{equation*}
$$

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# BOTTLENECKS IN GENERAL TYPE LOGICAL SISTEMS WITH UNRELIABLE ELEMENTS 

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In this paper a model of general type logical system with unreliable elements [1], [2] is considered. An asymptotic analysis of its work (failure) probability is made in appropriate conditions on work (failure) probabilities of the system elements. A concept of bottlenecks of this system is constructed on a suggestion that an increase (a decrease) of elements reliabilities lead to an increase (a decrease) of the system reliability.

A construction of general type logical system is founded on concepts of disjunctive and conjunctive normal forms (DNF and CNF) of a logical function. This approach allows obtaining main results in maximal general and convenient for engineering calculations form comparatively with recursive definitions of logical functions used in [3].

Denote $Z$ the set which consists of $|Z|$ independent random logical variables $z, I \subseteq\left\{1,2, \ldots 2^{|Z|}\right\}$. Consider the logical function $A$ represented in DNF

$$
\begin{equation*}
A=\bigvee_{i \in I}\left[\left(\bigwedge_{z \in Z_{i}} z\right) \wedge\left(\bigwedge_{z \in \bar{Z}_{i}} \bar{z}\right)\right] \tag{1}
\end{equation*}
$$

Here the family $\left\{\left(Z_{i}, \bar{Z}_{i}\right), i \in I\right\}$ consists of the sets pairs $Z_{i}, \overline{Z_{i}} \subseteq Z, \quad Z_{i} \cap \overline{Z_{i}}=\varnothing$, and for $i \neq j$ $\left(Z_{i}, \overline{Z_{i}}\right) \neq\left(Z_{j}, \overline{Z_{j}}\right)$. Suppose that $p_{z}=P(z=1), q_{z}=P(z=0), p_{z}+q_{z}=1$, and random variables $z \in Z$ are independent. The logical function $A$ with random arguments $z \in Z$ is denoted by $\mathbf{A}$ and called the logical system.

## Low reliable elements

Suppose that for $\forall z \in Z$

$$
\begin{equation*}
\exists c(z), c(z)>0: p_{z}=p_{z}(h) \sim \exp \left(-h^{-c(z)}\right), h \rightarrow 0 \tag{2}
\end{equation*}
$$

Denote $C=\min _{i \in I} \max _{z \in Z_{i}} c(z)$,
$I^{\prime}=\{i \in I: \max c(z)=C\}, S_{i}=\left\{z \in Z_{i}: c(z)=C\right\}, i \in I^{\prime}, S=\left\{S_{i}, \quad i \in I^{\prime}\right\}, N(S)=\min \left(\left|S_{i}\right|: S_{i} \in S\right)$,
$\mathcal{T}=\left\{\left\{z_{i} \in S_{i}, \quad i \in I^{\prime}\right\}\right\}, N(\mathcal{T})=\min (|T|: T \in \mathcal{T})$
and let $S^{\prime}, \mathcal{T}^{\prime}$ are families of minimal by an inclusion sets from the families $S, \mathcal{T}$,

$$
S^{\prime \prime}=\left\{S_{i} \in S^{\prime}:\left|S_{i}\right|=N(S)\right\}, \mathcal{T}^{\prime \prime}=\left\{T \in \mathcal{T}^{\prime}:|T|=N(\mathcal{T})\right\} .
$$

Theorem 1. If the formulas (1), (2) are true then

$$
\begin{equation*}
-\ln P(\mathbf{A}=1) \sim N(S) h^{-C}, h \rightarrow 0 \tag{3}
\end{equation*}
$$

Proof. Rewrite the logical function $A$ as follows

$$
A=\bigvee_{i \in I}\left[\left(\bigwedge_{z \in Z_{i}} z\right) \wedge A_{i}\right], A_{i}=\bigvee_{k \in J_{i}}\left({\widehat{z \in \in Z_{i}}}_{\bar{z}}^{\bar{z}}\right), J_{i}=\left\{k: Z_{k}=Z_{i}\right\} .
$$

The formula (2) leads to $p_{z}=P(z=1) \rightarrow 0, \quad h \rightarrow 0$, so

$$
P\left(\mathbf{A}_{\mathbf{i}}=0\right)=\prod_{z \in \bar{Z}_{k}: Z_{k}=Z_{i}} p_{z} \rightarrow 0, h \rightarrow 0 .
$$

If the obvious that

$$
\begin{align*}
& \sum_{i \in I} \prod_{z \in Z_{i}} p_{z} P\left(\mathbf{A}_{\mathbf{i}}=1\right)-\sum_{i, j \in I, i \neq j} P\left(\left(\mathbf{A}_{\mathbf{i}} \prod_{z \in Z_{i}} z=1\right) \cap\left(\mathbf{A}_{\mathbf{j}} \prod_{z \in Z_{j}} z=1\right)\right) \leq  \tag{4}\\
& \leq P(\mathbf{A}=1) \leq \sum_{i \in I} \prod_{z \in Z_{i}} p_{z} P\left(\mathbf{A}_{\mathbf{i}}=1\right)
\end{align*}
$$

As for $i \neq j$

$$
P\left(\mathbf{A}_{\mathbf{i}} \mathbf{A}_{\mathbf{j}}=1\right)=P\left(\sum_{k \in J_{i}, n \in J_{j}}\left(\prod_{z \in \overline{Z_{k}} \cup \bar{Z}_{n}} \bar{z}\right)=1\right) \geq \prod_{z \in \bar{Z}_{k} \cup \bar{Z}_{n}} q_{z} \rightarrow 1, h \rightarrow 0 \text { б }
$$

and

$$
\sum_{i, j \in I, i \neq j} P\left(\left(\mathbf{A}_{i} \prod_{z \in Z_{i}} z=1\right) \cap\left(\mathbf{A}_{\mathbf{j}} \prod_{z \in Z_{j}} z=1\right)\right)=P\left(\mathbf{A}_{\mathbf{i}} \mathbf{A}_{\mathbf{j}}=1\right) \prod_{z \in Z_{i} \cup Z_{j}} p_{z}
$$

So from the formula (4) obtain

$$
\begin{equation*}
P(\mathbf{A}=1) \sim \sum_{i \in I} \prod_{z \in Z_{i}} p_{z} \sim \sum_{i \in I} \exp \left(-\sum_{z \in Z_{i}} h^{c(z)}\right), h \rightarrow 0 \tag{5}
\end{equation*}
$$

Denote $C_{i}=\max _{z \in Z_{i}} c(z), K_{i}=\left\{z \in Z_{i}: c(z)=C_{i}\right\}$. The formulas

$$
\sum_{z \in Z_{i}} h^{-c(z)} \sim h^{-C_{i}}\left|K_{i}\right|, h \rightarrow 0,
$$

and (5) give

$$
P(\mathbf{A}=1) \sim \sum_{i \in I} \exp \left(-h^{C_{i}}(1+o(1))\left|K_{i}\right|\right), h \rightarrow 0 .
$$

Consequently,

$$
\begin{gathered}
P(\mathbf{A}=1) \sim \sum_{i \in I^{\prime}} \exp \left(-h^{-C}(1+o(1))\left|S_{i}\right|\right) \sim \sum_{i \in I^{\prime}:\left|S_{i}\right|=N(S)} \exp \left(-h^{-C}(1+o(1))\left|S_{i}\right|\right)= \\
=\exp \left(-h^{-C}(1+o(1)) N(s)\right)\left|\left\{i \in I^{\prime}:\left|S_{i}\right|=N(S)\right\}\right|
\end{gathered}
$$

As

$$
\begin{gathered}
\ln \left[\exp \left(-h^{-C}(1+o(1)) N(s)\right)\left|\left\{i \in I^{\prime}:\left|S_{i}\right|=N(s)\right\}\right|\right] \sim \ln \exp \left(-h^{-C}(1+o(1)) N(s)\right)= \\
=-h^{-C}(1+o(1)) N(s) \sim-h^{-C} N(s), h \rightarrow 0 .
\end{gathered}
$$

So formula (3) is true.
Remark 1. Suppose that $\tau(z)$ are independent random variables equal to life times of logical elements $z$, and $h=h(t)$ - is monotonically decreasing and continuous function, $h \rightarrow 0, t \rightarrow \infty$. Then the asymptotic

$$
P(\tau(z)>t)=p_{z}(h) \sim \exp \left(-h^{-c(z)}\right), t \rightarrow \infty .
$$

Is character for the Weibull distribution which is widely used in life time models of complex systems with old and so low reliable elements [4], [5].

## Highly reliable elements

Suppose that for $\forall z \in Z$

$$
\begin{equation*}
\exists c(z), c(z)>0: q_{z}=q_{z}(h) \sim \exp \left(-h^{-c(z)}\right), h \rightarrow 0 \tag{6}
\end{equation*}
$$

Consider the logical function $A$ represented in CNF

$$
\begin{equation*}
A=\bigwedge_{i \in I}\left[\left(\bigvee_{z \in Z_{i}} z\right) \bigvee\left(\bigvee_{z \in \bar{Z}_{i}} \bar{z}\right)\right] . \tag{7}
\end{equation*}
$$

Theorem 2. If the formulas (6), (7) are true then

$$
\begin{equation*}
-\ln P(\mathbf{A}=0) \sim N(s) h^{-C}, h \rightarrow 0 \tag{8}
\end{equation*}
$$

Remark 2. Suppose that $\tau(z)$ are independent random variables equal to life times of logical elements $z, u$ $h=h(t)$ - is monotonically increasing and continuous function, $h \rightarrow 0, t \rightarrow 0$. Then the asymptotic

$$
P(\tau(z) \leq t)=q_{z}(h) \sim \exp \left(-h(t)^{-c(z)}\right), t \rightarrow 0
$$

Is character for the Weibull distribution which is widely used in life time models of complex systems with young and so high reliable elements.

## Mixing case

Suppose that the sets $X_{i}, V_{i}, \bar{X}_{i}, \bar{V}_{i} \subseteq Z$ are nonintersecting. For $\forall z \in X_{i} \cup \bar{X}_{i}$ the formula (2) is true and for $\forall z \in V_{i} \cup \bar{V}_{i}$ the formula (6) taking place, $i \in I$. So low reliable and high reliable elements in the system $\mathbf{A}$ are present simultaneously

Theorem 3. Suppose that

$$
\begin{equation*}
A=\bigvee_{i \in I}\left[\left({\widehat{z \in X_{i}} \mathbf{U V _ { i }}} z\right) \bigwedge\left({\widehat{z \in X_{i} \cup \bar{V}_{i}}} \bar{z}\right)\right] \tag{9}
\end{equation*}
$$

Then for $Z_{i}=X_{i} \cup \bar{V}_{i} \neq \emptyset, \bar{Z}_{i}=V_{i} \cup \bar{X}_{i}, \quad i \in I$, the formula (3) is true .
Suppose that

$$
A=\bigwedge_{i \in I}\left[\left(\underset{z \in X_{i} \cup V_{i}}{\bigvee_{i}} z^{2}\right) \bigvee\left(\underset{z \in X_{i} \backslash \bar{V}_{i}}{\bigvee} \bar{z}\right)\right]
$$

Then for $Z_{i}=V_{i} \cup \overline{X_{i}} \neq \emptyset, \bar{Z}_{i}=X_{i} \cup \bar{V}_{i}, i \in I$, the formula (8) is true.

## Concept of bottlenecks

Define bottlenecks in logical system A
Theorem 4. Suppose that $\varepsilon_{0}=\min (|C-c(z)|>0: z \in Z)$.

1. For any $S_{i} \in S$ and each $\varepsilon, 0<\varepsilon<\varepsilon_{0}$, the property (B) is true: the replacement $c(z)$ by $c(z)-\varepsilon$ for all $z \in S_{i}$ leads to the replacement $C \rightarrow C-\varepsilon$.
2. If a set $S \subseteq Z$ and satisfies the condition (B), then $S_{*} \in S: S_{*} \subseteq S$.
3. For any $T \in \mathcal{T}$ and each $\varepsilon, 0<\varepsilon<\varepsilon_{0}$, the property $(C)$ is true: the replacement $c(z)$ by $c(z)+\varepsilon$ for all $z \in T$ leads to the replacement $C \rightarrow C+\varepsilon$.
4. If a set $T \subseteq Z$ and satisfies the condition ( $\mathbf{C}$ ), then $\exists T_{*} \in \mathcal{T}: T_{*} \subseteq T$.

Proof. Proof the statements 1, 3, as the statements 2, 4 are trivial.

1. If $c(z)$ is replace by $c(z)-\varepsilon, z \in S_{i}$, then
$\max _{z \in Z_{i}} c(z)=C-\varepsilon, \max _{z \in Z_{i}} c(z) \geq C-\varepsilon, j \neq i \Rightarrow \min _{i \in I} \max _{z \in Z_{i}} c(z)=C-\varepsilon$.
2. If $c(z)$ is replace by $c(z)+\varepsilon, z \in T$, then
$\max _{z \in Z_{i}} c(z)=C+\varepsilon, i \in I^{\prime}, \max _{z \in Z_{i}} c(z) \geq C+\varepsilon, j \neq I^{\prime} \Rightarrow \min _{i \in I} \max _{z \in Z_{i}} c(z)=C+\varepsilon$.

Corollary 1. The statements 2, 4 of the theorem 4 establish that the families $S^{\prime}, S^{\prime \prime}, \mathcal{T}^{\prime}, \mathcal{T}^{\prime \prime \prime}$ and the numbers $C, N(S), N(\mathcal{T})$ do not depend on a view of $D N F$ (of KNF) of the logical function $A$.

Proof. Suppose that the theorem 1 condition is true, all other case is considered analogically. Denote by $A_{1}, A_{2}-$ DNF, which define the logical function $A, S_{1}, S_{2}$ are families of subsets $Z$, created by $A_{1}, A_{2}$, and $S_{1}^{\prime}, S_{2}^{\prime}$ are families of minimal sets from the families $S_{1}, S_{2}$, correspondingly. If the set $S_{1} \in S_{1}^{\prime}$ then it satisfies the property (B) and so $\exists S_{2} \in S_{2}^{\prime}: S_{2} \subseteq S_{1}$. Analogously if $S_{2} \in S_{2}^{\prime}$ then there is $S_{1}^{*} \in S_{1}^{\prime}: S_{1}^{*} \subseteq S_{2}$. Consequently $S_{1}^{*} \subseteq S_{2} \subseteq S_{1}$ and the families $S_{1}^{\prime}, S_{2}^{\prime}$ definition leads to the equality $S_{1}^{*}=S_{2}=S_{1}$ and so $S_{1}^{\prime}=S_{2}^{\prime}$. Thus, the family $S^{\prime}$ does not depend on a view of logical function $A$ DNF. Similar statements may be proved for the families $\mathcal{T}^{\prime}, \mathcal{T}^{\prime \prime}, S^{\prime \prime}$. For the numbers $N(\mathcal{T}), C, N(S)$ the statements of the corollary 1 may be obtain from the formula (3).

Remark 3. The statements 1 (the statements 3) of the theorem 4 establishes that an increase of elements $z \in S$ reliabilities for any set $S \in S$ (a decrease of elements $z \in T$ reliabilities for any set $T \in \mathcal{T}$ ) leads to an increase (to a decrease) of system A reliability. The corollaryl allows to call sets from the families $S^{\prime}, S^{\prime \prime}, \mathcal{T}^{\prime}, \mathcal{T}^{\prime \prime}$ by bottlenecks in logical system $\mathbf{A}$.

Remark 4. Suppose that $\forall z \in Z$ the condition (2) or the condition (6) are replaced by

$$
\exists c(z), d(z), c(z)>0, d(z)>0: p_{z}=p_{z}(h) \sim \exp \left(-d(z) h^{-c(z)}\right), h \rightarrow 0
$$

or by

$$
\exists c(z), d(z), c(z)>0, d(z)>0: q_{z}=q_{z}(h) \sim \exp \left(-d(z) h^{-c(z)}\right), h \rightarrow 0
$$

correspondingly. Then to obtain the formula (3) or the formula (8) correspondingly it is enough to redefine $|S|$, $S \subseteq Z$, and put (besides of number of elements in a set $S$ ):
$|S|=\sum_{z \in S} d(z)$.

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# AN INTEGRAL MEASURE OF AGING/REJUVENATION FOR REPAIRABLE AND NON-REPAIRABLE SYSTEMS 

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Abstract - This paper introduces a simple index that helps to assess the degree of aging or rejuvenation of repairable systems and non-repairable systems (components). The index ranges from -1 to 1 . It is negative for the class of decreasing failure rate distributions and point processes with decreasing ROCOF and is positive for the increasing failure rate distributions and point processes with increasing ROCOF. The introduced index is distribution free.

Index Terms - aging, rejuvenation, homogeneity, non-homogeneity.

## ACRONYMS ${ }^{1}$

| CDF | cumulative distribution function |
| :--- | :--- |
| CFR | constant failure rate |
| CIF | cumulative intensity function |
| DFR | decreasing failure rate |
| GPR | G-renewal process |
| HPP | homogeneous Poison process |
| IFR | increasing failure rate |
| NHPP | non-homogeneous Poison process |
| PP | point process |
| ROCOF | rate of occurrence of failures <br> RP |
| renewal process |  |
| TTF | time to failure |

## I. INTRODUCTION

In reliability and risk analysis, the terms aging and rejuvenation are used for describing reliability behavior of repairable as well as non-repairable systems (components).

The repairable systems reliability is modeled by various point processes (PP), such as the homogeneous Poisson process (HPP), non-homogeneous Poisson process (NHPP), renewal process (RP), G-renewal process (GRP), to name a few. Among these PP, some special classes are introduced in order to model the so-called improving and deteriorating systems. An improving (deteriorating) system is defined as the system with decreasing (increasing) rate of occurrence of failures (ROCOF). It might be said that among the point processes used as models for repairable systems, the HPP (having a constant ROCOF) is a basic one.

Similarly, among the distributions used as models of time to failure (TTF) of non-repairable systems (components), the exponential distribution, which is the only distribution having a constant failure rate, plays a fundamental role. This distribution might be considered as the limiting between the class of increasing failure rate

[^0](IFR) distributions and the class of decreasing failure rate (DFR) distributions. The distribution is closely related to the above mentioned HPP. Indeed, in the framework of the HPP model, the distribution of the intervals between successive events observed during a time interval $[0, t]$ is the exponential one with parameter $\lambda$ equal to parameter $\lambda$ of the respective Poisson distribution with mean $\lambda t$.

In many practical situations, it is important to make an assessment how far a given point process deviates from the HPP, which can be considered as a simple and, therefore, strong competing model. Note that if the HPP turns out to be an adequate model, the respective system is considered as non-aging, so that it does not need any preventive maintenance (as opposed to the case, when a repairable system reveals aging).

The statistical tools helping to find out if the HPP is an appropriate model are mainly limited to statistical hypothesis testing, in which the null hypothesis is
$\mathrm{H}_{0}$ : "The times between successive events (interarrival times) are independent and identically exponentially distributed ", and the alternative hypothesis is
$\mathrm{H}_{1}$ : "The system is either aging or improving."
The most popular hypothesis testing procedures for the considered type of problems are the Laplace test (Rausand \& Hoyland, 2004) and the so-called Military Handbook test (AMSAA, 1981). It should be noted that these procedures do not provide a simple measure quantitatively indicating how different the ROCOF of a given point process is, compared to the respective constant ROCOF of the competing HPP model.

Analogously, for the non-repairable systems, some hypothesis testing procedures can be applied to help to determine if the exponential distribution is an appropriate TTF model. In such situations, in principle, any goodness-of-fit test procedure can be used. Some of these tests for the null-hypothesis: "the times to failure are independent and identically exponentially distributed" appear to have good power against the IFR or DFR alternatives (Lawless, 2003).

Among such goodness-of-fit tests, one can mention the G-test, which is based on the so-called Gini statistic (Gail \& Gastwirth, 1978). In turn, the Gini statistics originates from the so-called Gini coefficient used in macroeconomics for comparing an income distribution of a given country with the uniform distribution covering the same income interval. The Gini coefficient is used as a measure of income inequality (Sen, 1997). The coefficient takes on the values between 0 and 1 . The closer the coefficient value to zero, the closer the distribution of interest is to the uniform one. The interested reader could find the index values sorted by countries in (List of Countries by Income Inequality, 2007) that includes the UN and CIA data.

In the following sections, we first introduce a Gini-type coefficient for the repairable systems. The coefficient takes on the values between -1 and 1 . The closer it is to zero, the closer the PP of interest is to the HPP. A positive (negative) value of this coefficient will indicate whether a given repairable system is deteriorating (improving). Then, we introduce a similar coefficient for non-repairable systems. Again, the coefficient takes on the values between -1 and 1 . The closer the coefficient's value is to zero, the closer the distribution of interest is to the exponential distribution. A positive (negative) value of the coefficient indicates an IFR (DFR) failure time distribution. For the sake of simplicity, in the following, this Gini-type coefficient will be referred to as GT coefficient and denoted as $C$.

## II. GT COEFFICIENT FOR REPAIRABLE SYSTEMS

## A. Basic Definitions

A point process (PP) can be informally defined as a mathematical model for highly localized events distributed randomly in time. The major random variable of interest related to such processes is the number of events, $N(t)$, observed in time interval $[0, t]$. Using the nondecreasing integer-valued function $N(t)$, the point process $\{N(t), t \geq$ $0\}$ is introduced as the process satisfying the following conditions:

1. $N(t) \geq 0$
2. $N(0)=0$
3. If $t_{2}>t_{1}$, then $N\left(t_{2}\right) \geq N\left(t_{1}\right)$
4. If $t_{2}>t_{1}$, then $\left[N\left(t_{2}\right)-N\left(t_{1}\right)\right]$ is the number of events occurred in the interval $\left(t_{1}, t_{2}\right]$

The mean value $E[N(t)]$ of the number of events $N(t)$ observed in time interval [ $0, t]$ is called cumulative intensity function (CIF), mean cumulative function (MCF), or renewal function. In the following, term cumulative intensity function is used. The CIF is usually denoted by $\Lambda(t)$ :

$$
\Lambda(t)=E[N(t)]
$$

Another important characteristic of point processes is the rate of occurrence of events. In reliability context, the events are failures, and the respective rate of occurrence is abbreviated to ROCOF. The ROCOF is defined as the derivative of CIF with respect to time, i.e.,

$$
\lambda(t)=\frac{d \Lambda(t)}{d t}
$$

When an event is defined as a failure, the system modeled by a point process with an increasing ROCOF is called aging (sad, unhappy, or deteriorating) system. Analogously, the system modeled by a point process with a decreasing ROCOF is called improving (happy, or rejuvenating) system.

The distribution of time to the first event (failure) of a point process is called the underlying distribution. For some point processes, this distribution coincides with the distribution of time between successive events; for others it does not.

## B. GT Coefficient

Consider a PP having an integrable over [ $0, T$ ] cumulative intensity function, $\Lambda(t)$. It is assumed that the respective ROCOF exists, and it is increasing function over the same interval [ $0, T$ ], so that $\Lambda(t)$ is concave upward, as illustrated by Figure 1. Further consider the HPP with CIF $\Lambda_{H P P}(t)=\lambda t$ that coincides with $\Lambda(t)$ at $t=$ $T$, i.e., $\Lambda_{H P P}(T)=\Lambda(T),-$ see Figure 1.

Then, for a given time interval $[0, T]$ the GT coefficient is defined as

$$
\begin{equation*}
C(T)=1-\frac{\int_{0}^{T} \Lambda(t) d t}{0.5 T \Lambda(T)}=1-\frac{2 \int_{0}^{T} \Lambda(t) d t}{T \Lambda(T)} \tag{1}
\end{equation*}
$$



Figure 1. Graphical interpretation of GT coefficient for a point process with an increasing ROCOF.

The smaller the absolute value of the GT coefficient, the closer the considered PP is to the HPP; clearly, for the HPP, $C(T)=0$. GT coefficient satisfies the following inequality: $-1<C(T)<1$. It is obvious that for a PP with an increasing ROCOF, the GT coefficient is positive and for a PP with a decreasing ROCOF, the coefficient is negative. One can also show that the absolute value of GT coefficient $C(T)$ is proportional to the mean distance between the $\Lambda(t)$ curve and the CIF of the HPP.

For the most popular NHPP model - the power law model with the underlying Weibull CDF - the GT coefficient is expressed in a closed form:

$$
\begin{equation*}
C=1-\frac{2}{\beta+1}, \tag{2}
\end{equation*}
$$

where $\beta$ is the shape parameter of the underlying Weibull distribution.
Some examples of applying the GT coefficient to other PP commonly used in reliability and risk analysis are given in Table 1.

Table 1. GT coefficients of some PP over time interval [0, 2].
Weibull with scale parameter $\alpha=1$ is used as the underlying distribution.

| Stochastic <br> Point <br> Process | Shape parameter <br> of Underlying <br> Weibull Distribution | Repair <br> Effectiveness <br> Factor | GT <br> Coefficient |
| :---: | :---: | :---: | :---: |
| HPP | 1 | N/A | 0 |
| NHPP | 1.1 | 1 | 0.05 |
| NHPP | 2 | 1 | 0.33 |
| NHPP | 3 | 1 | 0.50 |
| RP | 2 | 0 | 0.82 |
| GRP | 2 | 0.5 | 0.21 |

Note: the GT coefficient for RP and GRP was obtained using numerical techniques.
Repair effectiveness factor in Table 1 refers to the degree of restoration upon the failure of a repairable system; see (Kijima \& Sumita, 1986), (Kaminskiy \& Krivtsov, 1998). This factor equals zero for an RP, one - for an NHPP and is greater-or-equal-to zero - for a GRP (of which the RP and the NHPP are the particular cases).

## III. GT COEFFICIENT FOR NON-REPAIRABLE SYSTEMS (COMPONENTS)

Consider a non-repairable system (component) whose TTF distribution belongs to the class of the IFR distributions. Denote the failure rate or the hazard function associated with this distribution by $h(t)$. The respective cumulative hazard function is then

$$
H(t)=\int_{0}^{t} h(\tau) d \tau
$$

and is concave upward - see Figure 2.
Consider time interval $[0, T]$. The cumulative hazard function at $T$ is $H(T)$, the respective CDF is $F(T)$ and the reliability function is $R(T)$. Now, introduce $h_{\text {eff }}$, as the failure rate of the exponential distribution with the CDF equal to the CDF of interest at the time $t=T$, i.e.,

$$
h_{e f f}(T)=-\frac{\ln (1-F(T))}{T}
$$

In other words, the introduced exponential distribution with parameter $h_{\text {eff, }}$, at $t=T$, has the same value of the cumulative hazard function as the IFR distribution of interest, see Figure 2.


Figure 2. Graphical interpretation of the GT coefficient for an IFR distribution.

The GT coefficient, $C(T)$, is then introduced as

$$
\begin{equation*}
C(T)=1-\frac{\int_{0}^{T} H(t) d t}{0.5 T h_{e f f}(T) T}=1-\frac{2 \int_{0}^{T} H(t) d t}{T H(T)}=1-\frac{2 \int_{0}^{T} \ln (R(t)) d t}{T \ln (R(t))} \tag{3}
\end{equation*}
$$

In terms of Figure $2, C(T)$, is defined as one minus the ratio of areas $A$ and $A+B$. It is easy to check that the above expression also holds for the decreasing failure rate (DFR) distributions, for which $H(t)$ is concave downward.

It is clear that $C(T)$ satisfies the following inequality: $-1<C(T)<1$. The coefficient is positive for the IFR distributions, negative - for the DFR distributions and is equal to zero for the constant failure rate (CFR), i.e., exponential distribution. Note that the suggested coefficient is, in a sense, distribution-free.

## A. GT Coefficient for the Weibull Distribution

For some TTF distributions, the GT coefficient can be expressed in a closed form. For example, in the most important (in the reliability context) case of the Weibull distribution with the scale parameter $\alpha$ and the shape parameter $\beta$, and the CDF of the form:

$$
F(t)=1-\exp \left(-\left(\frac{t}{\alpha}\right)^{\beta}\right)
$$

the GT coefficient can be found as

$$
\begin{equation*}
C=1-\frac{2}{\beta+1} \tag{4}
\end{equation*}
$$

It's worth noting that in this case, the GT coefficient depends neither on the scale parameter $\alpha$, nor on time interval T. Also note that (4) is exactly the same as (2). This is because NHPP's CIF is formally equal to the cumulative hazard function of the underlying failure time distribution; see, e.g., (Krivtsov, 2007).
Interestingly, $C(\beta)=-C\left(\frac{1}{\beta}\right)$, which is illustrated by Table 2 .

Table 2. GT coefficient for Weibull Distribution as Function of Shape Parameter $\beta$.

| Shape Parameter $\beta$ | GT Coefficient | TTF Distribution |
| :---: | :---: | :---: |
| 5 | $0.6(6)$ | IFR |
| 4 | 0.6 | IFR |
| 3 | 0.5 | IFR |
| 2 | $0.3(3)$ | IFR |
| 1 | 0 | CFR |
| 0.5 | $-0.3(3)$ | DFR |
| 0.3 | -0.5 | DFR |
| 0.25 | -0.6 | DFR |
| 0.2 | $-0.6(6)$ | DFR |

## B. GT Coefficient for the Gamma Distribution

Although not as popular as the Weibull distribution, the gamma distribution still has many important reliability applications. For example, it is used to model a standby system consisting of $k$ identical components with exponentially distributed times to failure; the gamma distribution is also the conjugate prior distribution in Bayesian estimation of the exponential distribution.

Let's consider the gamma distribution with the CDF given by

$$
F(t)=\frac{1}{\Gamma(k)} \int_{0}^{\lambda t} \tau^{k-1} e^{-\tau} d \tau=I(k, \lambda t)
$$

where $k>0$ is the shape parameter, $1 / \lambda>0$ is the scale parameter, and $I(k, x)=\int_{0}^{x} y^{k-1} e^{-y} d y$ is the incomplete gamma function. Similar to the Weibull distribution, the gamma distribution has the IFR, if the shape parameter $k$ $>1$; DFR, if $k<1$, and CFR, if $k=1$.

Using definition (3), the GT coefficient for the gamma distribution can be written as

$$
C(T)=1-\frac{2 \int_{0}^{T} \ln (1-I(k, \lambda \tau)) d \tau}{T \ln (1-I(k, \lambda T)}
$$

Table 3 displays $C(T)$ for the gamma distribution with $\lambda=1$ evaluated at $T=1$.

| Table 3. GT Coefficient for Gamma Distribution with $\lambda=1$ and $T=1$. |  |  |
| :---: | :---: | :---: |
| Shape Parameter $k$ | GT Coefficient | TTF Distribution |
| 5 | 0.623 | IFR |
| 4 | 0.543 | IFR |
| 3 | 0.428 | IFR |
| 2 | 0.258 | IFR |
| 1 | 0.000 | CFR |
| 0.5 | -0.196 | DFR |
| 0.3 | -0.285 | DFR |
| 0.25 | -0.338 | DFR |
| 0.2 | -0.375 | DFR |

## IV. CONCLUSIONS

We have introduced a parsimonious index that helps to assess the degree of aging or rejuvenation of a (non)repairable system. The index ranges from -1 to 1 . It is negative for the class of decreasing failure rate distributions and point processes with decreasing ROCOF and is positive for the increasing failure rate distributions and point processes with increasing ROCOF. The index can also be found useful in hypothesis testing for exponentiality of the TTF or failure inter-arrival times.

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# ANALYSIS OF ALTERNATING RENEWAL PROCESSES WITH DEPENDED COMPONENTS 

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#### Abstract

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In the terms of operational calculus the probability characteristics of direct and reverse residual renewal time of alternating renewal process, where renewal time depends on life-time, are found.


## 1. Definitions, Motivations and Formulation of the Problem

1.1 According to [1] let's consider a system which fails after random life-time X 1 and is fully renewed after the lapse of random time $\mathrm{Y}_{1}$. The renewed system again fails after random life-time $\mathrm{X}_{2}$ and is fully renewed after the lapse of random time $Y_{2}$ and so on (Fig.1). Time moments $T_{1}=X_{1}, T 2=X_{1}+Y_{1}+X_{2}, \ldots$. , when the system fails is called moments of failure or moments of 0-renewal while time moments $S_{1}=X_{1}+Y_{1}, S_{2}=X_{1}+Y_{1}+X_{2}+$ $\mathrm{Y}_{2} \ldots$, when renewals are ended are called the moments of renewal (or 1-renewal)


Fig.1. Realization of alternating renewal process

Definition. If $\left\{\mathrm{X}_{\mathrm{n}}, \mathrm{n} \geq 1\right\}$ and $\left\{\mathrm{Y}_{\mathrm{n}}, \mathrm{n} \geq 1\right\}$ are two sequences of equally distributed non-negative random variables when $\mathrm{n} \geq 1$, then sequence $\left\{\left(\mathrm{T}_{\mathrm{n}}, \mathrm{S}_{\mathrm{n}}\right), \mathrm{n} \geq 1\right\}$, as well as sequence $\left\{\left(\mathrm{X}_{\mathrm{n}}, \mathrm{Y}_{\mathrm{n}}\right), \mathrm{n} \geq 1\right\}$ are called the alternating renewal processes.

Renewal process given in definition 1 can be equivalently described with process $\{Z(t) t \geq 0\}$ using relation

$$
\mathrm{Z}(\mathrm{t})=\left[\begin{array}{ll}
0, & \text { if } \mathrm{t} \in\left[\mathrm{~T}_{\mathrm{K}}, \mathrm{~S}_{\mathrm{K}}\right] \\
1, & \text { otherwise }
\end{array}\right.
$$

as realizations of process $\left(X_{n}, Y_{n}\right)$ or $\left(T_{n}, S_{n}\right)$ are one-to-one determined by realizations of process $Z(t)$. According to definition, process $Z(t)$ gives system state in moment $t: Z(t)=1$, if the system is serviceable in moment $t$, and $\mathrm{Z}(\mathrm{t})=0$, if system is renewing (is non-serviceable) in moment t (Fig.2).

t
Fig.2. Realization of process $Z(t)$
For practical applications it is sometimes advisable to consider alternating processes $\left\{\mathrm{Y}_{1},\left(\mathrm{X}_{\mathrm{n}}+\mathrm{Y}_{\mathrm{n}}\right), \mathrm{n} \geq\right.$ $2\}$.
1.2 It is known that the existing renewal theory mainly studies alternating processes with independent components. This significantly limits its application, since in practice the problems modeled with alternating processes with depended (correlated) components arise very often and this dependence is so important that there is no possibility for its negligence.

In the well-known monograph of Cox [2], that has already become classic, such processes are not considered at all.

In two volumes by Feller on probability theory and its applications [3, 4] a rigorous mathematical investigation of the renewal processes is given. The alternating renewal processes (or renewal processes with two phases: active and passive) with independent components (phases) are also considered.

Alternating renewal processes are considered in the well-known monographs by Barlow and Proschan [5], Beichelt and Franken [1], etc. However, only in [1] the alternating renewal processes with dependent components are mentioned. The relations for different probability characteristics of alternating renewal processes with independent components are derived. It is shown that some of them are valid also if random variables $X_{n}$ and $Y_{n}$ (components of the process) are not independent and only the sums $\left(X_{n}+Y_{n}\right), n 1,2 \ldots$ are independent.

In the given work, as distinct from [2-5], the independence of $X_{n}$ and $Y_{n}$ is not required.
1.3 The sequence of random variables $\left\{\mathrm{S}_{\mathrm{n}}=\left(\mathrm{X}_{\mathrm{n}}+\mathrm{Y}_{\mathrm{n}}\right), \mathrm{n} \geq 1\right\}$ represents the ordinary renewal process, therefore all probability characteristics introduced and studied in the renewal theory are valid.

Sums $\left\{\mathrm{S}_{\mathrm{n}}=\left(\mathrm{X}_{\mathrm{n}}+\mathrm{Y}_{\mathrm{n}}\right), \mathrm{n} \geq 1\right\}$ divide time axis on regeneration cycles, at the same time, time moments $\mathrm{S}_{\mathrm{n}}, \mathrm{n}$ $\geq 1$ are the points of regeneration.

For alternating renewal processes the probabilities $\mathrm{P}\left(\mathrm{Z}(\mathrm{t})=\mathrm{i}, \mathrm{V}_{\mathrm{t}}^{(\mathrm{i})}>\mathrm{x}\right), \mathrm{P}\left\{\mathrm{Z}(\mathrm{t})=\mathrm{i}, \mathrm{R}_{\mathrm{t}}{ }^{(\mathrm{i})}>\mathrm{x}\right\}, \quad \mathrm{i}=0 ; 1$ are of essential interest, where $\mathrm{V}_{\mathrm{t}}{ }^{(1)}$ means residual life time (direct residual life time), while $\mathrm{V}_{\mathrm{t}}{ }^{(0)}$ is residual (direct) renewal time, $R_{t}^{(1)}$ is reverse residual life time and $R_{t}^{(0)}$ is reverse residual renewal time [1].

For these probabilities, in the case of independent components of alternating renewal process, the expressions are derived with the help of renewal functions for processes $\left\{\left(X_{n}+Y_{n}\right), n \geq 1\right\}$ or $\left\{Y_{1},\left(X_{n}+Y_{n}\right), n \geq\right.$ $2\}$, respectively.

On the other hand, the mentioned renewal functions are the solutions of the corresponding renewal equations [1-5]. In [1-5] the Laplace-Stielties transforms of renewal functions are obtained. Therefore, it can be supposed that analogous transforms for the mentioned probabilities are known. In particular, all the enumerated calculations are done in [1] conformably to $\mathrm{P}\left\{\mathrm{Z}(\mathrm{t})=1, \mathrm{~V}_{\mathrm{t}}^{(1)}>\mathrm{x}\right\}$.

Using the same methods, analogous results may be obtained conformably to $P^{`}\left\{Z(t)=1, R_{t}^{(1)}>x\right\}$.
It is noted in [1] that expression for $\mathrm{P}\left\{\mathrm{Z}(\mathrm{t})=1, \mathrm{~V}_{\mathrm{t}}^{(1)}>\mathrm{x}\right\}$ is also valid if random variables $\mathrm{X}_{\mathrm{n}}$ and $\mathrm{Y}_{\mathrm{n}}$ are not independent. Only the independence of $\operatorname{sums}\left(X_{n}+Y_{n}\right), n \geq 1$ is required.

On our part we will add, that after some transformations of calculations done in [1] we can receive analogous expressions for $\mathrm{P}\left\{\mathrm{Z}(\mathrm{t})=1, \mathrm{R}_{\mathrm{t}}{ }^{(1)}>\mathrm{x}\right\}$ as well, in the case of dependent $\mathrm{X}_{\mathrm{n}}$ and $\mathrm{Y}_{\mathrm{n}}, \mathrm{n} \geq 1$.

At the same time when components of alternating renewal process components are dependent, the expressions of probabilities $\mathrm{P}\left\{\mathrm{Z}(\mathrm{t})=0, \mathrm{~V}_{\mathrm{t}}^{(0)}>\mathrm{x}\right\}$ and $\mathrm{P}\left\{\mathrm{Z}(\mathrm{t})=0, \mathrm{R}_{\mathrm{t}}^{(0)}>\mathrm{x}\right\}$ are obtained neither directly with the help of the appropriate matching of renewal functions nor with the help of this probability reasoning that is used in [1] to derive the expressions of the mentioned renewal functions.

On the other hand, the latter probabilities have a considerable theoretical interest in the case of alternating renewal processes with depended (correlated) components. Besides, they have particular importance in practical applications, namely, within the framework of the structural and maintenance modeling of the technical systems.

Naturally, universally recognized scientist I. Ushakov distinguishes structural and maintenance models as one of the main directions of the modern Reliability Theory [5].
1.4 As a physical analog of alternating renewal process we can consider the above mentioned technical system if we assume that the wear and aging of the system's elements affect the system's reliability. Suppose such system fails when one of its elements loses serviceability. After such failure, as a rule, a detailed survey of the system is done and all elements that are in "critical" state by the signs of aging or wear are revealed. Repair (full renewal) of the system implies the substitution of all such elements, as well as of the failed element, with new ones. Besides, different operations are carried out directed to bring back the system to the initial state. In other words, as a result of such renewal, within the admissible errors, the renewed system can be considered as identical to the initial one.

At the same time, conformably to system life-time the quantity of worn-out and aged elements will be different. In particular, it is natural to suppose that this quantity increases with the increase of system life-time.

Thus, in such cases renewal time on average increases at the expense of the increase of the quantity of elements to be substituted.

The same can be said about the other renewal operations mentioned above.
As a result of the given reasoning we shall make a brief conclusion: the more the average life-time of the above described technical systems, the more the average renewal time or, in other words, renewal time depends on life-time.

Let's consider an example of technical system where the mentioned dependence shows up not as a result of physical and chemical processes of aging or wear but as a result of structural organization features of the standby system.

Let technical system contain main and standby elements undergoing failures.
In the system there functions continuous, reliable control of serviceability that instantly detects the failure of the main, as well as standby element. There also is one unit for replacement (switching over) and repair that immediately starts the necessary maintenance operation. After the failure of the main element it is replaced with a standby one if in this moment it is serviceable (I type failure).

If in the moment of main element failure the standby one is unserviceable and therefore is repaired, it replaces the failed main element after repair (II type failure). Thus, the functioning process of the standby system represents the sequence of regeneration cycles. Each cycle consists of life-time of the main element and downtime (idle time) equal to replacement time in case of I type failure and to the sum of durations of the remained renewal and following replacement in case of II type failure. If life-time of the main element in each cycle of regeneration denote through $\mathrm{X}_{1}, \mathrm{X}_{2} \ldots$., and idle time - through $\mathrm{Y}_{1}, \mathrm{Y}_{2}, \ldots$, then the sequence $\left\{\left(\mathrm{X}_{\mathrm{n}}, \mathrm{Y}_{\mathrm{n}}\right), \mathrm{n} \geq 1\right\}$ forms alternating renewal process. Thus, standby system is reduced to single-unit one and its subsequent functioning can be realized from this point of view.

As the duration of $n$-th idle time of such system $Y_{n}, n \geq 1$ depends on in what state is the standby element in the moment of the failure of the main one, it is easy to guess that it depends on $X_{n}, n \geq 1$. As we see, there exists an alternating renewal process with depended components and this dependence is "created" not with physicalchemical but with structural peculiarities of the considered system.

The reduction of two-unit technical system to single-unit one at different suppositions respective to statistic characteristics of life-time, replacement time and renewal time is given in [6]. Thus, alternating renewal process with depended components is "created".

Other kinds of technical systems with depended (correlated) life-time (failure time) and repair time are considered in [7, 8].

## 2. Probabilistic Analysis

Below we propose the method of determination of probabilities $\mathrm{P}\left\{\mathrm{Z}(\mathrm{t})=0, \mathrm{~V}_{\mathrm{t}}^{(0)}>\mathrm{x}\right\}$ and $\mathrm{P}\left\{\mathrm{Z}(\mathrm{t})=0, \mathrm{R}_{\mathrm{t}}{ }^{(0)}>\mathrm{x}\right\}$ in the terms of operational calculus, in the presumption, that $X_{n}$ and $Y_{n}$ are depended. Namely $Y_{n}$ depends on $X_{n}$, $\mathrm{n} \geq 1$ and this dependence is given with the conditional distribution function:

$$
\begin{equation*}
\mathrm{G}(\mathrm{t}, \mathrm{v})=\mathrm{P}\left\{\mathrm{Y}_{\mathrm{n}}<\mathrm{t} \mid \mathrm{X}_{\mathrm{n}}=\mathrm{v}\right\}, \mathrm{n} \geq 1 \tag{1}
\end{equation*}
$$

It is clear that distribution function (unconditional) of $\mathrm{Y}_{\mathrm{n}}$ is expressed through the following integral:

$$
\begin{equation*}
\mathrm{G}(\mathrm{t})=\int_{0}^{\mathrm{t}} \mathrm{G}(\mathrm{t}, \mathrm{v}) \mathrm{dF}(\mathrm{v}) \tag{2}
\end{equation*}
$$

where $\mathrm{F}(\mathrm{v})=\mathrm{P}\left\{\mathrm{X}_{\mathrm{n}}<\mathrm{v}\right\}, \mathrm{n} \geq 1$
As within the given article such notations will not be used in a different sense, in order to simplify the calculations these probabilities are denoted respectively with:

$$
\begin{align*}
& \mathrm{V}(\mathrm{t}, \mathrm{x})=\mathrm{P}\left\{\mathrm{Z}(\mathrm{t})=0, \mathrm{~V}_{\mathrm{t}}^{(0)}>\mathrm{x}\right\}  \tag{3}\\
& \mathrm{R}(\mathrm{t}, \mathrm{x})=\mathrm{P}\left\{\mathrm{Z}(\mathrm{t})=0, \mathrm{R}_{\mathrm{t}}^{(0)}>\mathrm{x}\right\} \tag{4}
\end{align*}
$$

Denote with $Q(u)$ the distribution function of sum $\left\{X_{n}+Y_{n}\right\}, n \geq 1$ :

$$
\begin{equation*}
\mathrm{Q}(\mathrm{u})=\mathrm{P}\left\{\mathrm{X}_{\mathrm{n}}+\mathrm{Y}_{\mathrm{n}}<\mathrm{u}\right\}, \quad \mathrm{n} \geq 1 \tag{5}
\end{equation*}
$$

Evidently:

$$
\begin{equation*}
\mathrm{Q}(\mathrm{u})=\int_{0}^{u} \mathrm{G}(\mathrm{u}-\mathrm{v}, \mathrm{v}) \mathrm{dF}(\mathrm{v}) \tag{6}
\end{equation*}
$$

Theorem 1. Function $\mathrm{V}(\mathrm{t}, \mathrm{x})$ is the solution of the following convolution type second order Volterra integral equation:

$$
\begin{equation*}
V(t, x)=\int_{0}^{t}[1-G(t-u+x, u)] d F(u)+\int_{0}^{t} V(t-u, x) d Q(u) \tag{7}
\end{equation*}
$$

Proof. Event $\mathrm{B}(\mathrm{t}, \mathrm{x})=\left\{\mathrm{Z}(\mathrm{t})=0, \mathrm{~V}_{\mathrm{t}}{ }^{(0)}>\mathrm{x}\right\}$ can be realized simultaneously with one of the two incompatible events: $\mathrm{A}(\mathrm{t})$ and $\overline{\mathrm{A}}(\mathrm{t})$.

1. $A(t)=\left\{\right.$ moment $t$ is covered with the first regeneration cycle of renewal process $\left.S_{n}=\left(X_{n}+Y_{n}\right), n \geq 1\right\}$;
2. $\overline{\mathrm{A}}(\mathrm{t})=\left\{\right.$ moment t is not covered with the first regeneration cycle of renewal process $\mathrm{S}_{\mathrm{n}}=\left(\mathrm{X}_{\mathrm{n}}+Y_{\mathrm{n}}\right), \mathrm{n} \geq$ $1\}$.

Simultaneous execution of events $\mathrm{A}(\mathrm{t})$ and $\mathrm{B}(\mathrm{t}, \mathrm{x})$ can be presented as: in interval $(0, \mathrm{t})$ in some moment of time $u$ system fails; renewal time is more than $t-u+x$; the probability of this event with consideration of all possible values of variable $u$ is:

$$
\begin{equation*}
\int_{0}^{t}[1-G(t-u+x)] d F(u) \tag{8}
\end{equation*}
$$

Simultaneous execution of events $\overline{\mathrm{A}}(\mathrm{t})$ and $\mathrm{B}(\mathrm{t}, \mathrm{x})$ can be presented as: in interval $(0, \mathrm{t})$ in some moment of time $u$ the first cycle of regeneration ends and event $B(t-u, x)$ is executed; the probability of this event with consideration of all possible values of variable $u$ is:

$$
\begin{equation*}
\int_{0}^{t} V(t-u, x) d Q(u) \tag{9}
\end{equation*}
$$

Sum of (8) and (9) gives the right part of (7) that proves the theorem.
Theorem 2. Function $R(t, x)$ is the solution of convolution type second order Volterra integral equation:

$$
R(t, x)= \begin{cases}\int_{0}^{t-x}[1-G(t-u, x)] d F(u)+\int_{0}^{t} R(t-u, x) d Q(u) & \text { if } x<t  \tag{10}\\ 0 & \text { if } x \geq t\end{cases}
$$

Proof. As in the initial moment of time the life time of the system begins, it is easy to guess that the probability of event $\left\{Z(t)=0, R_{t}^{(0)}>x\right\}$ for all $x \geq t$ is equal to zero.

Denote $\mathrm{C}(\mathrm{t}, \mathrm{x})=\left\{\mathrm{Z}(\mathrm{t})=0, \mathrm{R}_{\mathrm{t}}{ }^{(0)}>\mathrm{x}\right\}, \mathrm{x}<\mathrm{t}$. Event $\mathrm{C}(\mathrm{t}, \mathrm{x})$ can be realized simultaneously with one of events $A(t)$ and $\bar{A}(t)$ (see proof of theorem 1 ).

Simultaneous execution of events $\mathrm{A}(\mathrm{t})$ and $\mathrm{C}(\mathrm{t}, \mathrm{x})$ can be presented as: in interval ( $0, \mathrm{t}-\mathrm{x})$, in some moment of time $u$ the system fails; its renewal time is more than $t-u$; the probability of this event with consideration of all possible values of variable $u$ is:

$$
\begin{equation*}
\int_{0}^{t-x}[1-G(t-u, x)] d F(u) \tag{11}
\end{equation*}
$$

Simultaneous execution of events $\overline{\mathrm{A}}(\mathrm{t})$ and $\mathrm{C}(\mathrm{t}, \mathrm{x})$ can be presented as: in interval $(0, \mathrm{t})$ in some moment of time $u$ the first cycle of regeneration is ended; event $\left\{Z(t-u)=0, R_{t-u}{ }^{(0)}>x\right\}$ is executed; the probability of this event with consideration of all possible values of variable $u$ equals:

$$
\begin{equation*}
\int_{0}^{t} R(t-u, x) d Q(u) \tag{12}
\end{equation*}
$$

Sum of (11) and (12) gives the right part of (10) when $x<t$, that proves the theorem.
Applying Laplace transformation in respect to $t$ to (7) and (10) we get:

$$
\begin{equation*}
\bar{V}(s, x)=A(s, x)+\bar{Q}(s) \bar{V}(s, x) \tag{13}
\end{equation*}
$$

Here

$$
\begin{gathered}
\bar{V}(s, x)=\int_{0}^{\infty} e^{-s t} V(t, x) d t \\
A(s, x)=\int_{0}^{\infty} e^{-s t}\left(\int_{0}^{t}[1-G(t-u+x, u) d F(u)]\right) d t \\
\bar{Q}(s)=\int_{0}^{\infty} e^{-s t} d Q(u)
\end{gathered}
$$

From (7) we easily get:

$$
\begin{equation*}
\overline{\mathrm{V}}(\mathrm{~s}, \mathrm{x})=\mathrm{A}(\mathrm{~s}, \mathrm{x}) /(1-\overline{\mathrm{Q}}(\mathrm{~s})) \tag{14}
\end{equation*}
$$

Similarly, from (10) we get:

$$
\begin{equation*}
\bar{R}(s, x)=B(s, x) /(1-\bar{Q}(s)) \tag{15}
\end{equation*}
$$

Here

$$
\begin{gathered}
\bar{R}(s, x)=\int_{0}^{\infty} e^{-s t} R(t, x) d t \\
B(s, x)=\int_{0}^{\infty} e^{-s t}\left(\int_{0}^{t-x}[1-G(t-x, x)] d F(x)\right) d t
\end{gathered}
$$

As a rule, the reverse transformation of transforms (14) and (15) is rarely successful. However, all numerical characteristics of the considered events, random variables and processes can be obtained from them. Investigations in these directions will be the continuation of the given work.

## 3. Remarks and Conclusions

The main motivation for conducting this research is the statement, that residual life-time and residual renewal (repair) time are the important characteristics of renewable technical systems. Analysis of such characteristics is carried out within the framework of the renewal theory [1]. However, when life-time and renewal time of technical system are interdependent (correlated), probabilistic analysis of residual life-time and residual renewal time by the methods of classical renewal theory is difficult, if not impossible (subsection 1.2). At the same time the technical systems, in which there is no possibility for negligence of such correlation are widespread (subsections 1.2, 1.3, 1.4).In subsection 1.4 two kinds of such systems are described.

In section 2 the alternating renewal process in which the renewal time depends on the life-time is examined. Probabilistic characteristics of direct and reverse residual renewal time are studied. The method is worked out for obtaining the Laplace transforms of these characteristics (expressions (14) and (15)). Undoubtedly these results have a wide practical application.

Currently the author is working for obtaining analogous results for alternating renewal process in which the life-time depends on the previous renewal time.

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# COMPUTATION OF FAILURE/REPAIR FREQUENCY OF MULTI-STATE MONOTONE SYSTEMS 

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#### Abstract

The paper deals with calculation methods for failure and repair frequencies of multi-state monotone systems, both for the instantaneous and steady state cases. Being based on the binary representation of multi-state structure, new general formula for the failure/repair frequency is derived. This formula is used to obtain simple rules for the calculation of failure/repair frequency. In particular, the use of the algebra of dual numbers is presented.


## 1. Introduction

The failure frequency, called also the rate of occurrence of failures (ROCOF), is defined as the mean number of failures per unit time. Let $W(t)$ be the mean number of failures of an item (element or system) in timeinterval $(0, t]$. When $W(t)$ is absolutely continuous function in any finite time interval, then the failure frequency $w(t)$ is defined as the density of $W(t)$ with respect to the Lebesque measure on the real line, i.e.

$$
W(t)=\int_{0}^{t} w(s) \mathrm{d} s, w(t)=\mathrm{d} W(t) / \mathrm{d} t \text { (a.e.). }
$$

The limiting (or steady-state) failure frequency $w(\infty)$ is defined as the limiting value of $w(t)$ when $t$ tends to infinity. The failure frequency is an important reliability measure of repairable items, since it may be used to compute the expected number of failures in given interval. Furthermore, $w(\infty)$ is equal to the reciprocal of the mean time between failures, and the following well known expressions hold true:

MUT $=A(\infty) / w(\infty), \quad$ MDT $=(1-A(\infty)) / w(\infty)$,
where MUT = mean up-time, MDT = mean down time and $A(\infty)$ is the limiting (or steady state) availability of the item.

MUT and MDT are of practical importance, because they well enough characterise the reliability performance of repairable systems. Furthermore, these indices are often included into customer's requirements for reliability of newly designed systems, typically in industry and military areas. Therefore, calculations of MUT and MDT are needed during design and development phase in order to check if the requirements are met. According to the equation above, it is therefore important to calculate not only system availability, but also system failure frequency.

The repair (or restoration) frequency $v(t)$ of an item is defined similarly as the failure frequency, by replacing failures with restorations (i.e. completion of repairs) of the item. That is, by integrating $v(t)$ over given time-interval $[a, b]$, we obtain the mean number of restorations of the item in that interval.

Considerable efforts have been devoted to the problem of finding the efficient calculation methods for the failure/repair frequency of binary monotone systems composed of independent binary components. See Amari (2000, 2002), Chang et al. (2004), Pavlov \& Ushakov (1989), Schneeweiss (1999) and the references given therein.

The main objective of these researches was to obtain simple rules for transforming expressions of system availability/unavailability given in terms of element availability and unavailability into an expression for system failure frequency, and system repair frequency as well, both for time-dependent and steady-state cases.

In many real-life situations, however, the systems and their elements are capable of assuming a whole range of performance levels, varying from perfect functioning to the complete failure. A multi-state system (MSS) fails if its performance level is less than the desired performance level (demand). Beginning from the middle of 70s, the theory of binary systems is being replaced by the theory of MSS. The present state-of-art of the theory and practice of MSS may be found in recent monographs Kołowrocki (2004), Kuo and Zuo (2003), Levitin (2005), and Lisnianski and Levitin (2003).

In opposite to the binary case, rather little attention has been devoted to finding practical methods for computation of the frequency-type indices for MSS. Main results have been obtained by Murchland (1975), where very general relations for the computation of failure frequency and related indices were given. Similar relations were considered in Aven and Jensen (1999), Natvig and Streller (1984) and Franken et al. (1984) for the steadystate case of multi-state monotone systems (MMS). However the expressions obtained are stated in general form which is not very convenient for practical purpose due to its complexity. Another approach, based on the inclusionexclusion principle applied to the set of prime implicants of an MSS was suggested by Bossche (1984, 1986). This approach has however big computational complexity. None of the results mentioned so far has the form of simple rules converting availability expression to failure frequency expression, as in binary case.

The main aim of this paper is to show how to calculate the failure/repair frequency of multi-state systems using conversion rules being generalizations of the rules known from the binary systems. These multi-state conversion rules are obtained using a new general formula for the failure/repair frequency of MMS, which has very simple form. The presentation of these results is based on recent works of Korczak (2006, 2007), with some improvements. Moreover, is shown that the calculation can performed using the algebra of dual numbers.

## 2. Basic definitions and assumptions

### 2.1. Multi-state monotone systems and their binary representations

Let $<\boldsymbol{C}, \mathbf{K}, \mathbf{K}_{1}, \ldots, \mathbf{K}_{n}, \varphi>$ be a multi-state system consisting of $n$ multi-state elements with the index set $\boldsymbol{C}=$ $\{1,2, \ldots, n\}$, where $\mathbf{K}=\{g(0), g(1), \ldots, g(M)\} \subseteq[0,+\infty)$ is the set of the system states, $\mathbf{K}_{i}=\left\{g_{i}(0), g_{i}(1), \ldots\right.$, $\left.g_{i}\left(M_{i}\right)\right\} \subseteq[0,+\infty)$ is the set of the states of element $i \in \boldsymbol{C}$, and $\varphi: \mathbf{V} \rightarrow \mathbf{K}$ is the system structure function, where $\mathbf{V}=$ $\mathbf{K}_{1} \times \mathbf{K}_{2} \times \ldots \times \mathbf{K}_{n}$ is the space of element state vectors. We assume that the states of the system [element $i$ ] represent successive performance rates ranging from the perfect functioning level $g(M)\left[g_{i}\left(M_{i}\right)\right]$ down to the complete failure level $g(0)$ [ $\left.g_{i}(0)\right]$, that is $0 \leq g(0)<g(1)<\ldots<g(M)$ and $0 \leq g_{i}(0)<g_{i}(1)<\ldots<g_{i}\left(M_{i}\right)$. The system is a multi-state monotone system (MMS) if its structure function $\varphi$ is non-decreasing in each argument, $\varphi(\mathbf{g}(\mathbf{0}))=g(0)$ and $\varphi(\mathbf{g}(\mathbf{M}))=g(M)$, where $\mathbf{g}(\mathbf{0})=\left(g_{1}(0), g_{2}(0), \ldots, g_{n}(0)\right), \mathbf{g}(\mathbf{M})=\left(g_{1}\left(M_{1}\right), g_{2}\left(M_{2}\right), \ldots, g_{n}\left(M_{n}\right)\right)$. We refer to Kuo and Zuo (2003), Levitin (2005) and Lisnianski and Levitin (2003) for detailed description and numerous examples of MMS. Throughput the paper, we will consider MMS only.

A vector $\mathbf{y}=\left(y_{1}, y_{2}, \ldots, y_{n}\right) \in \mathbf{V}$ is said to be a path $[$ cut $]$ vector to level $c \in \mathbf{K}$ of an MMS if $\varphi(\mathbf{y}) \geq c[\varphi(\mathbf{y})<c]$. It is called a minimal path [cut] vector to level $c$ if in addition $\mathbf{x}<\mathbf{y}[\mathbf{x}>\mathbf{y}]$ implies $\varphi(\mathbf{x})<c[\varphi(\mathbf{x}) \geq c]$, where $\mathbf{x}<\mathbf{y}$ means $x_{i} \leq y_{i}$ for $i=1, \ldots, n$, and $x_{i}<y_{i}$ for some $i$. The set of all minimal path [cut] vectors to level $c$ is denoted by $\mathbf{U}_{c}$ $\left[\mathbf{L}_{c}\right]$, where $\mathbf{U}_{g(0)}=\{\mathbf{g}(\mathbf{0})\}$ and $\mathbf{L}_{g(0)}=\varnothing$.

The state (performance level) of element $i$ at time $t$ is represented by a (random) variable $X_{i}(t)$, which takes values in $\mathbf{K}_{i}$. The state (performance level) $X(t)$ of the system at time $t$ is fully determined by the states of the elements through the multi-state structure function $\varphi$, i.e., $X(t)=\varphi(\mathbf{X}(t))$, where $\mathbf{X}(t)=\left(X_{1}(t), X_{2}(t), \ldots, X_{n}(t)\right)$.

Let us introduce level indicator processes $X_{i}(e, t)=\mathbf{1}\left(X_{i}(t) \geq e\right)$ and $X(d, t)=\mathbf{1}(X(t) \geq d), e, d \geq 0$, where $\mathbf{1}($.$) is the$ indicator function. Let $\varphi_{d}=\mathbf{1}(\varphi \geq d), d \in \mathbf{K}-\{0\}$, be the system level indicators. They can be considered as functions of vector of binary variables $\underline{\underline{\mathbf{X}}}(t)=\left[X_{i}(r, t)\right.$ : $\left.i \in \boldsymbol{C}, r \in \mathbf{K}_{i}-\left\{g_{i}(0)\right\}\right]$, so that $\varphi_{d}(\underline{\underline{\mathbf{X}}}(t))=X(d, t)$, resulting in the binary representation of MMS; see Block and Savits (1982), Korczak (2005) and Lisnianski and Levitin (2003) for more details.

From the definition of minimal path and minimal cut vectors, we obtain so-called, min-path and min-cut forms:

$$
\begin{equation*}
\varphi_{d}(\underline{\underline{\mathbf{X}}}(t))=\max _{\mathbf{y} \in \mathbf{U}_{d}} \min _{i \in C: y_{i}>g_{i}(0)} X_{i}\left(y_{i}, t\right), \quad \varphi_{d}(\underline{\underline{\mathbf{X}}}(t))=\min _{\mathbf{z} \in \mathbf{L}_{d}} \max _{i \in C: z_{i}<g_{i}\left(M_{i}\right)} X_{i}\left(z_{i} \oplus_{i} 1, t\right) \tag{2.1}
\end{equation*}
$$

where $r \oplus_{i} 1=\min \left(\mathbf{K}_{i} \cap(r, \infty)\right)$, for $r \in \mathbf{K}_{i}-\left\{g_{i}\left(M_{i}\right)\right\}$, is the next state in $\mathbf{K}_{i}$ better than state $r$.
There are several algebraic forms of $\varphi_{d}$, which can be obtained from (2.1) using inclusion-exclusion, SDP or other methods, see Korczak (2005, 2007). For example, the pseudo-polynomial form is given by:

$$
\begin{equation*}
\varphi_{d}(\underline{\underline{\mathbf{X}}}(t))=\beta_{0}+\sum_{k=1}^{m} \beta_{k} B_{k}(\underline{\underline{\mathbf{X}}}(t)), \tag{2.2}
\end{equation*}
$$

where

$$
\begin{equation*}
B_{k}(\underline{\underline{\mathbf{X}}}(t))=\prod_{i \in C}\left(X_{i}(a(k, i), t)-X_{i}(b(k, i), t)\right), \tag{2.3}
\end{equation*}
$$

$\beta_{k}$ are integer coefficients, $a(k, i), b(k, i) \in \mathbf{K}_{i} \cup\left\{g_{i}\left(M_{i}\right)+1\right\}, a(k, i)<b(k, i)$ for all $i$ and $k$, and the products $B_{k}$ are nontrivial. The term $X_{i}(a(k, i), t)-X_{i}(b(k, i), t)$ reduces to $X_{i}(a(k, i), t)$ if $b(k, i)=g_{i}\left(M_{i}\right)+1$, to $1-X_{i}(b(k, i), t)$ if $a(k, i)=g_{i}(0)$, and to 1 , if $b(k, i)=g_{i}\left(M_{i}\right)+1$ and $a(k, i)=g_{i}(0)$.

When reliability structure of a binary system is complex, the Shannon decomposition is frequently used to simplify the structure. The corresponding multi-state Shannon decomposition (or pivotal decomposition, or factoring) formulae are:

$$
\begin{align*}
& \varphi_{d}(\underline{\underline{\mathbf{X}}}(t))=\sum_{r \in \mathbf{K}_{i}}\left(X_{i}(r, t)-X_{i}\left(r \oplus_{i} 1, t\right)\right) \varphi_{d}\left(\underline{\mathbf{e}}_{i}(r), \underline{\underline{\mathbf{X}}}(t)\right), \\
&=\varphi_{d}\left(\underline{\mathbf{e}}_{i}\left(g_{i}(0)\right), \underline{\underline{\mathbf{X}}}(t)\right)+\sum_{r \in \mathbf{K}_{i}} X_{i}(r, t)\left[\varphi_{d}\left(\underline{\mathbf{e}}_{i}(r), \underline{\underline{\mathbf{X}}}(t)\right)-\varphi_{d}\left(\underline{\mathbf{e}}_{i}\left(r-{ }_{i} 1\right), \underline{\underline{\mathbf{X}}}(t)\right)\right], \tag{2.4}
\end{align*}
$$

where $\left((r)_{i}, \mathbf{X}(t)\right)=\left(X_{1}(t), \ldots, X_{i-1}(t), r, X_{i+1}(t), \ldots, X_{n}(t)\right), \quad \underline{\mathbf{e}}_{i}(r)=\left(\mathbf{1}(u \leq r): u \in \mathbf{K}_{i}-\left\{g_{i}(0)\right\}\right), r \in \mathbf{K}_{i}, g_{i}\left(M_{i}\right) \oplus_{i} 1=$ $g_{i}\left(M_{i}\right)+1$, and $r-_{i} 1=\max \left(\mathbf{K}_{i} \cap(-\infty, r)\right)$, for $r \in \mathbf{K}_{i}-\left\{g_{i}(0)\right\}$, is the best state preceding state $r$, and $g_{i}(0)-{ }_{i} 1=g_{i}(0)$, so that $\underline{\mathbf{e}}_{i}\left(g_{i}(0)-_{i} 1\right)=\underline{\mathbf{e}}_{i}\left(g_{i}(0)\right)$. Observe that for $r=g_{i}(k), \underline{\mathbf{e}}_{i}(r)=\underline{\mathbf{e}}_{i}\left(g_{i}(k)\right)=(\underbrace{1, \ldots,}_{k}, \underbrace{0, \ldots, 0}_{M_{i}-k}), \underline{\mathbf{e}}_{i}\left(g_{i}(0)\right)=$ $(\underbrace{0, \ldots, 0}_{M_{i}}), \underline{\mathbf{e}}_{i}\left(g_{i}\left(M_{i}\right)\right)=(\underbrace{1, \ldots, 1}_{M_{i}})$.

Note that $\varphi\left((r)_{i}, \mathbf{x}\right)$ is an extended structure function (i.e. it can be degenerated), taking its values in the set $\left\{\varphi\left((r)_{i}, \mathbf{g}(\mathbf{0})\right), \ldots, \varphi\left((r)_{i}, \mathbf{g}(\mathbf{M})\right)\right\}$. When $r>g_{i}(0)$, it may happen that $\varphi\left((r)_{i}, \mathbf{g}(\mathbf{0})\right)>g(0)=\varphi(\mathbf{g}(\mathbf{0}))$. However, if $\varphi\left((r)_{i}, \mathbf{g}(\mathbf{0})\right)<\varphi\left((r)_{i}, \mathbf{g}(\mathbf{M})\right)$, then $\varphi\left((r)_{i}, \mathbf{x}\right)$ fits our definition of MMS with the lowest performance levels being not necessarily 0 .

Unless otherwise stated, we make the following assumptions regarding stochastic properties of the elements of an MMS.
Assumption 2.1. The system's elements, that is, the stochastic processes $\left\{X_{i}(t)\right\}, i \in \boldsymbol{C}$, are mutually s-independent.

Assumption 2.2. $\left\{X_{i}(t)\right\}, i \in \boldsymbol{C}$, are regular jump processes, i.e.: have jump right-continuous sample paths with leftside limits, and have finite expected number of jumps in bounded intervals.

For any $r, s \in \mathbf{K}_{i}, r \neq s$, let $N_{i}^{r \rightarrow s}(t)$ be the number of transitions of element $i$ from its state $r$ to its state $s$ in time interval $(0, t]$. Its expected value is denoted by $W_{i}^{r \rightarrow s}(t)=E\left[N_{i}^{r \rightarrow s}(t)\right]$. When $W_{i}^{r \rightarrow s}(t)$ is locally absolutely continuous on $[0, \infty)$, its density $w_{i}^{r \rightarrow s}(t)$ is called the instantaneous frequency of transitions from $r$ to $s$ :

$$
\begin{equation*}
W_{i}^{r \rightarrow s}(t)=\int_{0}^{t} w_{i}^{r \rightarrow s}(s) \mathrm{d} s, \quad w_{i}^{r \rightarrow s}(t)=\mathrm{d} W_{i}^{r \rightarrow s}(t) / \mathrm{d} t \tag{2.5}
\end{equation*}
$$

Assumption 2.3. (transient, or instantaneous case) For any $i \in \boldsymbol{C}$, all the functions $W_{i}^{r \rightarrow s}(t), r, s \in \mathbf{K}_{i}, r \neq s$, are locally absolutely continuous on $[0, \infty)$, i.e. the interstate frequencies $w_{i}^{r \rightarrow s}(t)$ exist.

Let $p_{i}(r ; t)=\operatorname{Pr}\left\{X_{i}(t)=r\right\}$. The steady state frequency $w_{i}^{r \rightarrow s}(\infty)$ of transitions of element $i$ from its state $r$ to its state $s$ and the steady state probability $p_{i}(r ; \infty)$ that element $i$ is in $r$ are defined by:

$$
\begin{equation*}
w_{i}^{r \rightarrow s}(\infty)=\lim _{t \rightarrow \infty} w_{i}^{r \rightarrow s}(t), \quad p_{i}(r ; \infty)=\lim _{t \rightarrow \infty} p_{i}(r ; t) \tag{2.6}
\end{equation*}
$$

Assumption 2.4. (steady state, or asymptotic case) For any $i \in \boldsymbol{C}$ and $r, s \in \mathbf{K}_{i}, r \neq s$, the steady state frequencies $w_{i}^{r \rightarrow s}(\infty)$ and steady state probabilities $p_{i}(r ; \infty)$ exist.

From assumptions 2.1 and 2.3, it follows that the processes $\left\{X_{i}(t)\right\}, i \in \boldsymbol{C}$, have no common jump times with probability 1. And in consequence, any change of the system's state is caused, with probability one, by a jump of exactly one element.

### 2.2. Basic reliability measures of MMS

For an MMS, one can define various reliability and performance measures, see Aven and Jensen (1999), Levitin (2005), and Lisnianski and Levitin (2003). In this paper we will consider basic reliability indices only, which, however, may be used to calculate many other measures. For any fixed performance level $d, g(0)<d \leq g(M)$, we define the system reliability measures like for binary systems, considering the sets $\mathbf{G}(d)=\mathbf{K} \cap[d, \infty)$ and $\mathbf{F}(d)=$ $\mathbf{K}-\mathbf{G}(d)$ as up and down states respectively. The system availability to level $d$ (or to demand $d$ ) is defined as $A(d, t)$ $=\operatorname{Pr}\{X(t) \geq d\}=E[X(d, t)]=E\left[\varphi_{d}(\underline{\underline{\mathbf{X}}}(t))\right]$. The system unavailability to level $d$ (or to demand $d$ ) is defined as $U(d, t)=\operatorname{Pr}\{X(t)<d\}=1-A(d, t)$. A transition from $\mathbf{G}(d)$ to $\mathbf{F}(d)$ is called $d$-failure, and the reverse transition is called $d$-repair. The instantaneous failure [repair] frequency to level $d$ (shortly: $d$-failure [repair] frequency) is denoted by $w(d, t)$ [ $v(d, t)]$ and defined as the density of the function $W(d, t)[V(d, t)]$, the expected number of $d$ failures [ $d$-repairs] in ( $0, t$ ], i.e.:

$$
\begin{equation*}
W(d, t)=\int_{0}^{t} w(d, s) d s, \quad V(d, t)=\int_{0}^{t} v(d, s) d s \tag{2.7}
\end{equation*}
$$

We set $A(g(0), t) \equiv U(u, t) \equiv 1$ and $w(g(0), t) \equiv w(u, t) \equiv A(u, t) \equiv U(g(0), t) \equiv 0$ for $u>g(M)$. Binary-like reliability indices of the system elements are defined similarly, and are denoted as $A_{i}(u, t), U_{i}(u, t), w_{i}(u, t)$ and $v_{i}(u, t)$
for $i \in \boldsymbol{C}$ and $g_{i}(0)<u \leq g_{i}\left(M_{i}\right)$, with $A_{i}\left(g_{i}(0), t\right) \equiv U_{i}(s, t) \equiv 1, w_{i}\left(g_{i}(0), t\right) \equiv w_{i}(s, t) \equiv A_{i}(s, t) \equiv U_{i}\left(g_{i}(0), t\right) \equiv 0$ for $s>g_{i}\left(M_{i}\right)$. They can be calculated from known state probabilities $p_{i}(r ; t)$ and interstate frequencies $w_{i}^{r \rightarrow s}(t)$ :

$$
\begin{equation*}
A_{i}(u, t)=\sum_{\substack{r \in \mathbf{K}_{i} \\ r \geq u}} p_{i}(r ; t), \quad w_{i}(u, t)=\sum_{\substack{r, s \in \mathbf{K} \\ r \leq u \\ s<u, r \geq u}} w_{i}^{r \rightarrow s}(t), \quad v_{i}(u, t)=\sum_{\substack{r, s \in \mathbf{K}_{i} \\ s<u, r \geq u}} w_{i}^{s \rightarrow r}(t) . \tag{2.8}
\end{equation*}
$$

The steady state (or limiting, asymptotic) reliability indices are defined as the limiting values of the corresponding instantaneous indices, by letting $t \rightarrow \infty$, if the limits exist. In steady state, the failure and repair frequencies of system, and each of its element as well, are equivalent:

$$
\begin{equation*}
w(d, \infty)=v(d, \infty), \quad w_{i}(r, \infty)=v_{i}(r, \infty) . \tag{2.9}
\end{equation*}
$$

The steady state system failure frequency is important for applications, since under rather mild assumptions, see Cocozza-Thivent (1997) and Cocozza-Thivent and Roussignol (2000) (for example when system's elements are modelled by irreducible time-continuous Markov chains, or by its functions), we have the following familiar relations:

$$
\begin{equation*}
\operatorname{MUT}(d)=A(d, \infty) / w(d, \infty), \quad \operatorname{MDT}(d)=U(d, \infty) / w(d, \infty), \tag{2.10}
\end{equation*}
$$

where $\operatorname{MUT}(d)[\operatorname{MDT}(d)]$ is the mean up-time [down-time] to level $d$ of the system.
In the simplest case, when the stochastic evolution of element $i$ is described by a homogeneous timecontinuous Markov chain with transition rate matrix $\left[\lambda_{i}(r, s): r, s \in \mathbf{K}_{i}\right]$, we have:

$$
\begin{equation*}
w_{i}^{r \rightarrow s}(t)=p_{i}(r ; t) \lambda_{i}(r, s), \tag{2.11}
\end{equation*}
$$

where $t \geq 0$ or $t=\infty$ (for the limiting case).

### 2.2. System availability calculation

For any fixed $i \in \mathrm{C}$, stochastic processes $\left\{X_{i}(e, t)\right\}, e \in \mathbf{K}_{\mathrm{i}}-\left\{g_{i}(0)\right\}$, are dependent, as $1 \geq X_{i}\left(g_{i}(1), t\right) \geq$ $X_{i}\left(g_{i}(2), t\right) \geq \ldots \geq X_{i}\left(g_{i}\left(M_{i}\right), t\right) \geq 0$. However, by stochastic independence of elements, the processes belonging to different elements are independent. Therefore, having $\varphi_{d}(\underline{\underline{\mathbf{X}}}(t))$ written in a suitable form, and knowing availability/unavailability of independent elements, calculation of the system availability is very easy. For example, if $\varphi_{d}(\underline{\underline{\mathbf{X}}}(t))$ is given in the form (2.2), then:

$$
\begin{equation*}
A(d, t)=\beta_{0}+\sum_{k=1}^{m} \beta_{k} B_{k}(\underline{\underline{\mathbf{A}}}(t)), \tag{2.12}
\end{equation*}
$$

where $A_{i}\left(g_{i}(0), t\right) \equiv 1, A_{i}(u, t) \equiv 0$ for $u>g_{i}\left(M_{i}+1\right)$, and

$$
\begin{equation*}
B_{k}(\underline{\underline{\mathbf{A}}}(t))=E\left[B_{k}(\underline{\underline{\mathbf{X}}}(t))\right]=\prod_{i \in C}\left(A_{i}(a(k, i), t)-A_{i}(b(k, i), t)\right) . \tag{2.13}
\end{equation*}
$$

with $\underline{\underline{\mathbf{A}}}(t)=\left[A_{i}(r, t): i \in \boldsymbol{C}, r \in \mathbf{K}_{i}-\left\{g_{i}(0)\right\}\right]$.
Applying the factoring formula (2.4), we get:

$$
\begin{gather*}
A(d, t)=\sum_{r \in \mathbf{K}_{i}}\left(A_{i}(r, t)-A_{i}\left(r \oplus_{i} 1, t\right)\right) A^{(i, r)}(d, t) \\
=A^{\left(i, g_{i}(0)\right)}(d, t)+\sum_{r \in \mathbf{K}_{i}} A_{i}(r, t)\left[A^{(i, r)}(d, t)-A^{(i, r-i)}(d, t)\right], \tag{2.14}
\end{gather*}
$$

where $A^{(i, r)}(d, t)=\operatorname{Pr}\left\{\varphi(\mathbf{X}(t)) \geq d \mid X_{i}(t)=r\right\}=\operatorname{Pr}\left\{\varphi\left((r)_{i}, \mathbf{X}(t)\right) \geq d\right\}=E\left[\varphi_{d}\left(\mathbf{e}_{i}(r), \underline{\underline{\mathbf{X}}}(t)\right)\right]$ is the availability to level $d$ of the system with indicator structure function $\varphi_{d}\left(\underline{\mathbf{e}}_{i}(r), \underline{\underline{\mathbf{X}}}(t)\right)$, or in other words, $A^{(i, r)}(d, t)$ is the availability of the system with structure function $\varphi$, given that element $i$ is strapped in state $r$.

## 3. Failure and repair frequency calculation

### 3.1. The main formula

According to general results obtained by Murchland (1975), we have:

$$
\begin{align*}
& w(d, t)=\sum_{i \in C} \sum_{\substack{r, s \in \mathbf{K}_{i} \\
r \neq s}} \operatorname{Pr}\left\{\varphi\left((r)_{i}, \mathbf{X}(t)\right) \geq d, \varphi\left((s)_{i}, \mathbf{X}(t)\right)<d\right\} w_{i}^{r \rightarrow s}(t)  \tag{3.1}\\
& v(d, t)=\sum_{i \in C} \sum_{\substack{r, s \in \mathbf{K}_{i} \\
r=s}} \operatorname{Pr}\left\{\varphi\left((r)_{i}, \mathbf{X}(t)\right) \geq d, \varphi\left((s)_{i}, \mathbf{X}(t)\right)<d\right\} w_{i}^{s \rightarrow r}(t) \tag{3.2}
\end{align*}
$$

for both monotone and non-monotone systems. For monotone systems considered in this paper we have $\operatorname{Pr}\left\{\varphi\left((r)_{i}, \mathbf{X}(t)\right) \geq d, \varphi\left((s)_{i}, \mathbf{X}(t)\right)<d\right\}=\mathbf{1}(r>s) \cdot\left(A^{(i, r)}(d, t)-A^{(i, s)}(d, t)\right)$, hence these general expressions reduce to the following:

$$
\begin{align*}
& w(d, t)=\sum_{i \in C} \sum_{\substack{r, s \in \mathcal{K}_{i} \\
\text { rrs }}}\left(A^{(i, r)}(d, t)-A^{(i, s)}(d, t)\right) w_{i}^{r \rightarrow s}(t)  \tag{3.3}\\
& v(d, t)=\sum_{i \in C} \sum_{\substack{r, s \in \in_{i} \\
r>s}}\left(A^{(i, r)}(d, t)-A^{(i, s)}(d, t)\right) w_{i}^{s \rightarrow r}(t) \tag{3.4}
\end{align*}
$$

The main disadvantages of these formulae are that they depend on a number of element's inter-state transition frequencies, and that the format of input data $\left\{w_{i}^{r \rightarrow s}(t)\right\}$ is different from the format of output data $\{w(d, t), v(d, t)\}$. As a result, recursive application of these formulae for complex systems with hierarchical structure is difficult, or even impossible. More convenient are formulae stated in terms of element's failure/repair frequencies, $w_{i}(r, t)$ and $v_{i}(r, t)$. Observe that for $r>s, r, s \in \mathbf{K}_{i}$ :

$$
\begin{equation*}
A^{(i, r)}(d, t)-A^{(i, s)}(d, t)=\sum_{\substack{u \in \in_{i}: \\ s<u \leq r}}\left(A^{(i, u)}(d, t)-A^{\left(i, u u_{i}\right)}(d, t)\right) . \tag{3.5}
\end{equation*}
$$

Substituting (3.5) into (3.3) and (3.5), interchanging the order of summation and using relations (2.8), we obtain:

$$
\begin{align*}
& w(d, t)=\sum_{i \in \boldsymbol{C}} \sum_{r \in \mathbf{K}_{i}-\left\{g_{i}(0)\right\}}\left[A^{(i, r)}(d, t)-A^{\left(i, r_{i}\right)}(d, t)\right] w_{i}(r, t),  \tag{3.6}\\
& v(d, t)=\sum_{i \in \boldsymbol{C}} \sum_{r \in \mathbf{K}_{i}-\left\{g_{i}(0)\right\}}\left[A^{(i, r)}(d, t)-A^{\left(i, r_{-i}\right)}(d, t)\right] v_{i}(r, t) . \tag{3.7}
\end{align*}
$$

According to (2.14),

$$
\begin{equation*}
A^{(i, r)}(d, t)-A^{\left(i, r_{-}\right)}(d, t)=\frac{\partial A(d, t)}{\partial A_{i}(r, t)}, \tag{3.8}
\end{equation*}
$$

where we consider any $U_{i}(r, t)$ appearing in the expression for $A(d, t)$ as $1-A_{i}(r, t)$, so that $\partial U_{i}(r, t) / \partial A_{i}(r, t)=$ $\partial\left(1-A_{i}(r, t)\right) / \partial A_{i}(r, t)=-1$. Thus we have proved the following main result:

$$
\begin{align*}
& w(d, t)=\sum_{i \in C} \sum_{r \in \mathbf{K}_{i}-\left\{g_{i}(0)\right\}} w_{i}(r, t) \frac{\partial A(d, t)}{\partial A_{i}(r, t)},  \tag{3.9}\\
& v(d, t)=\sum_{i \in C} \sum_{r \in \mathbf{K}_{i}-\left\{g_{i}(0)\right\}} v_{i}(r, t) \frac{\partial A(d, t)}{\partial A_{i}(r, t)} . \tag{3.10}
\end{align*}
$$

Now the input and output data are of the same format as the failure/repair frequencies. Moreover, the expressions obtained are easy to remember, and are very similar to the formulae known from the binary system theory.

Since the expressions for $w(d, t)$ and $v(d, t)$ are similar, we will restrict our consideration to $w(d, t)$. Repair frequency formulae can be obtained from failure frequency formulae by replacing $w$ with $v$. Furthermore, for sake of brevity, we will time parameter $t$ in what follows. Thus we will write:
$A_{i}(r), U_{i}(r), w_{i}(r), A(d), U(d), w(d)$ instead of $A_{i}(r, t), U_{i}(r, t), w_{i}(r, t), A(d, t), U(d, t), w(d, t)$.
Let us consider some alterative forms of (3.9). When the system unavailability $U(d)$ is given as a function of $U_{i}(r)$ and $A_{i}(r)$, then:

$$
\begin{equation*}
w(d, t)=\sum_{i \in C} \sum_{r \in \mathbf{K}_{i}-\left\{g_{i}(0)\right\}} w_{i}(r) \frac{\partial U(d)}{\partial U_{i}(r)}, \tag{3.11}
\end{equation*}
$$

where we consider any $A_{i}(r)$ appearing in the expression for $U(d)$ as $1-U_{i}(r)$, so that $\partial A_{i}(r) / \partial U_{i}(r)=$ $\partial\left(1-U_{i}(r)\right) / \partial U_{i}(r)=-1$.

When $U_{i}(r)$ and $A_{i}(r)$ appearing in the expression for $A(d)$ or $U(d)$ are considered as independent variables, so that $\partial U_{i}(r) / \partial A_{i}(r)=\partial A_{i}(r) / \partial U_{i}(r)=0$, then according to the chain rule of differentiation, we may write (3.9) and (3.11) as:

$$
\begin{align*}
& w(d)=\sum_{i \in C} \sum_{r \in \mathbf{K}_{i}-\left\{g_{i}(0)\right\}} w_{i}(r)\left(\frac{\partial A(d)}{\partial A_{i}(r)}-\frac{\partial A(d)}{\partial U_{i}(r)}\right),  \tag{3.12}\\
& w(d)=\sum_{i \in C} \sum_{r \in \mathbf{K}_{i}-\left\{g_{i}(0)\right\}} w_{i}(r)\left(\frac{\partial U(d)}{\partial U_{i}(r)}-\frac{\partial U(d)}{\partial A_{i}(r)}\right) . \tag{3.13}
\end{align*}
$$

### 3.2. Conversion rules for the failure frequency calculation

Fairly general conversion rule that converts an availability expression of an MMS into its failure frequency expression can be described as follows. Suppose that the availability $A(d)$ is given in the following sum of products form:

$$
\begin{equation*}
A(d)=\beta_{0}+\sum_{k=1}^{L} \beta_{k} \prod_{m \in \mathbf{E}_{k}} G_{k, m}(\underline{\underline{\mathbf{A}}}) \tag{3.14}
\end{equation*}
$$

where $\mathbf{E}_{k}$ is a non-empty index set, and $G_{k, m}(\underline{\underline{\mathbf{A}}}), m \in \mathbf{E}_{k}$, are functions having no common relevant variable belonging to the same system's element, i.e. if $G_{k, m}(\underline{\underline{\mathbf{A}}})$ depends on the variable $A_{i}(r)$ (belonging to element $i$ ), then other functions $\left.G_{k, l}(\underline{\underline{\mathbf{A}}})\right), l \neq m$, do not depend on variables $A_{i}(s), s \in \mathbf{K}_{i}-\left\{g_{i}(0)\right\}$. This relevant variable disjointness property relates to each product separately. We assume that $G_{k, m}(\underline{\underline{\mathbf{A}}})$ are differentiable with respect to each variable (the derivatives being 0 for of non-relevant variable). By applying (3.9) to $A(d)$ given by (3.14), and using usual algebra and calculus, we obtain:

$$
\begin{equation*}
w(d)=\sum_{k=1}^{L} \beta_{k} \sum_{m \in \mathbf{E}_{k}}\left(\prod_{l \in \mathbf{E}_{k}-\{m\}} G_{k, l}(\underline{\underline{\mathbf{A}}})\right) \cdot w_{k, m}=\sum_{k=1}^{L} \beta_{k}\left(\prod_{m \in \mathbf{E}_{k}} G_{k, m}(\underline{\underline{\mathbf{A}}})\right) \sum_{m \in \mathbf{E}_{k}} \frac{w_{k, m}}{G_{k, m}(\underline{\underline{\mathbf{A}}})} \tag{3.15}
\end{equation*}
$$

where, by convention, $a / 0=0$ for any $a$, and

$$
\begin{equation*}
w_{k, m}=\sum_{i \in C} \sum_{r \in \mathbf{K}_{i}-\left\{g_{i}(0)\right\}} \frac{\partial G_{k, m}(\underline{\underline{\mathbf{A}})}}{\partial A_{i}(r)} \cdot w_{i}(r) . \tag{3.16}
\end{equation*}
$$

According to (3.9), when $G_{k, m}(\underline{\underline{\mathbf{A}})}$ is the availability [unavailability] to a given level of a multi-state subsystem, then $w_{k, m}\left[-w_{k, m}\right]$ is its failure frequency to this level, and, in the steady state, $w_{k, m} / G_{k, m}(\underline{\underline{\mathbf{A}}})=1 / \mathrm{MUT}_{k, m}$ $\left[-1 / \mathrm{MDT}_{k, m}\right]$ of that subsystem to the given level.

By a suitable choice of functions $G_{k, m}(\underline{\underline{\mathbf{A}})}$ in the above general rule, we may obtain several special cases, being multi-state generalizations of conversion rules known for binary systems. By applying (3.15) with $A(d)$ given by (2.12), we obtain:

$$
\begin{align*}
w(d) & =\sum_{k=1}^{m} \beta_{k} \sum_{z \in C}\left[w_{z}(a(k, z))-w_{z}(b(k, z))\right] B_{k}^{(z)}(\underline{\underline{\mathbf{A}}}) \\
& =\sum_{k=1}^{m} \beta_{k} B_{k}\left(\underline{\underline{\mathbf{A}})} \sum_{i \in C} \frac{w_{i}(a(k, i))-w_{i}(b(k, i))}{A_{i}(a(k, i))-A_{i}(b(k, i))}\right. \tag{3.17}
\end{align*}
$$

where, $a / 0=0$ for any $a$, and

$$
\begin{equation*}
B_{k}^{(z)}(\underline{\underline{\mathbf{A}}})=\prod_{i \in C-\{z\}}\left(A_{i}(a(k, i))-A_{i}(b(k, i))\right) . \tag{3.18}
\end{equation*}
$$

As a very simple example, let us consider an MMS with structure function $\varphi(\mathbf{X})=\min \left(X_{1}, \max \left(X_{2}, X_{3}\right)\right)$. Then $\left.\varphi_{d}(\underline{\underline{\mathbf{X}}})\right)=X_{1}(d)\left[1-\left(1-X_{2}(d)\right)\left(1-X_{3}(d)\right)\right]$ and:

$$
\begin{equation*}
A(d)=A_{1}(d)\left[1-U_{2}(d) U_{3}(d)\right]=A_{1}(d)-A_{1}(d) U_{2}(d) U_{3}(d) \tag{3.19}
\end{equation*}
$$

Using (3.15)-(3.16) yields:

$$
\begin{equation*}
w(d)=w_{1}(d)\left[1-U_{2}(d) U_{3}(d)\right]+A_{1}(d)\left[w_{2}(d) U_{3}(d)+U_{2}(d) w_{3}(d)\right], \tag{3.20}
\end{equation*}
$$

or, in steady state:

$$
\begin{align*}
& w(d)=A_{1}(d)\left[1-U_{2}(d) U_{3}(d)\right] / \operatorname{MUT}_{1}(d)+A_{1}(d) U_{2}(d) U_{3}(d)\left[1 / \operatorname{MDT}_{2}(d)+1 / \operatorname{MDT}_{3}(d)\right] \\
& \quad=A_{1}(d) / \operatorname{MUT}_{1}(d)-A_{1}(d) U_{2}(d) U_{3}(d)\left[1 / \operatorname{MUT}_{1}(d)-1 / \operatorname{MDT}_{2}(d)-1 / \operatorname{MDT}_{3}(d)\right] \tag{3.21}
\end{align*}
$$

As a more general example, observe that Shannon's decomposition formulae (2.14) for the availability $A(d)$ are of the form (3.14). Hence, by using (3.15) with $A(d)$ given by (2.14), we obtain at once the following Shannon's decomposition formulae for the failure frequency $w(d)$ :

$$
\begin{gather*}
w(d)=\sum_{r \in \mathbf{K}_{i}}\left[\left(w_{i}(r)-w_{i}\left(r \oplus_{i} 1\right)\right) A^{(i, r)}(d)+\left(A_{i}(r)-A_{i}\left(r \oplus_{i} 1\right)\right) w^{(i, r)}(d)\right] \\
=\sum_{r \in \mathbf{K}_{i}}\left(A_{i}(r)-A_{i}\left(r \oplus_{i} 1\right)\right) A^{(i, r)}(d)\left(\frac{w_{i}(r)-w_{i}\left(r \oplus_{i} 1\right)}{A_{i}(r)-A_{i}\left(r \oplus_{i} 1\right)}+\frac{w^{(i, r)}(d)}{A^{(i, r)}(d)}\right) \\
=w^{\left(i, g_{i}(0)\right)}(d)+\sum_{r \in \mathbf{K}_{i}} A_{i}(r)\left(A^{(i, r)}(d)-A^{\left(i, r_{i} l\right)}(d)\right)\left(\frac{w_{i}(r)}{A_{i}(r)}+\frac{w^{(i, r)}(d)-w^{(i, r-1)}(d)}{A^{(i, r)}(d)-A^{\left(i, r_{i}\right)}(d)}\right), \tag{3.22}
\end{gather*}
$$

where $w^{(i, r)}(d)$ is the failure frequency to level $d$ of the system with indicator structure function $\varphi_{d}\left(\underline{\mathbf{e}}_{i}(r), \underline{\underline{\mathbf{X}}}\right)$, or in other words, $w^{(i, r)}(d)$ is the failure frequency to level $d$ of the system with structure function $\varphi$, given that element $i$ is strapped in state $r$.

### 3.3. Application of dual numbers

A dual number is one of the form $a+\varepsilon b$, where $a$ and $b$ are real numbers and $\varepsilon$ is an algebraic (imaginary) unit having the formal property that $\varepsilon^{2}=0$. The set $\mathbb{D}$ of all dual numbers is a commutative ring with basic algebraic operations defined by:

$$
(a+\varepsilon b)+(c+\varepsilon d)=a+c+\varepsilon(b+d), \quad(a+\varepsilon b) \cdot(c+\varepsilon d)=a b+\varepsilon(a d+b c)
$$

Since $\varepsilon^{2}=0$, a pure dual number $\varepsilon d$ has no inversion, so the ring $\mathbb{D}$ is not a field (has zero divisors). However, if $c$ $\neq 0$, then $1 /(c+\varepsilon d)=1 / c-\varepsilon d / c^{2}$.

The concept of the dual number was introduced by Clifford (1873) and the name was given by Study (1903). Application areas of dual numbers include kinematics, geometry, mechanics, robotics, etc. We refer to (Angeles 1998), (Dimentberg 1978), (Fischer 1999), (Veretennikov and Sinitsyn 2006) and (Yaglom 1968, 1979) for more detailed historical account, discussion of properties and applications of dual numbers.

Observe that

$$
\begin{equation*}
\prod_{k=1}^{n}\left(x_{k}+\varepsilon y_{k}\right)=\prod_{k=1}^{n} x_{k}+\varepsilon \sum_{k=1}^{n} y_{k} \prod_{\substack{m=1 \\ m \neq k}}^{n} x_{m} . \tag{3.23}
\end{equation*}
$$

Let $f\left(x_{1}, x_{2}, \ldots, x_{n}\right)=x_{1} \cdot x_{2} \cdot \ldots \cdot x_{n}$, where $x_{1}, x_{2}, \ldots, x_{n}$ are real variables. By replacing each $x_{i}$ by dual number $x_{i}+\varepsilon y_{i}$, we obtain dual function $F$ of $n$ dual numbers. According to (3.23), we have the following representation for the function $F$ :

$$
\begin{equation*}
F(\mathbf{x}+\varepsilon \mathbf{y})=\prod_{k=1}^{n}\left(x_{k}+\varepsilon y_{k}\right)=f(\mathbf{x})+\varepsilon \sum_{k=1}^{n} y_{k} \frac{\partial f(\mathbf{x})}{\partial x_{k}}, \tag{3.24}
\end{equation*}
$$

where $\mathbf{x}+\varepsilon \mathbf{y}=\left(x_{1}+\varepsilon y_{1}, \ldots, x_{n}+\varepsilon y_{n}\right), \mathbf{x}=\left(x_{1}, \ldots, x_{n}\right), \mathbf{y}=\left(y_{1}, \ldots, y_{n}\right)$.
It is easy to see that this representation holds true for dual functions $F$ defined in the above way using a real analytic functions $f(\mathbf{x})$. In particular, it holds for polynomial, multi-linear and more general functions $f(\mathbf{x})$ defined by elementary algebraic expressions.

Let $A(d)=A(d ; \underline{\underline{\mathbf{A}}})$ be given in appropriate algebraic form. e.g. in the form (2.12) or (3.14). Replacing each variable $A_{i}(r)$ by dual variable $A_{i}(r)+\varepsilon w_{i}(r)$ in $A(d ; \underline{\underline{\mathbf{A}}})$ we obtain dual function $A^{\circ}(d ; \underline{\underline{\mathbf{A}}}+\varepsilon \underline{\underline{\mathbf{w}}})$, where $\underline{\underline{\mathbf{w}}}=\left(w_{i}(r)\right.$ : $\left.i \in \boldsymbol{C}, r \in \mathbf{K}_{i}-\left\{g_{i}(0)\right\}\right)$. According to representation (3.24) and formula (3.9), we have:

$$
\begin{equation*}
A^{\circ}(d ; \underline{\underline{\mathbf{A}}}+\varepsilon \underline{\underline{\mathbf{w}}})=A(d ; \underline{\underline{\mathbf{A}}})+\varepsilon w(d ; \underline{\underline{\mathbf{A}}}, \underline{\underline{\mathbf{w}}})=A(d)+\varepsilon w(d) . \tag{3.25}
\end{equation*}
$$

This formula leads to simple calculation method of failure frequency using dual number algebra:
(1) write $A(d)$ in appropriate algebraic form,
(2) replace all $A_{i}(r)\left[U_{i}(r)\right]$ by dual variables $A_{i}(r)+\varepsilon w_{i}(r)\left[1-\left(A_{i}(r)+\varepsilon w_{i}(r)\right)=U_{i}(r)-\varepsilon w_{i}(r)\right]$,
(3) perform calculation using dual number algebra to obtain dual number $a+\varepsilon b$,
(4) $w(d)=b$.

For example, if $A(d)=A_{1}(d)\left(1-U_{2}(d) U_{3}(d)\right)$, then

$$
\begin{aligned}
& A^{\circ}(d ; \underline{\underline{\mathbf{A}}}+\varepsilon \underline{\underline{\mathbf{w}}})=\left\{A_{1}(d)+\varepsilon w_{1}(d)\right\}\left(1-\left\{U_{2}(d)-\varepsilon w_{2}(d)\right\}\left\{U_{3}(d)-\varepsilon w_{3}(d)\right\}\right) \\
& =\left\{A_{1}(d)+\varepsilon w_{1}(d)\right\}\left\{\left[1-U_{2}(d) U_{3}(d)\right]+\varepsilon\left[w_{2}(d) U_{3}(d)+U_{2}(d) w_{3}(d)\right]\right\} \\
& =A_{1}(d)\left(1-U_{2}(d) U_{3}(d)\right)+\varepsilon\left\{w_{1}(d)\left[1-U_{2}(d) U_{3}(d)\right]+A_{1}(d)\left[w_{2}(d) U_{3}(d)+U_{2}(d) w_{3}(d)\right]\right\} \\
& =a+\varepsilon b=A(d)+\varepsilon w(d) ;(\operatorname{compare} \text { with }(3.20)) .
\end{aligned}
$$

## 4. Extension to random demand rate

Many real technical systems operate under demand randomly changing in time. Examples of such systems are power generating systems, transportation systems, distributed computer networks and production systems. We refer to Levitin (2005) and Lisnianski and Levitin (2003) for more examples and further discussion.

Let $D(t)$ be the demand rate at time $t$. The fixed (time-independent) demand rate $D(t) \equiv d$ was considered in section 3. Now we consider randomly changing in time demand rate $\{D(t)\}$. We assume that the process $\{D(t)\}$ takes its state in finite set $\mathbf{D} \subseteq[0, \infty)$ and that it satisfies Assumptions 2.2 and 2.3. The system with performance process $X(t)=\varphi(\mathbf{X}(t))$ is operating at time $t$, if $X(t) \geq D(t)$. Otherwise the system is failed. It is assumed that the processes $\{X(t)\}$ and $\{D(t)\}$ are independent. We show how to apply the results of section 4 to the case of randomly changing demand.

Let the demand states in $\mathbf{D}$ be indexed in decreasing order:
$0 \leq d(m)<d(m-1)<\ldots<d(1), m \geq 1$.

Let $\mathbf{H}=\{1,2, \ldots, m\}$ be the index set of demand levels and let $L(t)$ be the index of demand level at time $t$, so that $D(t)=d(L(t))$. Define a function $\psi: \mathbf{H} \times \mathbf{K} \rightarrow\{0,1\}$ by:

$$
\begin{equation*}
\psi(k, x)=\mathbf{1}(x \geq d(k)) \tag{4.1}
\end{equation*}
$$

Since $d(k)$ is decreasing in $k, \psi(k, x)$ is a monotone increasing binary structure, which can be considered as the structure function of a binary system consisting of two multi-state elements. The first element corresponds to the demand level index and its stochastic behaviour is described by $\{L(t)\}$. The second element corresponds to the original system with stochastic behaviour described by $\{X(t)\}$. We have:

$$
\begin{equation*}
\left.\psi(L(t), X(t))=\mathbf{1}(X(t) \geq d(L(t)))=\sum_{k=1}^{m}(L(k, t)-L(k+1, t)) X(d(k), t)\right), \tag{4.2}
\end{equation*}
$$

where $L(k, t)=\mathbf{1}(L(t) \geq k), L(m+1, t) \equiv 0$ and $X(c, t)=\mathbf{1}(X(t) \geq c)$. Therefore:

$$
\begin{equation*}
A(t)=\sum_{k=1}^{m} \operatorname{Pr}\{L(t)=k\} A(d(k), t)=\sum_{k=1}^{m}\left(A_{L}(k ; t)-A_{L}(k+1 ; t)\right) A(d(k), t), \tag{4.3}
\end{equation*}
$$

where $A_{L}(k ; t)=\operatorname{Pr}\{L(t) \geq k\}=\operatorname{Pr}\{L(k, t)=1\}$.
Let $w_{L}^{j \rightarrow l}(t)$ be the frequency of transitions of the process $\{L(t)\}$ from state $j$ to state $l$ at time $t$ (assumed to exist). Then we can apply the factoring formula (3.22) to obtain the failure frequency $w(t)$ and the repair frequency $v(t)$ of the system operating under random demand $\{D(t)\}$ :

$$
\begin{gather*}
w(t)=\sum_{k=1}^{m}\left(w_{L}(k, t)-w_{L}(k+1, t)\right) A(d(k), t)+\sum_{k=1}^{m}(L(k, t)-L(k+1, t)) w(d(k), t),  \tag{4.4}\\
v(t)=\sum_{k=1}^{m}\left(v_{L}(k, t)-v_{L}(k+1, t)\right) A(d(k), t)+\sum_{k=1}^{m}(L(k, t)-L(k+1, t)) v(d(k), t), \tag{4.5}
\end{gather*}
$$

where

$$
\begin{equation*}
w_{L}(k, t)=\sum_{j=k}^{m} \sum_{l=1}^{k-1} w_{L}^{j \rightarrow l}(t), \quad v_{L}(k, t)=\sum_{j=1}^{k-1} \sum_{l=k}^{m} w_{L}^{j \rightarrow l}(t), \tag{4.6}
\end{equation*}
$$

with $w_{L}(1, t) \equiv w_{L}(m+1, t) \equiv 0$ and $v_{L}(1, t) \equiv v_{L}(m+1, t) \equiv 0$.
The failure [repair] frequency of system with variable demand rate has two contributors, designated by $w^{(L)}(t)$ and $w^{(X)}(t)\left[v^{(L)}(t)\right.$ and $\left.v^{(X)}(t)\right]$ :

1) $w^{(L)}(t)$ and $v^{(L)}(t)$ are related to failures caused by changes of demand rate, and correspond to the first sum in equations (4.4) and (4.5) respectively, and
2) $w^{(X)}(t)$ and $v^{(X)}(t)$ are related to failures caused by changes of the state of the system of elements, and correspond to the second sum in these equations.

Of course, for the steady state $(t=\infty)$, the failure and repair frequencies, and their two separate contributors as well, coincide: $w(\infty)=v(\infty), w^{(L)}(\infty)=v^{(L)}(\infty), w^{(X)}(\infty)=v^{(X)}(\infty)$.

Notice that the results presented in this section also include, as a special case, the random demand, which does not change in time: $D(t) \equiv D$ and consequently, $L(t) \equiv L$ ( $D$ and $L$ are just random variables). Then all $w_{L}^{j \rightarrow l}(t)$
are equal to 0 , and thus the demand related contributors $w^{(L)}(t) \equiv v^{(L)}(t) \equiv 0$, i.e. the first sum in each right-hand side of each equation (4.4) and (4.5) disappear.

## 5. Conclusions

New general formula for the failure/repair frequency of a multi-state monotone system was derived in the paper. Using this formula simple conversion rules from an availability or unavailability expression into an expression for failure/repair frequency, were obtained. Application of dual number algebra was also discussed.

As further investigations in the area, we may mention:

1) developing other efficient algorithms, considering, for example, application of the Universal Generating Function (UGF) technique, Levitin (2005);
2) considering some statistical dependencies among system's elements (e.g. common-cause failures), and between demand rate and elements performance processes as well;
3) obtaining approximations useful for analysing very complex and large systems;
4) generalisation to multi-state systems which are not necessarily monotone (thought in this case not so simple conversion rules are expected).

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# LIFETIME ANALYSIS OF INCANDESCENT LAMPS: THE MENON-AGRAWAL MODEL REVISITED 

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#### Abstract

The use of the Weibull distribution to model lifetimes of incandescent lamps was originally suggested by Leff (1990). Following this suggestion, Agrawal and Menon have offered and investigated, in a series of papers, an improved model constructed from physical considerations and laws of mathematical statistics. In the present paper we offer supplementary thoughts concerning the Agrawal-Menon model and its several modifications. In addition, we discuss the use of Pinelis's l'Hospital-type calculus rules in the analysis of ageing properties of lifetime distributions.


Keywords: Survival function, hazard rate function, mean residual life function, Weibull distribution, normal distribution, truncated normal distribution, lognormal distribution.

## 1. Introduction

The laws of physics are commonly taught using incandescent lamps (see, e.g., Evans, 1978; Leff, 1990; MacIsaac et al., 1999; Menon and Agrawal, 2003). Interestingly, statistical analysis of the lifetime of incandescent lamps does not appear to be an old science despite the fact that lamps have been around for more than two centuries: H. Davy created the first incandescent lamp in 1802, and T. Edison created the first practical incandescent lamp in 1879 (see, e.g., Wikipedia, 2007). Recently, Agrawal and Menon (1998), and Menon and Agrawal (2003, 2006, 2007, 2008) have analyzed their reliability characteristics based on theoretical models and experimental data.

Leff (1990) argues that since the hazard rate (HR) function $h(t)=-S^{\prime}(t) / S(t)$ of the exponential survival function,

$$
S_{E X P}(t)=S_{E X P}(t \mid \beta)=e^{-t / \beta},
$$

is constant (i.e., $h(t)=1 / \beta$ ), the lifetimes of incandescent lamps cannot follow the exponential law, unlike radioactive decay. To include the necessary dependence on history and thus improve upon the model's fit to experimental data, Leff (1990) therefore suggested using the Weibull survival function
$S_{W}(t)=S_{W}(t \mid \alpha, \beta)=e^{-(t / \beta)^{\alpha}}$,
where $\alpha, \beta>0$ are unknown parameters. The Weibull HR function $h(t)=(\alpha / \beta)(t / \beta)^{\alpha-1}$ is increasing for every $\alpha>1$. Leff (1990) notes that $\alpha=5$ has given a good fit to his data. It is interesting to note that when $\alpha=5$ the Weibull survival function is close to the normal survival function (see, e.g., Johnson et al., 1994, p. 632), which hints that the latter may be the basis for an alternative hazard formulation.

Among other things, Leff (1990) also notes that the 'average life' indicated on bulb's package is actually the 'median life'. From the mathematical point of view, the mean and median lifetimes are different, respectively:
$t_{a v}=\int_{0}^{\infty} S(t) d t$ and $t_{\text {med }}=F^{-1}\left(\frac{1}{2}\right)$,
where $F^{-1}(u)$ is the inverse of the cumulative distribution function $F(t)=1-S(t)$. Leff (1990) observes that despite being mathematically different, $t_{a v}$ and $t_{\text {med }}$ are nearly equal in practice, thus hinting at the symmetric nature of the lifetime distributions of incandescent lamps. Menon and Agrawal (2003) corroborate this observation.

In a series of papers, Menon and Agrawal (2006, 2007, 2008) suggest and investigate an improved model for the survival function based on laws of physics and the normal approximation to the binomial distribution. Specifically, Menon and Agrawal (2007) argue that on the unit-less scale of an argument $\tau$ (see below) the survival function is
(1) $\quad S(\tau)=\frac{1+\operatorname{erf}(\gamma(1-\tau))}{2}$ with $\operatorname{erf}(t)=\frac{2}{\sqrt{\pi}} \int_{0}^{t} e^{-y^{2}} d y$,
where $\gamma$ is a parameter associated with variability (see below). We note that the survival function $S(\tau)$ can be written as $\Phi_{0,1}(\gamma(1-\tau) \sqrt{2})$, and we thus have the equation

$$
\begin{equation*}
S(\tau)=1-\Phi_{\mu, \sigma^{2}}(\tau), \text { where } \mu=1 \text { and } \sigma^{2}=1 /\left(2 \gamma^{2}\right) \tag{2}
\end{equation*}
$$

where $\Phi_{\mu, \sigma^{2}}$ denotes the normal distribution function. Thus, the unit-less $\tau$ scale has been chosen in such a way that the mean lifetime $\mu$ is equal to 1 , and thus we have the equation

$$
\begin{equation*}
\tau=\frac{t}{t_{a v}} . \tag{3}
\end{equation*}
$$

Hence, on the $t$-scale we have the following representation for the Agrawal-Menon survival function:
(4) $\quad S(t)=1-\Phi_{\mu, \sigma^{2}}(t)$, where $\mu=t_{a v}$ and $\sigma^{2}=t_{a v}^{2} /\left(2 \gamma^{2}\right)$.

We shall find it convenient to use the notation

$$
S_{N}(t)=S_{N}(t \mid \mu, \sigma)=\int_{t}^{\infty} \frac{1}{\sqrt{2 \pi \sigma^{2}}} e^{-(y-\mu)^{2} /\left(2 \sigma^{2}\right)} d y
$$

for the normal survival function $1-\Phi_{\mu, \sigma^{2}}(t)$.

Since (4) is the normal survival function, it is strictly smaller than 1 for all $t \in(-\infty, \infty)$, even though we expect the survival function to be exactly 1 for all $t \leq 0$. When the mean lifetime is notably larger than the variance, the survival function is close to 1 for all $t \leq 0$. In practical terms this justifies the use of the normal distribution for modeling the lifetime of lamps under the aforementioned caveat. Nevertheless, from the rigorous point of view we expect lamp lifetimes to follow distributions whose survival functions are exactly 1 at $t=0$. Menon and Agrawal (2006) scaled the distribution to have unit mass on the positive half-line, which is the truncated normal survival function
$S_{T N}(t)=S_{T N}(t \mid \mu, \sigma)=\int_{t}^{\infty} \frac{1}{\sqrt{2 \pi \sigma^{2} a^{2}}} e^{-(y-\mu)^{2} /\left(2 \sigma^{2}\right)} d y$,
where the normalizing constant is $a=\Phi_{0,1}(\mu / \sigma)$. Note also that the constant $a$ is practically equal to 1 when the mean $\mu$ is larger than, say, $3 \sigma$ (see Table 1 below) and so we have that $S_{T N}(t) \approx S_{N}(t)$. The latter observation and our numerical findings in Table 1 below do indeed justify the use of the normal distribution in the current context, as is done by Menon and $\operatorname{Agrawal}(2007,2008)$.

Given the practical performance of the Menon and Agrawal $(2006,2007)$ models, we expect that any candidate survival function should be close to the normal survival function. For this reason we suggest considering the lognormal survival function

$$
S_{L N}(t)=S_{L N}(t \mid \mu, \sigma)=\int_{t}^{\infty} \frac{1}{\sqrt{2 \pi \sigma^{2} y^{2}}} e^{-(\log y-\mu)^{2} /\left(2 \sigma^{2}\right)} d y
$$

This is defined for $t \geq 0$ and is equal to 1 at $t=0$. Limpert et al. (2001) provide a discussion on which distribution - normal or lognormal - should be preferred in various situations, accompanied with numerous illustrative examples. The fundamental difference between the two is that, while both are based on a variety of forces acting independently, in the former the effects are additive, while in the lognormal case they are multiplicative. Lamps can fail from a variety of causes, any one of which is sufficient, even though the other factors may not impede the lamps functionality. Thus lamps can be modeled as a series system, where the reliability function is a product of individual factors. Hence the lognormal with its multiplicative interpretation is an attractive alternative. When the coefficient of variation is small, it is difficult to distinguish the lognormal distribution from the normal distribution (Limpert et al., 2001). The major observable difference between them is that the lognormal is non-symmetric, i.e., the median and mean may differ.

Interestingly, in a survey of published data sets, Limpert et al. (2001) found that the only ones that were not fitted satisfactorily by the lognormal consisted of differences, sums, means or other functions of original measurements. However, for many data sets where a lognormal distribution was acceptable, a normal distribution was statistically rejected. We note also that Xie and Pecht (2003) selected a lognormal distribution to model the reliability of semiconductor light emitting devices.

## 2. Analysis of the data set of Menon and Agrawal (2008)

Menon and Agrawal (2008) provide the data of the failure times of 50 new Phillips (India) lamps, which we will use to examine the fit of the following four survival functions: $S_{W}(t), S_{N}(t), S_{T N}(t)$ and $S_{L N}(t)$. The lamps were monitored at regular time intervals of twelve hours to count the fused lamps. The instants when at least one fused lamp was found were recorded and there were thirty-two such instances. The minimal recorded value was 840 and the maximal one was 2568 . Naturally, several fused lamps were found at some instances.

Hence, we have 'grouped data' with each failure time that has occurred during a twelve-hour period ( $t_{i-1}, t_{i}$ ] recorded as $t_{i}$. To simplify the estimation procedure, we follow the obvious course, and instead of randomly 'dispersing' the observations throughout the corresponding time periods $\left(t_{i-1}, t_{i}\right.$ ] we replace them by the mid-values $t_{i-1}+\left(t_{i}-t_{i-1}\right) / 2$. Hence, the fifty failure times have been reassigned one of the values $6+12 k$ hours, for $k=0,1,2, \ldots$ Denoting these fifty 'observations' by $t_{1}^{*}, \ldots, t_{50}^{*}$, we fit the survival functions using the maximum likelihood method. The numerical results are presented in Table 1 with the corresponding survival functions shown in Figure 1.

Table 1. Fitted distributions for the data set from Menon and Agrawal (2008)

| Distributions | Parameters |  |  | LL | $t_{a v}$ |
| :--- | :--- | :--- | :--- | :---: | :---: |
| Weibull | $\alpha=4.2556$ | $\beta=1541.4$ | -364.25 | 1402.1 | 1414.3 |
| Normal | $\mu=1407.8$ | $\sigma=343.10$ | -362.84 | 1407.8 | 1407.8 |
| Truncated normal | $\mu=1407.8$ | $\sigma=343.10$ | -362.84 | 1407.8 | 1407.8 |
| Lognormal | $\mu=7.2198$ | $\sigma=0.24968$ | -362.06 | 1409.5 | 1366.3 |

Note that the Weibull distribution has estimated shape parameter $\alpha=4.2556$, less than the value $\alpha=5$ suggested by Leff (1990), but still making the Weibull distribution fairly close to the normal (see, e.g., Johnson et al., 1994, p. 632). The log-likelihood values in Table 1 show that the Weibull distribution has poorest fit of the four, the normal and truncated normal are tied for second place, and the lognormal is slightly superior. The difference between the mean $t_{a v}$ and median $t_{\text {med }}$ is largest for the lognormal distribution. Not surprisingly, the mean and median lifetimes of the normal and truncated normal are same.


Figure 1. The empirical (stepwise) and four fitted survival functions: Weibull (dotted), normal (solid), truncated normal (dot-dashed, coinciding with solid), and lognormal (dashed).

Here we see that the reason the lognormal fits better is that it models the skewness, in particular the right-hand tail. May et al. (2000) note that for samples with $n>30$, data fit well by the normal has significantly smaller skewness than that not well fit by the normal, and suggest the use of the Shapiro-Wilk (or Ryan-Joiner) test for normality to identify the better of the normal and lognormal distributions. The lamp failure data has skewness of 0.51 , compared to -0.21 after taking logs. The Ryan-Joiner test for normality provides P -values of 0.075 and " $>0.1$ " for the raw and logged data, respectively. Both of these results are evidence in favor of the lognormal over the normal distribution. We note that it is common when data is skewed to reject some observations as outliers, leaving a symmetric distribution. The question, which can only be resolved by the collection of more data, is whether this tail is a real phenomenon.

We will now look at some other characteristics of the four candidate distributions. In the two panels of Figure 2 we show graphs of the estimated hazard rate (HR) functions (top panel) and the mean residual life (MRL) functions
$\mu(t)=\int_{t}^{\infty} S(x) d x / S(t)$
(bottom panel).


Figure 2. The fitted HR (top panel) and MRL (bottom panel) functions: Weibull (dotted), normal (solid), truncated normal (dot-dashed, coinciding with solid), and lognormal (dashed). The saw-like function in the bottom panel is the empirical MRL function.

Note that $\mu(0)$ is the mean $t_{a v}$ whose numerical values are recorded in Table 1. The set of Menon and Agrawal (2008) data shows that the last fused lamp was found at 2568 hours. This explains our choice of the range [0, 3000] in the plots of Figures 1 and 2. A visual assessment of Figure 2 suggests that all the fitted HR functions are increasing and the fitted MRL functions are decreasing. Theoretical results (see Section 3 below) confirm these observations for the Weibull, normal, and truncated normal distributions. In the case of the lognormal distribution, however, the HR function is upside-down bathtub-shaped and the MRL function is bathtub-shaped. (We refer to Section 3 for the definition of UBT and BT shapes.) Thus we have plotted the lognormal HR and MRL functions on the interval [0, 10000] in Figure 3 using the same parameters as those in the bottom line of Table 1. Naturally, as in classical regression analysis, fitting distributions well beyond the data range - the interval [840, 2568] for the Menon and Agrawal (2008) data set - can be, and indeed frequently is, parlous. Hence, although they may be real, rather than artificial, we should not be too disturbed by the decreasing nature of the HR function and the increasing nature of the MRL function outside the interval [0, 3000]. In fact, even the monotonically increasing and decreasing, respectively, HR and MRL functions beyond the interval [0,3000] may not be natural even for the Weibull, normal, and truncated normal distributions. Indeed, the discussion in Agrawal and Menon (1998) indicates that more realistic monotonicity patterns of the HR and MRL functions should likely be more pronounced on the right-hand tails, due to physical properties such as nearly instantaneous fusing of light bulbs at the end of their lifetimes.


Figure 3. The HR (solid) and MRL (dashed) functions corresponding to the lognormal distribution with the parameters as in Table 1

## 3. Shapes of HR and MRL functions and Pinelis's calculus rules

We have described above the shapes of the HR and MRL functions of the Weibull, normal, truncated normal, and lognormal distributions. These shapes have been known for a long time (see, e.g., Lai and Xie, 2006, and references therein). The shapes of HR functions are usually identified using the result of Glaser (1980) concerning the similarity of the shapes of the ratios $f(t) / S(t)$ and $f^{\prime}(t) / S^{\prime}(t)$. In the case of MRL function, the results of Gupta and Akman (1995) have frequently been employed. In this section we will describe how to achieve the same goals using general results, reported by I. Pinelis in a series of papers, concerning the similarity of the shapes of generic ratio-functions $u(t) / v(t)$ and $u^{\prime}(t) / v^{\prime}(t)$. We believe that this is the first instance of Pinelis's calculus rules being utilized in the context of reliability engineering and, specifically, for analyzing the ageing properties of lifetime distributions.

Determining the shape of the Weibull HR function does not actually require sophisticated tools of analysis, since the function is easy to calculate and possesses the simple form:
$h_{W}(t)=\frac{\alpha}{\beta^{\alpha}} t^{\alpha-1}$.

Hence, we have that
(i) the HR function $h_{W}(t)$ is decreasing when $\alpha<1$, constant when $\alpha=1$ and increasing when $\alpha>1$.

The corresponding MRL function, on the other hand, is difficult to derive explicitly and is therefore challenging to analyze. We shall therefore employ one of Pinelis's results:

Theorem 1. (Pinelis, 2001; see Proposition 1.1). Let $u(t)$ and $v(t)$ be differentiable functions on the interval $(a, b)$ where $-\infty \leq a<b \leq \infty$. Assume that $v(t)$ and its derivative $v^{\prime}(t)$ are non-zero on the interval $(a, b)$ and $v^{\prime}(t)$ does not change its sign on $(a, b)$. Furthermore, assume that $u(b-)=0=v(b-)$. The following two statements hold:
(1) If $u^{\prime}(t) / v^{\prime}(t)$ is increasing on $(a, b)$, then $u(t) / v(t)$ is also increasing on $(a, b)$.
(2) If $u^{\prime}(t) / v^{\prime}(t)$ is decreasing on $(a, b)$, then $u(t) / v(t)$ is also decreasing on $(a, b)$.

We can now finish our discussion of the Weibull distribution by determining the shape of its MRL function. Note that the condition $u(b-)=0=v(b-)$ is satisfied for the functions $\int_{t}^{\infty} S(x) d x$ and $S(t)$ when $t \uparrow b=\infty$. Let us write the equation
$\mu_{W}(t)=\frac{u_{W}(t)}{v_{W}(t)}$, where $u_{W}(t)=\int_{t}^{\infty} S_{W}(x) d x$ and $v_{W}(t)=S_{W}(t)$.

The derivative $v_{W}^{\prime}(t)=-f_{W}(t)$ is always negative on $(0, \infty)$ and thus neither vanishes nor changes its sign on $(0, \infty)$. Hence, according to Theorem 1, whatever monotonicity we have for the ratio $u_{W}^{\prime}(t) / v_{W}^{\prime}(t)$, the same monotonicity holds for the ratio $u_{W}(t) / v_{W}(t)$ as well. Writing the former ratio as follows:

$$
\begin{equation*}
\frac{u_{W}^{\prime}(t)}{v_{W}^{\prime}(t)}=\frac{-S_{W}(t)}{S_{W}^{\prime}(t)}=\frac{1}{h_{W}(t)}, \tag{5}
\end{equation*}
$$

we see that monotonicity of $u_{W}^{\prime}(t) / v_{W}^{\prime}(t)$ is just the opposite to that of the HR function $h(t)$ which has already been determined in (i) above. We thus have that
(ii) the MRL function $\mu_{W}(t)$ is increasing when $\alpha<1$ and decreasing when $\alpha>1$. When $\alpha=1$, then the function $\mu_{W}(t)$ is constant; easy to check.

We shall also use statement (1) of Theorem 1 to investigate the HR functions of the normal and truncated normal distributions. Note that the condition $u(b-)=0=v(b-)$ is satisfied for the functions $f(t)$ and $S(t)$ when $t \uparrow b=\infty$. Furthermore, we shall use statement (2) of Theorem 1 to investigate the MRL functions of the normal and truncated normal distributions. Note that the condition $u(b-)=0=v(b-)$ is satisfied for the functions $\int_{t}^{\infty} S(x) d x$ and $S(t)$ when $t \uparrow b=\infty$.

Consider now the normal distribution. We start with the equation

$$
\begin{equation*}
h_{N}(t)=\frac{u_{N}(t)}{v_{N}(t)}, \text { where } u_{W}(t)=f_{N}(t) \text { and } v_{N}(t)=S_{N}(t) \tag{6}
\end{equation*}
$$

The derivative $v_{N}^{\prime}(t)=-f_{N}(t)$ is negative on $(-\infty, \infty)$ and thus neither vanishes nor changes its sign on $(-\infty, \infty)$. Hence, according to Theorem 1, whatever monotonicity we have for the ratio $u_{N}^{\prime}(t) / v_{N}^{\prime}(t)$, the same monotonicity holds for the ratio $u_{N}(t) / v_{N}(t)$ as well. Note that
$\frac{u_{N}^{\prime}(t)}{v_{N}^{\prime}(t)}=\frac{t-\mu}{\sigma^{2}}$,
which is an increasing function of $t$. Hence, we have that
(iii) the HR function $h_{N}(t)$ is increasing.

As to the shape of the normal MRL function, we proceed analogously to the Weibull case (see equation (5) in particular). Since we have already established that $h_{N}(t)$ is increasing, we therefore have that
(iv) the MRL function $\mu_{N}(t)$ is decreasing.

From the mathematical point of view, the only difference between the normal and truncated normal survival functions on $[0, \infty)$ is the constant $a$, which does not influence the shapes of the HR and MRL functions. Hence, from (iii) and (iv), we conclude that
(v) the HR function $h_{T N}(t)$ is increasing, and
(vi) the MRL function $\mu_{T N}(t)$ is decreasing.

Finally, we will check the shapes of the lognormal HR and MRL functions. With $u_{L N}(t)=f_{L N}(t)$ and $v_{L N}(t)=S_{L N}(t)$, we have that

$$
\begin{equation*}
\frac{u_{N}^{\prime}(t)}{v_{N}^{\prime}(t)}=\frac{\log t+\sigma^{2}-\mu}{t \sigma^{2}} \tag{7}
\end{equation*}
$$

The derivative of ratio (7) is $\left(1+\mu-\sigma^{2}-\log t\right) /\left(t \sigma^{2}\right)$ and so the ratio must be increasing on $(0, c)$ and decreasing on $(c, \infty)$, where $c=\exp \left\{1+\mu-\sigma^{2}\right\}$. Such functions are called upside-down bathtub (UBT) shaped. Similarly, if a function is first decreasing and then increasing, it is called bathtub (BT) shaped. (We understand 'increasing' and 'decreasing' in the strict sense throughout the current paper.) The following result of Pinelis tells us what to expect from the ratio $u(t) / v(t)$ when $u^{\prime}(t) / v^{\prime}(t)$ is either UBT or BT.

Theorem 2. (Pinelis, 2006; third and fourth lines of Table 4.1). Let $u(t)$ and $v(t)$ satisfy same assumptions as in Theorem 1, including $u(b-)=0=v(b-)$. Then the following two statements hold:
(1) If $u^{\prime}(t) / v^{\prime}(t)$ is UBT, then $u(t) / v(t)$ is either decreasing or $U B T$.
(2) $u^{\prime}(t) / v^{\prime}(t)$ is $B T$, then $u(t) / v(t)$ is either increasing or $B T$.

Since ratio (7) is UBT, part (1) of Theorem 2 implies that the lognormal HR function must be either decreasing or UBT. In turn, the latter statement together with part (2) of Theorem 2 (see also equations (5) above) imply that the lognormal MRL function must be either increasing or BT. To find out which of these alternatives are true, we appeal to Proposition 4.4 in Pinelis (2006), which requires us to calculate the ratio
$\Delta=\lim _{t \downarrow 0} v^{2}(t) \frac{(u / v)^{\prime}(t)}{\left|v^{\prime}(t)\right|}$.
Theorem 3. (Pinelis, 2006; lower half of Table 4.2). Let $u(t)$ and $v(t)$ satisfy same assumptions as in Theorem 1, including $u(b-)=0=v(b-)$. Then the following two statements hold:
(1) If $\Delta \leq 0$, then the ratio $u(t) / v(t)$ in part (1) of Theorem 2 is decreasing, and if $\Delta>0$, then the ratio is UBT.
(2) If $\Delta \geq 0$, then the ratio $u(t) / v(t)$ in part (2) of Theorem 2 is increasing, and if $\Delta<0$, then the ratio is $B T$.

Consider the lognormal HR function $h_{L N}(t)$. The corresponding ratio $\Delta$ is:

$$
\Delta=\lim _{t \downarrow 0} v_{L N}^{2}(t) \frac{h_{L N}^{\prime}(t)}{\left|v_{L N}^{\prime}(t)\right|}=\lim _{t \downarrow 0} \frac{f_{L N}^{\prime}(t)}{f_{L N}(t)}=-\lim _{t \downarrow 0} \frac{\log t+\sigma^{2}-\mu}{t \sigma^{2}}=\infty .
$$

Hence, according to part (1) of Theorem 3, we have that
(vii) the HR function $h_{L N}(t)$ is UBT.

Consider next the lognormal MRL function $\mu_{L N}(t)$. The corresponding ratio $\Delta$ is:
$\Delta=\lim _{t \downarrow 0} v_{L N}^{2}(t) \frac{\mu_{L N}^{\prime}(t)}{\left|v_{L N}^{\prime}(t)\right|}=\lim _{t \downarrow 0} \frac{\mu_{L N}^{\prime}(t)}{\left|v_{L N}^{\prime}(t)\right|}=-\lim _{t \downarrow 0} \frac{-S_{L N}(t)}{f_{L N}(t)}=-\infty$.
Hence, according to part (2) of Theorem 3, we have that
(viii) the MRL function $\mu_{L N}(t)$ is $B T$.

## 4. Concluding remarks

We have shown that the (truncated) normal and lognormal distributions satisfactorily describe the lifetimes of incandescent lamps. The latter is slightly superior statistically and easier to interpret in terms of failure being due to any one of a number of possible causes. The main difference resulting from the use of the lognormal is that the mean and median lifetimes diverge, but the available data set appears to be too small to distinguish this or, equivalently, the tail behavior of the distribution.

The shapes of the HR and MRL functions corresponding to the four distributions employed above to analyze the lifetime of incandescent lamps were analyzed using general calculus results for the ratios of functions and derivatives. Reflecting on these derivations in the context of the existing literature (see, e.g., Lai and Xie, 2006, and references therein), we note that except in some trivial cases such as the Weibull HR function, monotonicity of the HR and MRL functions has been investigated mainly using Glaser's (1980) $\eta$-function $\eta(t)=-f^{\prime}(t) / f(t)$. (It is of course the l'Hospital-type ratio $f^{\prime}(t) / S^{\prime}(t)$ corresponding to the HR function $f(t) / S(t)$.) In this sense, therefore, Glaser (1980) is a precursor to Pinelis's research of the past decade, and so are Gupta and Akman (1995) in the context of determining MRL shapes in terms of the corresponding HR shapes. We again refer to Lai and Xie (2006) for various uses of, and further theoretical developments related to, Glaser's $\eta$-function and Gupta and Akman's (1995) results; see also Bebbington et al. (2008) for a recent application of $\eta(t)$ to the BirnbaumSaunders distribution. We conclude the discussion with a passage from Pinelis (2004) which refers to the shapes of the generic ratio-functions $u(t) / v(t)$ and $u^{\prime}(t) / v^{\prime}(t)$ :
"...In contrast, the argument just presented is straightforward and rather mechanical. This is exactly the point that we wish to make in this paper. Now a wide class of inequalities become almost trivial in that ad hoc creativity is no longer needed for many such problems. But then is there any excitement left? Yes, what is exciting now is to have such general rules for monotonicity!" (Pinelis, 2004, p. 908)

We concur.

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    ${ }^{1}$ The singular and plural of an acronym are always spelled the same.

