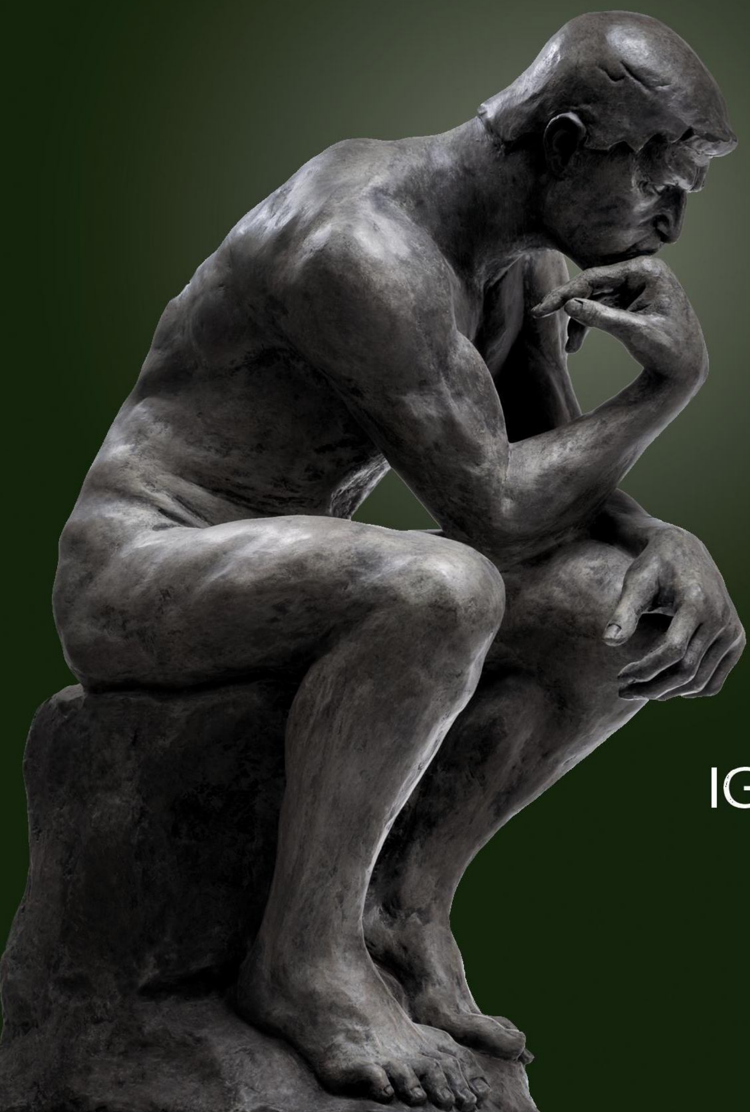
IN THE BEGINING WAS THE NUMBER...

STORIES AND LEGENDS ABOUT MATHEMATICAL INSIGHTS 2



IGOR USHAKOV

Tales and Stories
about Mathematical & Scientific Insights

Igor Ushakov

In the Beginning Was the Number…

San Diego
2012

<u>Series "Stories and Legends about Mathematical Insights"</u>

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Preface

The subject of mathematics is so serious that nobody should miss an opportunity to make it a little bit entertaining.

Blaise Pascal [1]**.**

What is this series of books about? For whom is it written? Why is this series written in this manner, not in another? Discussion about geometry, algebra and similar topics definitely hint that this is about mathematics. On the other hand, you cannot find within a proof of any statement or strong chronology of facts. Thus, these books are not tutorial. This is just a collection of interesting and sometimes exciting stories and legends about human discoveries in one or another way connected to mathematics…

These book are open for everybody who likes to enrich their intelligence with the stories of genius insights and great mistakes (mistakes also can be great!), and with biographies of creators of mathematical thinking and mathematical approaches in the study of the World.

Who are the readers of the proposed books? We believe that there is no special audience in the sense of education or age. The books could be interesting to schoolteachers and university professors (not necessarily mathematicians!) who would like to make their lectures more vivid and intriguing. At the same time, students of different educational levels – from middle school up to university – as well as their parents may find here many interesting facts and ideas. We can imagine that the book could be interesting even for state leaders whose educational level is enough to read something beyond speeches prepared for them by their advisors.

[1] **Blaise Pascal** (1623 –1662) was a French mathematician, physicist, inventor, writer and philosopher.

Summarizing, we have the courage to say: These books are destined for everybody!

Trust us: we tried to write the book clearly! Actually, it is non-mathematical book around mathematics.

This book is not intended to convert you to a "mathematical religion". Indeed, there is no need to do this: imagine how boring life would be if everybody were a mathematician? Mathematics is the world of ideas, however any idea needs to be realized: integrals cannot appease your hunger, differential equations cannot fill gas tank of your car....

However, to be honest, we pursued the objective: we tried to convince you, the reader, that without mathematics *homo erectus* would never transform into *homo sapiens*.

Now, let us travel into the very interesting place: Terra Mathematica. We'll try to make this your trip interesting and exciting.

What in particular is particular book about?

Here you, Reader, find many interesting facts about development of numerical systems, about some very special numbers. We even touch a bit numerology.

At the end you will be introduces with biographies of some genius in At the end you will be introduces with biographies of some genius in the area of knowledge that is a subject of this book.

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Igor Ushakov

San Diego
2012

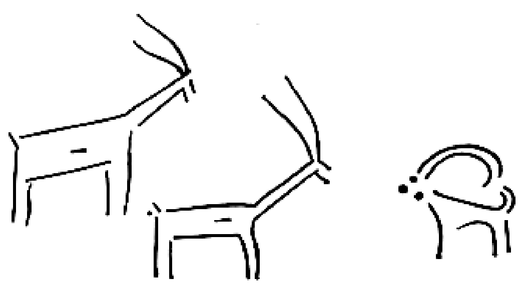
Contents

1 WHERE ARITHMETIC CAME FROM

Moreover, number, the most excellent
Of all inventions, I for them devised,
And gave them writing that retaineth all,
The serviceable mother of the Muse
Aeschylus*[2] *«Prometheus Bound»

The word "arithmetic" comes from the Greek word "arithmys," which means "number," though the concept of numbers and operations are much older than ancient Greece.

Aeschylus in his tragedy "Prometheus Bound", following an ancient Greek myth, wrote that the Greek hero Prometheus gifted to people flame and taught them to use numbers. Probably, our foremother Eve got numbers with the apple from the Tree of Knowledge… Well, at least, for certain it is known that 15-20 thousand years ago cave-dwellers (much before the Creation of the World?) recorded their successful hunting by drawing two or three deer on the wall.



[2] **Aeschylus** (525-456 BC.), ancient Greek dramatist who is considered as the founder of the genre of tragedy. He introd1uced dialogs, music and hours in his plays. Only seven his plays came to us, one of them is «*Prometheus Bound*», which is translated on all languages.

There, in caves of the Paleolithic era numerical notation – naming and symbols for numbers and operations with them – had been created. Let us now delve into the history of the development of numerical notation … Let us follow in ancient civilizations the development of mathematics. This is always so interesting and so exciting!

1.1 Sumerian Numbers

To be the first is always difficult even
if you are enigmatic ancient Sumerians…
Unknown author

The oldest known documents with arithmetic texts have been found on tablets in Sumeria[3]. Those tablets were made from clay, upon which symbols were written with wooden sticks, and which were dried under the sun or in an oven. Because the Sumerian symbols resembled straight strokes, archeologists called this style *cuneiform.*

Sumerians used a hexadecimal system based on 60. To represent any number only two symbols were used: "𒁹" for "one" and "𒌋" for ten:

Number	1	2	3	4	5	6	7	8	9	10	11	…
Cuneiform	𒁹	𒈫	𒐈	𒐉	𒐊	𒐋	𒑂	𒑄	𒑆	𒌋	𒌋𒁹	…

These two symbols were used for writing numbers from 1 to 60. For numbers larger than 60 they used these same symbols

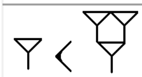
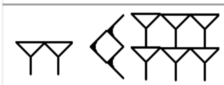
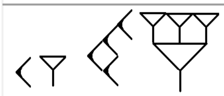
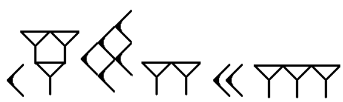
[3] **Sumerian civilization** existed in the state, which included independent cities, built by the people came in IV-V millenniums BC to Mesopotamia (the Southern part of Iraq between Tigris and Euphrates rivers). That people left to the manhood such scientific heritage that some scientist (linguists, astronomers, mathematicians, philosophers, and others) established a special commission, which is trying to prove that Sumerians were under aliens influence.

More details about Sumerian civilization see in 4.12 of Book 1.

(like we do in our decimal system) in positions to the left. Of course, while in our case one the next position means 10, the Sumerian symbol "𒁹" on the next position meant 60.

It is necessary to note that the Sumerian numeral system was the first position dependent system in human history, which was very much ahead of the rest of contemporary civilization.

Let us look at some samples of numbers in Sumerian notation:

Cuneiform	«Reading»	Explanation	Value
	**1 . 13**	**60+13**	**73**
	**2 . 36**	**2×60+36**	**156**
	**11 . 44**	**11×60+44**	**704**
	**13 . 52 . 23**	**$13\times60^2+52\times60+23$**	**49946**

In pure cuneiform writing it is difficult to distinguish positioning of the symbols. For instance, what does "𒁹𒁹" mean: 2 =1+1, or 61=1×60+1, or even $3601=1\times60^2+0\times60+1$? Sometimes the meaning becomes understandable from context; sometimes there were some comments within the text. How Sumerians decided what is right in those cases nobody knows.

Some historians assume that arithmetic operations were performed on an abacus, a device also invented by Sumerians. In an abacus each horizontal line corresponds to a position (the equivalent of our digit). In the left side, each stone represents 𒁹, and in the right side a stone represents 𒌋. In such interpretations there is no ambiguity. In astronomical tables where huge numbers appear, in the late Sumerian period a separating "empty" space was used. So, "𒁹𒁹" meant 2, but "𒁹_𒁹" meant 61, and "𒁹_ _𒁹" meant

3601. Since the beginning of the II millennium BC, a special sign – two diagonal arrows – was used for the same purpose[4].

Sumerians used this hexadecimal system widely. Sumerians had no coinage; they evaluated everything in weighted silver. For example, if the quoted price was 10 shekels, it meant 10 shekels weight of silver[5]. The money scale consisted of a shekel, min (60 shekels) and talent (60 minas)[6]. The circle was divided into 360 degrees. An hour included 60 minutes, and a minute included 60 seconds.

Why did these numbers arise? Most often historians suggest that it was connected with the monetary system (more exactly, with the system of weights). Mathematicians suggest that, on the contrary, the numerical system was the source of these numbers. A popular hypothesis is that the circle was divided into 360^0 because that corresponds to the division of a year into 360 days (1^0 per day). All those hypotheses contradict historical precedents: astronomy never was ahead arithmetic, and the 360-day calendar comes from with Egypt not Mesopotamia.

Authors of the book join those who believe that Sumerian mathematicians were “seduced” by the many divisors of the number 60: 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, and 30. Developed agriculture, as well as crafts and trade needed an adequate system of measurement, and there was a practical need was to divide an initial quantity into simple portions: 1/2, 1/3, 1/4, 1/5, 1/6, 1/10, 1/12, 1/15, 1/20, and 1/30. The hexadecimal system delivers all these opportunities. (The next “convenient” number is 420 but it is too large for a practical system.)

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4 There is no accurate reference about that innovation: who introduced it, Sumerians or Babylonians because on the edge of III and II millenniums Sumerians left the scene of history and were assimilated with their neighbors – Babylonians, Assyrians and Israelis who accepted all their scientific heritage.

5 They have also other shekels: 1 Shekel of Silver = 60 Shekels of Copper and 1 Shekel of Gold = 12 Shekels of Silver.

6 The Sumerian formula of justice: “A man of mina has no right to oppress a man of shekel”.

Notice that number 12 was widely used in Sumeria, but this number has no divisor of 5. Nevertheless, counting by dozens was very popular in Sumeria: there were 12 hours in a day time and 12 hours in a night time; and 12 months in a year. That focus on the number 12 has made it to our present time: we have tea and dinner sets of 12 items, pencils and flow-masters sets are also of 12 items, etc.

This seemingly clumsy hexadecimal system permitted Sumerians to make complex arithmetic operations: calculate fractions, multiply numbers up to million, extract roots and raise numbers to a power[7].

1.2 Numbers of Ancient Egypt

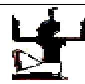
Ancient Egyptians used a decimal system; however it was in a very special form. They denoted a unit by a vertical line and all numbers less that 10 were marked by a corresponding number of such lines.

Number	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Hieroglyph	I	II	III	II II	III II	III III	IIII III	IIII IIII	III III III

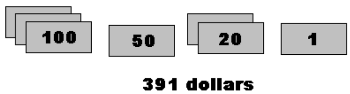
For 10 and its powers 100, 1000, 10000, etc., the Egyptian numerical system had special hieroglyphs.

Number	**10**	**100**	**1,000**	**10,000**	**100,000**	**1,000,000 or "very many"**

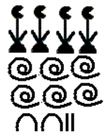
7 Sumerian numbers are not a real numerical system: by modern definition N-based system should have N unique signs. Factually, the Sumerian system is non-position up to 60 (say, for "simple people") and more complex positioning system over 60 (for "scientists").

Hieroglyph						
Description	**yoke**	**rope**	**lotus**	**finger**	**frog**	**Man**

Any number is compiled in the same way as we form payment by cash choosing minimal number of banknotes of various nominal:



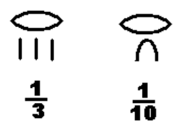
Such a system allows the writing of any number with a minimum number of symbols of different value, as in the making of change from a set of specific coin values. For instance, in the so-called Rhind, or Ahmes papyrus[8] the number 4622 is given.



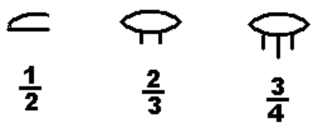
It is written vertically and the last row is read from left to right.

Egyptians had special symbols for fraction notation: a hieroglyph called Khor's Eye was located above the number to make a fraction.

Two examples are given on the right.



There were also special symbols for fractions 1/2, 2/3 and 3/4:



Egyptians worked with small numbers as well. For instance, fraction 1/331 has the form:

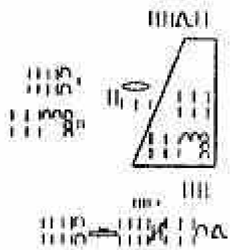
[8] This papyrus has size of 544 by 33 cm named after the Scottish Egyptologist Alexander Henry Rhind (1833-1863) who found it, or by the name of the scribe of the 17th century BC Ahmes (his name was on the papyrus). This is one of the first arithmetic textbooks. The papyrus is now in the British Museum in London.

 $= \frac{1}{331}$

The Rhind papyrus is a very interesting document of ancient Egypt. It contains 84 arithmetical and geometrical problems with solutions. Below are examples of some problems (we leave solutions for curious readers.)

- A quantity plus one-seventh of it becomes 19. What is the quantity?
- A quantity together with its two-thirds has one-third its sum taken away to yield 10. What is the quantity?
- The sum of a certain quantity together with its two-third, its half, and its one-seventh becomes 37. What is the quantity?

Another ancient Egyptian mathematical textbook is the Moscow Papyrus, or Golenischev Papyrus[9].



Practically everything that is known about the mathematics of ancient Egypt comes from these two papyri.

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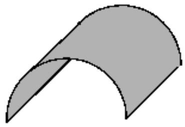
9 This papyrus of XIX century BC is the first ever known mathematical textbook (its size is only 549 на 8 cm). It named after its owner, Russian Egyptologist V. S. Golenishchev (1856-1947). The name of the scriber is not known because he had not recorded his name. The papyrus is now in the State Museum of Fine Arts in Moscow.



It contains 25 practical problems and their solutions. One of the problems is the earliest estimation of curvilinear area: Compute the surface area of a Quonset type hut roof.

(From this problem one can guess that ancient Egyptians knew something about the value of “pi”.)

1.3 Ancient Greek numerals

There was no single Greek notation for numbers in the first millennium BC because the country consisted of many independent microstates with their own currency, weights and measures. These led to some differences in the number system between different states. Naturally, it led to differences in the written symbols for numbers.

The oldest Greek number system is attic, which originated in Athens. This system, also called “*acrophonic*”[10], was use in the first millennium BC. Symbols for the main decimal numbers were as follows:

Γ	Δ	Η	Χ	Μ
Pente	Deka	Hekaton	Khilioi	Murioi
5	10	100	1000	10000

The only puzzle, probably, is the symbol for 5: why it is Γ (“gamma”), not Π (“pi”)? The explanation is that in that time “Pente was” originally “Gente” (the word began with letter “*gamma*”).

Notation for 1 was natural for almost all ancient people: First numbers were written in the style reminiscent of the Roman additive principle.

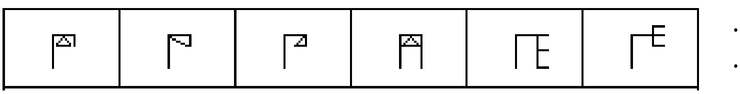
10 “Acrophonic” means that the symbols for the numerals come from the first letter of the number name, so the symbol has come from an abbreviation of the word that is used for the number.

I	II	III	IIII	Γ	ΓI	ΓII	ΓIII	ΓIIII	Δ
1	2	3	4	5	6	7	8	9	10

Of course, to use the additive principle for large numbers was very cumbersome. For instance, the number 9999 would require 36 symbols. So, the Greeks invented a system that was a mixture of base 10 and additive. Thus, they had intermediate symbols for 50, 500, 5000, and 50000 that were composite symbols made from 5 and the symbols for 10, 100, 1000, 10000 respectively.

Δ	𐅄	Η	𐅅	Χ	𐅆	Μ	𐅇
10	50	100	500	1000	5000	10000	50000

This is not the only way in which such composite symbols were created. Below are some examples how the number 50 was denoted in different Greek states.



It is worth noting that the Greeks were one of the first to adopt a system of writing based on an alphabet. (Actual inventors of the alphabetical system were the Phoenicians.)

In the end of the 4th century and beginning of the 3rd, Greeks had changed numbers notation, using both the upper case and lower case versions of the 24 classical letters. For all units, tens and hundreds they needed 27 symbols though there were only 24 letters in the language at that time. Thus, they were forced to include three more obsolete letters. This system was "quasi-decimal" Ionic[11], or alphabetic system: letters of alphabet denoted

[11] Ionia was the Western part of Asia Minor (now Turkey), which in XXI-XIX BC was settled by Greeks. That part of Greece was a cradle of ancient science: the first Greek philosopher, Anaximandr (610-546 BC), Pythagoras (580-500 BC), Herodotus (490-425 BC) and others came from Ionia.

numbers. Sometimes this system is called Milesian after Miletus, the capital and main city-port of the province. It is believed that the Ionic system had been created even earlier because it could be found in some form in the works by Thales of Miletus[12] and Pythagoras[13].

Α	Β	Γ	Δ	Ε	Ϛ	Ζ	Η	Θ
α	β	γ	δ	ε	ς	ζ	η	θ
1	2	3	4	5	6	7	8	9

..

For notations of the numbers, Greeks used both capital and small letters.

Notice that number 6 is represented by the symbol for the obsolete letter "*digamma*" (or "*stigma*"). For the next level of decimal numbers Greeks had:

Ι	Κ	Λ	Μ	Ν	Ξ	Ο	Π	Ϙ
ι	κ	λ	μ	ν	ξ	ο	π	ϙ
10	20	30	40	50	60	70	80	90

Notice that 90 is represented by the symbol for the obsolete letter "koppa". The remaining eight letters plus an obsolete one were taken as the symbols for larger numbers.

12 **Thales of Miletus** (625-547 BC), ancient Greek philosopher and mathematician from Miletus (Ionia). The founder of Miletus school that started developing cosmology, physics, geography, meteorology, astronomy and biology. It is believed that Thales is the founder of the antique science and philosophy. He was referred as the first of Seven Men of Wisdom.

13 **Pythagoras of Samos** (580-500 BC), great ancient Greek philosopher and mathematician.

Ρ	Σ	Τ	Υ	Φ	Χ	Ψ	Ω	ͲϠ
ρ	σ	τ	υ	φ	χ	ψ	ω	ϡ
100	200	300	400	500	600	700	800	900

Notice that 900 is represented by the symbol for the obsolete letter "sanpi". Sometimes when these letters are written to represent numbers, a bar was put over the symbol to distinguish it from the corresponding letter.

Composite symbols were created to overcome this problem. Adding a superscript letter "iota" to the symbols for 1 to 9 formed the numbers between 1000 and 9000.

͵Α	͵Β	͵Γ	͵Δ	͵Ε	͵Ϛ	͵Ζ	͵Η	͵Θ
1000	2000	3000	4000	5000	6000	7000	8000	9000

Their largest number was the "*myriad*" equal to 10000 and denoted by M (Greek letter "mu"). The symbol M with small numerals for a number up to 9999 written above meant that the number in those small numerals was multiplied by 10000. For instance 20000 was written as:

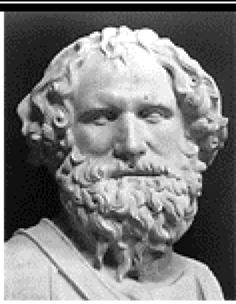
$\overset{\beta}{\mathrm{M}}$
20000

Of course writing a large number above the M was rather difficult, so mathematicians in such cases wrote the small numeral in front of the M rather than above it. An example from Aristarchus gives us the following writing for number 71755875:

͵ΖΡΟΕΜ ͵ΕΩΟΕ
71755875

Thus, ancient Greeks left the Romans a not so bad number system, but, while the Romans took much from Greek culture (even all of Greek mythology), they did not take this numbering system.

Two of the greatest mathematicians of that time – Archimedes and Apollonius[14] played a big role in the propagation of the alphabetic numeral system. Both of them made a substantial contribution to the development of a system for notation of large numbers.



Archimedes
(287-212 BC)

Great mathematician of antiquity, founder of mathematical physics, inventor. By legend, Archimedes pondered over a graph drawn on the sand, when a Roman soldier intruded into his house. The old man irritatingly said to the soldier: "Move out, you'll spoil my graph!" In response the soldier killed the scientist…

Scientists and engineers were taught Archimedes' works. In 9th – 11th centuries they were translated into Arabic; then in the 13th century they were translated from Arabic into Latin, and in the 16th – 18th centuries his works were translated into all European languages. The influence of his works cannot be overestimated.

For more details see Chapter "Pantheon".

Archimedes in his work "*Psammite*" ("*Sandreckoner*") introduced the concept of octad[15]. The first octad contains numbers from 1 to myriad (i.e. 10,000), as it was in an ordinary Ionian numerology system. In the second octad a myriad was taken

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[14] **Apollonius Pergaeus** (262 – 190 BC), ancient Greek mathematician was born in Perg (province Pamphilia). Contemporaries called him The Great Geometrician. His main work glorified his name in history, "Konica" ("The Conics") was written as a development of the earlier work by Euclid "Beginning of Conics". Apollonius introduced the parabola, hyperbola and ellipse along with their theoretical description. Other works of his are known only by titles, as they were referenced in later sources.

[15] A group or sequence of eight.

as a new unit and the same alphabetical numbers from α to ,θλϙθ give a set from a myriad to myriad myriad, i.e. 100,000,000. The third octad is again similar to the first one though the role of unit here plays 10^8. In such a manner, one can construct arbitrarily large numbers.

Of course, having such a system Archimedes was able to count all the grains of sand that fill a sphere of radius equal to the distance from the Earth to the Sun! Archimedes' system was almost perfect: he missed the smallest detail – the null, zero…

After Alexander the Great built his huge Empire, alphabetic number systems spread throughout the ancient World. Because the Egyptians, Syrians, Aramaics, Arabs, Jews, Armenians, Georgians, Indians and others were under Greek domination, this system was called Alexandrian, even though Greek letters were replaced by local ones. Byzantine religious influence on Northern and Eastern countries brought the Alexandrian number system into Russia and to Slavic people.

In this connection, it is interesting to look and compare various symbols of numerals of different origins. A sample of such symbols is given below.

Number	Armenian	Hebrew	Sanskrit	Arabic	Granolithic	Cyrillic	Georgian
1	Ա	א	१	ا	Ⰰ	А	ა
2	Բ	ב	२	ب	Ⰱ	Б	ბ
3	Գ	ג	३	ج	Ⰲ	В	გ
4	Դ	ד	४	د	Ⰳ	Г	დ
5	Ե	ה	५	ه	Ⰴ	Д	ე
6	Զ	ו	६	و	Ⰵ	Є	ვ
7	Է	ז	७	ز	Ⰶ	Ж	ზ

8	[illegible]	ח	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
9	[illegible]	ט	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

1.4 Roman numerals

It is said that ancient Rome accepted its numbering system from Etruscans[16]. Roman numerals in a sense are close to the Greek Attic system, though they had special symbols not only for 1, 10, 100 and 1000, but also for 5, 50 and 500; in addition a line over the letter meant multiplication on 1000.

Roman Numeral	Value
I	1
V	5
X	10
L	50
C	100
D	500
M	1 000
$\bar{V}$	5 000
$\bar{X}$	10 000
$\bar{L}$	50 000
$\bar{C}$	100 000
$\bar{D}$	500 000

From where did the images of the Roman numerals come? Probably, the symbol V meant a hand with 5 fingers and X showed two hands.

16 Etruria – ancient country located in Western part of Central Italy, now forming Tuscany and part of Umbria. It was the territory of the Etruscans, who in the 6th century BC spread Etruscan civilization throughout much of Italy. They were later ultimately assimilated with other peoples who populated Apennine peninsula.

Symbols "C" and "M" obviously originated from Roman names for 100 and 1000: "*centum*" and "*mille*", correspondingly.

The Roman number system is additive, i.e. the number equals to the sum of its constituents rather than on the position of the symbols (though the order of symbols exists). However, within the group of symbols in Roman numerology there is a possibility not only to add "weights" of symbols but also to extract them. For instance, number "4" is written as IV, i.e. =5-1. The same situation exists for tens, hundreds and thousands.

Units		**Tens**		**Hundreds**		**Thousands**	
1	I	10	X	100	C	1000	MM
2	II	20	XX	200	CC	2000	MMM
3	III	30	XXX	300	CCC	3000	MMM
4	IV	40	XL	400	CD	4000	$\mathrm{M}\overline{\mathrm{V}}$
5	V	50	L	500	D	5000	$\overline{\mathrm{V}}$
6	VI	60	LX	600	DC	6000	$\overline{\mathrm{V}}\mathrm{M}$
7	VII	70	LXX	700	DCC	7000	$\overline{\mathrm{V}}\mathrm{MM}$
8	VIII	80	LXXX	800	DCCC	8000	$\overline{\mathrm{V}}\mathrm{MMM}$
9	IX	90	XC	900	CM	9000	$\mathrm{M}\overline{\mathrm{X}}$

An interesting hybrid of additive and positioning systems, isn't it? On the one hand, IV and VI have different values though each is composed of the same symbols; on the other hand, VI definitely belongs to an additive system. Of course, the Roman numeral system is far from being a compact representation of numbers. For instance, 1844 looks like MDCCCXLIV. Moreover, do you have any idea how to multiply CDXCVIII by DCCLXXIV?

1.5 Arabic numerals… from India

The system of numeration employed throughout the greater part of the world today was developed in India.

The term "Arabic numerals" is a misnomer, since they were neither invented nor widely used by the Arabs. Arabs themselves even call the numerals they use "*Indian numerals*" ("*arqam hindiyyah*"). In the Arab World (until modern times) the Arabic numeral system was used only by mathematicians. Muslim scientists used the Babylonian numeral system, and merchants used a numeral system similar to the Greek and Hebrew numeral systems. However, because the Arabs took this Indian system from Persia and then brought it to the West, the numeral system became known as "Arabic".

First, the Indian numeral system was, as in many other countries, alphabetical. One of the latest improvements of the Indian alphabetical numeral system came from Aryabhata[17] and is found in his main work "*Aryabhatiya*". This system consisted of numerals, which were denoted by the 33 consonants of the Indian alphabet and represented numerals 1, 2, 3... 25, 30, 40, 50, 60, 70, 80, 90, and 100. The higher numbers are denoted by these consonants followed by a vowel to obtain 1000, 10000, and so on. In fact, the Aryabnata's system allowed recounting numbers up to 10^{18}.

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17 **Aryabhata** (476-550), earliest great Indian astronomer and mathematician. Known as Aryabhata I, or the Elder, to distinguish him from an Indian mathematician of the same name lived in X century. His book "*Aryabhatiya*" presented heliocentric astronomical theories in which the Earth was taken to be spinning on its axis and the periods of the planets were given with respect to the sun (in other words, it was heliocentric). The mathematical part of the book covers arithmetic, algebra, plane trigonometry and spherical trigonometry. It also contains continued fractions, quadratic equations, sums of power series and a table of sinus. He also gave a very good approximation of number "pi". In honor of Aryabhata role in astronomy, India has named its first satellite by his name. "

Actually that system was very similar to the Ionic numeral system, but customized to Sanskrit alphabet.

Approximately at that time, around the 5th century, some unknown genius made a great invention by introducing multiplying one of nine numerals by ten in corresponding power. Instead of using special symbols for 10, 20, 30 ..., one could use 1×10^1, 2×10^1, 3×10^1..., instead of special symbols for 100, 200, 300 ... one could use 1×10^2, 2×10^2, 3×10^2..., and so on. With such method any number could be easily written, for instance, such a huge number like 706005000400003, could be written by a simple rule with the help only 9 symbols and powers of ten: $7\times10^{14} + 6\times10^{12} + 5\times10^9 + 4\times10^5 + 3\times10^0$, and the power of ten was equivalent to the position of the corresponding numeral. Such system in contrast to additive ones (like the ancient Greek system) is called multiplicative.

This invention cannot be overestimated: due to it, arithmetic operations with large numbers became much easier. Pierre Laplace[18] in his article "*Overview of Indian mathematics*" wrote, "The ingenious method of expressing every possible number using a set of ten symbols (each symbol having a place value and an absolute value) emerged in India. The idea seems so simple nowadays that its significance and profound importance is no longer appreciated. Its simplicity lies in the way it facilitated calculation and placed arithmetic foremost amongst useful inventions. The importance of this invention is more readily appreciated when one considers that it was beyond the two greatest men of Antiquity, Archimedes and Apollonius".

One could ask: Why is 10 chosen as the basis of the number system? There is no actual logic; it is mostly a matter of historic customs: for thousands of years fingers were the only "computer"

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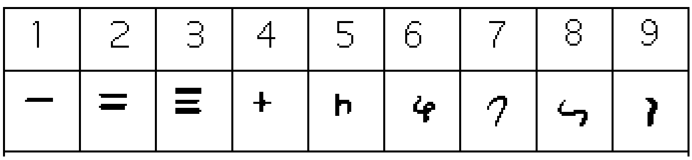
18 **Pierre Simon Laplace** (1749-1827), great French mathematician, physicist and astronomer, founder of probability theory, mathematical physics and celestial mechanics. He made outstanding contributions in differential equations, algebra, acoustics, thermodynamics and geodesy. During the French Revolution he was Chair of the Bureau of Longitude, under Napoleon, he was a Counselor and Minister, and after the Bourbon Kings were restored, he became a marquis of France. ***For more details see page 4.***

for people. To be fair, one should mention that the Sumerians had introduced the first multiplicative numeral system, though they had chosen base 60 instead of base 10.

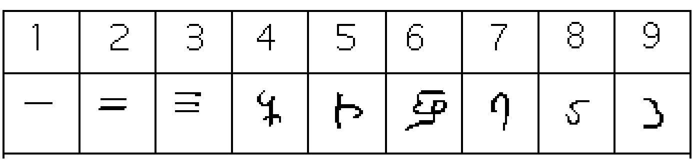
Notice that during the Great French revolution the Bureau of Longitude tried to introduce a numeral system with basis 12, though

without success. Probably, the reason was in the fact that there were ten numerals used in the current system and introducing two more symbols for 10 and 11 additionally would produce a mess in all financial and economical institutions.

The road to the modern form of our numerals – 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 – was long. Most historians and scientists believe that numerals we use now appeared as the result of the transformation of Sanskrit letters. It is known that in the 1st century AD there were the Brahmin numerals:



The Brahmin numerals were replaced by the Gupta numerals. This numeral system got its name from the Gupta dynasty, which ruled over the Magadha state in North-Eastern India from the early 4th century to the late 6th century. The Gupta numerals were developed from the Brahmin numerals, and they were spread over large areas by the Gupta Empire as they conquered territory.



The Gupta numerals evolved into the Nagari numerals, sometimes called the Devanagari numerals. This name literally means the "*writing of the gods*". This happened around the 7th century and continued to develop from the 11th century onward. By the way, this system already included zero, which was a significant step forward in numerology.

1	2	3	4	5	6	7	8	9	0
१	२	३	४	५	६	७	८	९	०

One can see that these images are close enough to those we used (probably, not in the same order), though we should remember that these numerals came to us through Persian and Arabs, so they were influenced by their alphabetical system.

In parallel, an analogous system developed in Tamili (now Sri Lanka) with their own symbols. Actually, the decimal numeral system existed simultaneously (as it was with the earlier alphabetical systems) in several regions with different symbols for numerals.

Modern	0	1	2	3	4	5	6	7	8	9
Indo-Aryan	٠	١	٢	٣	٤	٥	٦	٧	٨	٩
Persian and Urdu	۰	۱	۲	۳	۴	۵	۶	۷	۸	۹
Devanagari (Hindi)	०	१	२	३	४	५	६	७	८	९
Tamil		௧	௨	௩	௪	௫	௬	௭	௮	௯

The Indo-Aryan languages form a subgroup of the Indo-Iranian languages, thus belonging to the Indo-European family of languages.

After extending throughout the Middle East, the Arabs began to assimilate the science and cultures of the peoples they lived with. Baghdad became a great center of learning, where Arab, Greek, Persian, Jewish, and other scholars pooled heritages of their civilizations. The Arab scientist al-Qifti[19] in his book "*Chronology of*

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[19] **Ibn al-Qifti** (1172-1248), Arabic scientist worked in grammar, jurisprudence, logic, astronomy, mathematics, history, and medicine.

the scholars" gave the following description of the appearance of Indian numerals: "... a person from India presented himself before the Caliph al-Mansur[20] in the year 776 who was well versed in the method of calculation related to the movement of the heavenly bodies... Al-Mansur ordered the book to be translated into Arabic, and based on the translation, to give the Arabs a solid base for calculating the movements of the planets ..." The book referred in this citation is said to be "*Brahmasphutasiddhanta*" ("*The Opening of the Universe*"), which was written by the Indian mathematician Brahmagupta[21]. In this book he used the Hindu numerals, including the symbol for zero.

Thus, the decimal numeral system became known to the Persian mathematician Al-Khwarizmi[22], who wrote the book "*On the Calculation with Hindu Numerals*" around the year 825, and the Arab mathematician Al-Kindi[23], who wrote four volumes of "*On the Use of the Indian Numerals*" in about the year 830. It is said that these two books principally are responsible for the diffusion of the Indian decimal system of numeration to the Middle East and the West.

It is not known exactly when the new number system first came to Europe. The evidence is that it came many times. The oldest dated European manuscript containing Arabic numbers was

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20 **Abu Jafar abd-Allah al-Mansur** (754–75), brother and successor of Abu al-Abbas, who established dynasty of Abbasids, founder of the city of Baghdad.

21 **Brahmagupta** (598 - 670), the foremost Indian mathematician of his time. He made advances in astronomy and most importantly in number systems including algorithms for square roots and the solution of quadratic equations.

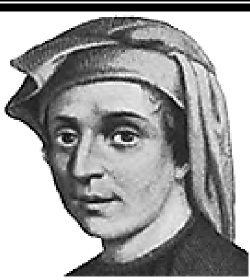
22 **Abu Jafar Muhammad ibn Musa al-Khwarizmi,** or **Algoritmi** in Latin (790-840), Uzbek scientist, mathematician and astronomer. He is often cited as "the father of algebra". His book "*Al-Kitab al-mukhtasar fi hisab al-jabr wa'l-muqabala*" ("*The Compendious Book on Calculation by Completion and Balancing*") was translated in Europe in XII century. The part of the book title ("*al-gabr*") gave the name for "*algebra*", and the Latin name of the author gave name for the mathematical term "*algorithm*".

23 **Abu Yusuf Yaqub ibn Ishaq al-Sabbah Al-Kindi** (805-873), the father of Islamic Philosophy, a scientist of high caliber: a gifted mathematician, astronomer, physician and a geographer as well as a talented musician.

written in Spain in 976. However, even though the French monk and mathematician Gerbert (940-1003), who became Pope Sylvester II in 999, also used Arabic numerals in several of his writings, they did not yet come into common use.

Thus, even the Pope of Rome was powerless to introduce a new number system into Europe! The Greek system of numerals remained popular for many years more among scientists and merchants, the latter who continued to use Roman numerals in keeping their books.

Leonardo Fibonacci, an Italian mathematician, who had studied in Algeria, promoted the Arabic numeral system in Europe with his book "*Liber Abaci*" ("*The Book of Counting*"), which was published in 1202.



Leonardo of Pisa
(1175 – 1240)

The greatest figure of the Medieval mathematics. Possessing Arabic arithmetic, he made calculations faster than professional counters with the help of abacus.

Fibonacci introduced a decimal system to Europe.

His books on arithmetic, algebra and geometry became textbooks in all European universities during next four centuries.

For more details see Chapter "Pantheon".

European scientists began to use the decimal system; however, it did not come into wide use until the invention of the printing press by Johannes Gutenberg[24].

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[24] **Johannes Gensfleisch zur Laden zum Gutenberg** (1398 – 1468) was a German blacksmith, goldsmith, printer, and publisher who introduced printing to Europe inventor printing technology. His family name was Gensfleisch zur Laden and Gutenverg was only nickname after the place where he lived. In 1455 Gutenberg printed two-volume "*Bible*" ("*Biblia Sacra*"). Handwritten Bible could take a single monk several years to transcribe.

1.6 What kind of numeration system may be

We used to decimal numeration system: units digit, tens digit, hundreds digit, etc. Where did it originated? How did it develop? Why does this system exist in different countries? Do there other numeration systems exist?

One of usual answers on the first question is: “A man used to count with the use of fingers, and there are 10 fingers on the both hands”. Indeed, this suggestion is, probably, very close to the truth. However, immediately a question arises: “Were there six-fingered people on Britain islands where the duodecimal numerical system deeply penetrate in common life?” Indeed, There are the following measures of length in English system, coming to us from Meddle Ages: an inch consists of 12 lines, a foot consists of 12 inches[25]… A similar situation with weight measurements where Britons use sexadecimal system:16 drams form one ounce, and 16 ounces form one pound...

If we did deeper (and into the farer past – almost 4 millenniums from now), we find even more exotic numerical systems. For instance, Sumerians and after them Babylonians widely used sexagesimal numerical system. Strange? Not so strange: we use this numerical system even nowadays☺: 60 seconds compose a minute, and 60 minutes compose one hour. Moreover, remember about 360 degrees and 360 days a year (approximately).

At the same time, modern computers use simplest numerical system – binary one. (We will consider this more detailed in the book of this series “From finger count to computer”.)

Now, let us “design” ourselves, for instance, quintary numerical system. First, recollect that many people of the past used repetition of a symbol for visual presentation of increased row of numbers. For instance, first Roman numbers were I for 1, II for 2, III for 3, IIII for 4, and only for 5 they used a new sign – Γ. So, let us call some imaginary unit in our new quintary system by symbol “A” and call it “Abrik”. For number “5” we introduce a unit of the next rank that is denoted by “B” and called “Babrik”. Units of next

[25] One line equals 2,1167 mm, and, correspondingly, inch and foot are equal to 2/54csm and 30.48 cm.

ranks will be "Cabrik" ("C"=25), Dabrik" ("D"=125), etc. Using these notations, let us compile the following self-explanatory table.

Decimal	0	1	2	3	4
Quintary		A	AA	AAA	AAAA
Decimal	5	6	7	8	9
Quintary	B	BA	BAA	BAAA	BAAAA
			.		
Decimal	10	11	12	13	14
Quintary	BB	BBA	BBAA	BBAAA	BBAAAA
			.		
Decimal	15	16	17	18	…
Quintary	BBB	BBBA	BBBAA	BBBAAA	..
	. . . .				

Thus, B = AAAA+A, C = BBBBAAAA+A, D = CCCCBBBBAAAA+A, and so on.

Quintary system	Decimal	Quintary system	Decimal
A×A = A A×AA = AA A×AAA = AAA A×AAAA = AAAA	1×1=1 1×2=2 1×3=3 1×4=4	AA×A = AA AA×AA = AAAA AA×AAA = BA AA×AAAA = BAAA	2×1=2 2×2=4 2×3=6 2×4=8
Quintary system	Decimal	Quintary system	Decimal
AAA×A = AAA AAA×AA = BA AAA×AAA = BAAA AAA×AAAA = BBAA	3×1=3 3×2=6 3×3=9 3×4=12	AAAA×A = AAAA AAAA×AA = BAAA AAAA×AAA = BBAA AAAA×AAAA =BBBAAA	4×1=4 4×2=8 4×3=12 4×4=16

Of course this table can be continued.

For transforming a number written in our new quintary notation as DDCCCBAAA into common decimal numerical system, one needs to perform the following calculations:

AA×D + AAA×C + A×B + AAA×A = 2×D + 3×C + 1×B + A×1 = 2×125 + 3×25 + 1×5 + 3×1 = 433.

This number in "Quintarians" language that we introduced above should be pronounced like:

"**(AA times D)-and-(AAA times C)-and-(A times B)-and-(AAA)**".

Of course, there is no needs to invent "abriks-babriks". One can use images of decimal figures, however correctly understands what new "tens", "hundreds" and "thousands" actually mean.

By the way, if one decides to use numerical system with larger number of figures of the 1st rank than in decimal system, then anyway it is necessary to introduce a new notation. For instance, duodecimal system needs new symbols for 10 and 11, say, A=10 and B=11. Then decimal 12 in duodecimal system is denoted as 10 and decimal 25 denoted as 21. Further, for decimal 12×12=144, in duodecimal system one has to write 100 (i.e. unit of the third order.

Different numerical systems can help you to make jokes. Assume that you are 9 years old. Somebody asks you: "How old are you?" You answer nothing though write "100"on a sheet of paper and show it. "O, you are a joker!" – "No, it is my age but written in ternary system…"

Indeed, using decimal figures, you can write the following table:

Decimal	0	1	2
Ternary		1	2
Decimal	3	4	5
Ternary	10	11	12
Decimal	6	7	8
Ternary	20	21	22
Decimal	9	10	11
Ternary	100	101	102
Decimal	12	13	14
Ternary	1000	1001	1002
…			

1.7 A Modest Fascination with Zero

Zero, cipher, nil, naught… This assumingly auxiliary numeral has the most intrigues and even scandalous history among its brethrens. In numeral systems, zero has two duties: one is to denote a "missed" numeral on some position, and another to be just a number that stands between positive and negative numbers and means "nothing".

From where did the word "zero" came? The Arabic name for zero is "*as-sifr*" that itself is, in turn, a phonetic carbon-paper of Sanskrit word "*sunya*" that means "*empty*".

Fibonacci wrote the word as it sounds in Latin as "*zephyrum*". This word later became the word "*zero*" in English and French. The same word became French "*chiffre*", German "*ziffer*", and English "*cipher*".

The history of zero is intriguing. This mystical shadowy number appeared only to vanish again: it seems that mathematicians were searching for it yet did not recognize its significance even when they saw it.

Ancient Greeks began their contributions to mathematics around the time when Babylonian mathematics began to use zero as an empty place indicator. However, the Greeks did not adopt a positional number system.

It would be incorrect to say that ancient Greeks did not think about zero: even Pythagoreans held discussions on the theme. However, Greek philosophers, who always tried to find geometrical sense in everything, rejected zero because a line with no length or a polygon with no vertices does not exist.

However, Greek astronomers worked with such huge numbers that they were forced to invent zero. By the way, they began to use for these purposes the symbol "O". There are many guesses and speculations why this particular notation was used. Some historians believe that it is due to omicron, the first letter of the Greek word for nothing ("*ouden*"). However, the Greeks already used omicron as a number (it represented number 70). Other explanations offered include the fact that it stands for "*obol*", a coin of almost no value that was used by counters who put it on a sand board as a symbol (it left a depression in the sand which looked like "O").

At least, Ptolemy in his "*Almagest*" written around 130 AD used the Babylonian sexagesimal system together with the empty place holder "O". He used the symbol both between digits and at the end of a number. However, only a few exceptional astronomers used the notation and it would fall out of use…

In turn, Roman philosophers formulated their attitude to zero in an elegant proverb: "*Ex nihilo nihil*" that means "*nothing comes from nothing*".

Nevertheless, for practical calculations with the use of abaci where numbers were expressed by rows of stones there was a special round stone with the hole in the center to denote nothing. (By the way, a stone in Latin is "*calculus*", from where English "*calculate*" originated.)

The idea of the zero place holder makes its next appearance in Indian mathematics. It is fair to say that the modern numerals and number system were born in India. Indians were not confused by the enigmatic properties of zero, probably, because in contrast to Greece or
Rome they always accepted infinity and emptiness as a part of their religious images.

By around 650AD the use of zero as a number came into Indian mathematics. The Indians as the Babylonians also used a place-value system and zero was used to denote an empty place.

In around 500AD Aryabhata devised a number system, which has no zero yet was a positional system. He used the word "*kha*" for position and it would be used later as the name for zero. In his book "*Arybhatia*" he formulated properties of this unusual number: if one adds to or extracts from any number a number zero, the initial number is not changed; if one multiplies any number by zero, the result will be zero. Aryabhata tactfully kept silence about division by zero… Indeed, the impossibility to divide by zero was understood only in the 12th century.

In earlier Indian manuscripts, a dot was used to denote an empty place in positional notation. Later Indian mathematicians had names for zero in positional numbers yet had no symbol for it. The first record of the Indian use of zero is dated in the year 876:

on a stone tablet among other numbers, 270 and 50 were written with a symbol looked that like a small "o".

Eventually zero gained use as a number. Let us first note that it was not a trivial task to include zero as a number: numbers were referred to collections of objects. Only when the idea of a number became abstract, did it become possible to consider zero and negative numbers, concepts which do not arise as properties of collections of physical objects.

The main conceptual work was put forth in the book of Indian mathematician Brahmagupta. In the 7th century he stated the rules for arithmetic involving zero and negative numbers that were more advanced than those introduced by Aryabhata. He explained that a number subtracted from itself gives zero as a result. He gave the following rules for addition with zero: The sum of zero and a negative number is negative, the sum of a positive number and zero is positive; the sum of zero and zero is zero.

It was a brilliant attempt from the first person that we know who tried to extend arithmetic to negative numbers and zero.

Perhaps it is the time to mention that there was another great civilization that developed a place-value number system with a zero. This was the Mayan people who lived in Central America and whose civilization flourished between 250 and 900 AD. It is known that by 665 AD they used a place-value number system with a base of 20 and with a symbol for zero. This was a remarkable achievement that, unfortunately, did not influence other peoples.

The brilliant work of the Indian mathematicians was transmitted to Arabic mathematicians. Al-Khwarizmi wrote a book "*Hindu Art of Reckoning*", which describes the Indian place-value system of numerals based on 1, 2, 3, 4, 5, 6, 7, 8, 9, and zero.

Ibn Ezra[26], in the 12th century, wrote three treatises on numbers, which helped to bring the Indian symbols and ideas of decimal fractions to the attention of some of the learned people in Europe. His "*Book of the Number*" described the decimal system for

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26 **Abraham ben Meir ibn Ezra** (1092-1167), magnificent Jewish scholar, philosopher, biblical exegete, astronomer, astrologist, scientist, poet, and Hebrew grammarian. He introduced the decimal system to Jews living in the Christian world. He used the Hebrew alef to tet for 1–9, but added a special sign to indicate zero. He then placed the tens to the left of the digits in the usual way.

integers with place values from left to right. In this work he used zero, which he called "*galgal*" (meaning "wheel" or "circle").

The Indian ideas spread east to China as well. In 1247 the Chinese mathematician Ch'in Chiu-Shao[27] wrote his "*Mathematical Treatise in Nine Sections*" ("*Shu-shu chiu-chang*") in which the symbol "O" was used for zero.

While there were ongoing theoretical/philosophical discussions about zero, the use of zero, along with the decimal system, penetrated everyday life. The Latin word "*nullus*" was adopted to denote zero.

Fibonacci was one of the main people to bring these new ideas about the number system to Europe. In his "*Liber Abaci*" he described the nine Indian symbols together with the sign 0 for Europeans in around 1200 but it was not widely used for a long time after that. It is significant that Fibonacci is not bold enough to treat 0 in the same way as the other numbers 1, 2, 3, 4, 5, 6, 7, 8, 9; he speaks of the "sign" zero while calling the other symbols numbers. In his treatment of zero he did not reach the sophistication of Brahmagupta nor of the Arabic mathematician al-Samawal[28].

The introduction of zero into Europe met with resistance. The appearance of Leonardo Fibonacci's book began almost half a millennium long discussion about and around zero. Philosophers insisted that zero is the number of elements in an empty set and that such set did not exist. (Some even hinted that zero had Satanic origins!)

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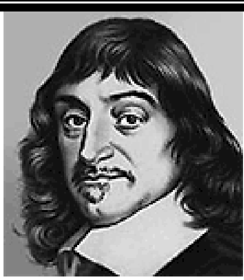
[27] **Ch'in Chiu-Shao**, or **Qin Jiushao** (1202-1261),Chinese mathematician, now regarded as one of the greatest mathematicians of the 13th century. This is particularly remarkable that he did not devote his life to mathematics and was accomplished in many other fields.

[28] **Ibn Yahya al-Maghribi Al-Samawal** (1130-1180), son of a Jewish scholar of religion and literature who wished his son to become a physician. At age about 13 years old, he began a serious study of Hindu methods of calculation. His most famous treatise "The Brilliance in Algebra" ("*al-Bahir fi'l-jabr*") was written when he was only 19 years old. It is a work of great importance both for the original ideas, which it contains, and also for presentation of mathematics of the time.



Rene Descartes
(1596 - 1650)

Outstanding French philosopher, mathematician, physicist and physiologist. Got his education at the Jesuit college. Fought during the 30-Year War. Since age 35 he lived in Holland, where he published his main works. At age 55 he was invited to Sweden by the Queen. He caught a cold in Sweden and died.

He founded analytical geometry (with the Cartesian coordinates). The name of philosophical teaching – Cartesianism – originated from the Latin spelling of his name *Renatus Cartesius*. This philosophical school used Descartes famous motto: "C*ogito ergo sum*" ("*I think therefore I am*").

With the advent of paper and inexpensive accounting books in 13th and 14th centuries, the process of accepting Arabic numerals became irreversible; the convenience of this system was obvious. Nevertheless, Europe accepted the concept of zero only in the 17th century after Rene Descartes' works. (By the way, initially Descartes actively rejected zero.)

2. ABOUT NUMBERS

2.1. Six and Six Hundred Sixty Six

Here is wisdom. Let him who has understanding calculate the number of the beast, for it is the number of a man: His number is 666.

The Bible, Rev.13:18.

Six, six, six... 666... It happens that this is also a special number! And this special meaning is hidden in the last chapter of the Bible, in Revelation: "Here is wisdom. Let him that hath understanding count the number of the beast: for it is the number of a man; and his number is Six hundred threescore and six." This hardly can be interpreted in understandable way, however…

Is there a meaning here? What knowledge does one need to understand the number 666? And why is the number of the beast also the number for a man? Anyway, this number has teased human minds for many centuries…

As we already have seen, in ancient times many people used letters of the alphabet as numbers. So, some tried to find mysticism in names. For instance, the number 666 is produced by the names of Caesar, Nero, and Martin Luther. In one way or another, we find the Devil or Anti-Christ... However, it is too early to be excited by such: some have looked at the "King of Israel" written in Hebrew and got… 666! Sorry: no Devil, no Anti-Christ!

Pythagoras considered number 6 as a perfect number because the sum of its dividers equals to the number itself (1+2+3=6). The number 666 is not perfect in a Pythagorean sense. However, modern numerologists try to find something exceptional in this number. And they have succeeded! They found that 666 is equal to the sum of the first 36 numbers! And $36=6\times 6$! So what? – You might be tempted to ask. However, arithmetical mysticism has an answer for you: in 36 years a man tames his passions and finds his mind.

Other found that 666 is the sum of squares of first seven prime numbers:

$$2^2 + 3^2 + 5^2 + 7^2 + 11^2 + 13^2 + 17^2 = 666.$$

However, we certainly will disappoint those who are so engaged with 666: in some religious manuscripts the number of the Beast was written as 616…

2.2. Seven, seven, seven...

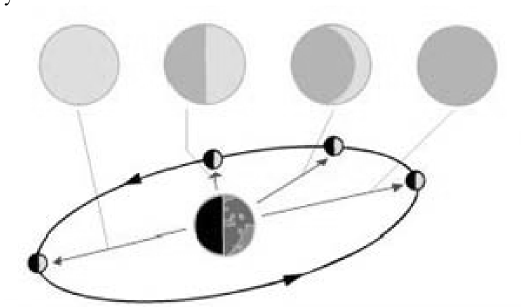
Everything is number seven.
Pythagoras.

There are no uninteresting numbers. However, number 7 is a very interesting number! Who of us did not play in childhood the following game? One asked another: "Think of a number and tell me. Fast! Fast!" More often than not the answer was: "Seven…" Or we asked each other: "What is your favorite number?" And again a usual answer was: "Seven…"

What is special about this number? There is no comprehensive answer on this question, though a number of scientists throughout history have tried to answer this question. There are many historical examples of the use of seven as a special number; this number was sacred amongst Oriental and European civilized people.

There are Seven colors in a rainbow… Seven notes in music…

One of the explanations for a cult of the number 7, of course, ties into astronomy. A man from time immemorial idolized the sky. The biggest and brightest heavenly bodies were believed as the strongest and most influential powers in human life. In ancient times, there were seven habitants of the heaven: Sun, Moon, Venus, Mars, Mercury, Jupiter and Saturn, i.e. those that can be seen by naked eye.



Lunar cycle.

Furthermore, let us notice that almost all ancient civilizations had a 7-day week because a month was chosen as a cycle of the Moon – 28 days. During this period the Moon changes 4 phases: New Moon plus Waxing Crescent Moon; First Quarter Moon plus Waxing Gibbous Moon; Full Moon plus Waning Gibbous Moon; Last Quarter Moon plus Waning Crescent Moon. "Waxing" means growing and "waning" means shrinking; an appropriate definition of "gibbous" is "swollen on one side".

So we see that in ancient time seven heavenly bodies were personified: Egyptians and Phoenicians had seven major gods; Persians had seven angels and seven countervailing demons... Priests of many Oriental religions were divided into seven levels of hierarchy…Seven footsteps lead to the altar…

Christian processions travel seven times around the church… Before giving a promise to God, a believer has to kneel seven times…

Seven, seven, seven…

Ancient China was divided into seven provinces; ancient Persia consisted of seven principalities ruled by independent satraps.

In ancient Greece the number seven was a symbol for Apollo. Apollo was born on the seventh day of the month; his lyre had seven strings. Citizens of the Athens yearly sacrificed seven young men and women to the mythological bull-headed man Minotaur living in the Crete Labyrinth. The Nymph Calypso kept Odysseus a prisoner for seven years. Atlas, who held the Earth on his shoulders had seven daughters; the Pleiades were transformed into seven stars to form the constellation Taurus. In legends, one might find seven circles of Hell, seven Cyclops, seven tubes of the Pan's flute, and so on…

Seven, seven, seven…

Pythagoreans, who considered numbers as the basis of everything, especially honored number 7. They called it a symbol of sanctity, health and mind. Why not? Number seven, in a sense, is a separator within the first decade:

$1\times2\times3\times4\times5\times6\times7 = 7\times8\times9\times10 = 5040$...

And there is the following curious property of decimal fractions which are result when dividing by 7:

1/7=0.(142857)
2/7=0.(285714)
3/7=0.(428571)
4/7=0.(572428)
5/7=0.(714285)
6/7=0.(857142)
7/7=1
8/7=1 + (1/7) = 1.(142857)
9/7= 1 + 2/7 = 1.(285714)
10/7= 1 + 3/7 = 1.(428571)
and so on.

Here after each period we see a cyclic changing of the same 6 figures: 142857!

(However, ancient Greeks had no relations to this property of number 7: they did not know decimal fractions!)

Ancient Greeks called 7 "a virgin number" because there is no number less than 7, which is its divider, and there is no number under 10, which being divided could produce 7, i.e. 7 was considered to be "a number closed within itself". Indeed, $2 = 6/3 = 8/4 = 10/5$, $3 = 6/2 = 9/3$, $4 = 2\times2 = 8:2$; $5 = 10/2$; $6 = 3\times2$; $8 = 4\times2$; $9 = 3\times3$, but 7 cannot be gotten in a similar way!

Seven, seven, seven…

Rome stands on seven hills and was called "the City on Seven Hills", or "the City of Seven Towers". According to Islamic sources, Rome was besieged seven times and surrendered after a 7-week siege by the seventh sultan of the Ottoman Empire.

In medieval Europe there were 7 "free arts" (grammar, rhetoric, dialectics, arithmetic, geometry, music and astronomy).

In those times, oath was made in the presence of 7 witnesses and those who made the oath were 7 times besprinkled by his own blood.

The number 7 has a special meaning in Judaism: "Everything seventh is beloved". Very significant are the seventh day of a week, Sabbath, and the seventh year. They even have their special names: Sabbath for Man and Sabbath for Land.

The Menorah – Jewish candlestick – has seven branches…

Menora and the coat of arms of the State of Israel

In accordance with Holy Bible, number 7 is an exclusive one; it rules time and space. Number 7 is mentioned in the Old and New Testaments about 600 (!) times. Some samples: "And all the days of Lamech were seven hundred seventy and seven years: and he died...", "And it came to pass after seven days, that the waters of the flood were upon the earth…", " And thou shalt number seven Sabbaths of years unto thee, seven times seven years…", "And he took the seven aloaves and the fishes, and gave thanks, and brake them…", "… he appeared first to Mary Magdalene, out of whom he had cast seven devils…"

Even God – who is almighty – created the World in 7 days (including a day of rest)!

The Jewish New Year, Rosh HaShanah, and Yom Kippur are in the seventh months counted from Nisan, the first month of Jewish year.

The Talmud mentions seven prophetesses (Sara, Miriam, Deborah, Hanna, Abigail, Hulda and Esther) and seven archangels (Michael, Raphael, Gabriel, Uriel, Sariel, Raguel, and Remiel).

Seven, seven, seven…

In Islam there are seven brides and seven lands, seven gates to Heaven and seven footsteps to Hell. There are also seven prophets (Adam, Noah, Moses, David, Jesus and Mohammed). During the Hajj to Mecca, pilgrims have to go around Kaaba during praying seven times. A newcomer is given the name on the seventh day.

2.3. Enigmatic number Nine

This trick was very popular in my childhood. You write in advance number 9 on a piece of paper and put it in your pocket. Then you offer your friend: "Think of a two-digit number. Add one digit to another. Extract the obtained sum of digits from the initial number. Did it?" At this moment you take out of your pocket a piece of paper with number nine and declare: "Your new number can be divided by this number!"

You became a real magician in eyes of your friends!

Everybody begins to try: 16 – (1+6) = 9; 93 – (9+3) = 81… and so on!

Let us explain this mathematical trick. At the beginning, let us compile the following table:

0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
10	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**
20	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**
30	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**
40	**41**	**42**	**43**	**44**	**45**	**46**	**47**	**48**	**49**
50	**51**	**52**	**53**	**54**	**55**	**56**	**57**	**58**	**59**
60	**61**	**62**	**63**	**64**	**65**	**66**	**67**	**68**	**69**
70	**71**	**72**	**73**	**74**	**75**	**76**	**77**	**78**	**79**
80	**81**	**82**	**83**	**84**	**85**	**86**	**87**	**88**	**89**
90	**91**	**92**	**93**	**94**	**95**	**96**	**97**	**98**	**99**

You can observe that all numbers on the shadowed diagonal have the form of 9×*n,* where *n* is equal 1, or 2, or 3, …, or 10. Then you take notice of a strange thing: if you will perform operations described above, you get only numbers belonging to that shadowed diagonal. Indeed, all numbers of the first line give always number 9: indeed, 11 – (1+1) = 9; 12 – (1+2) = 9; …; 19 – (1+9) = 9. Then all numbers on the second line give always number 18: indeed, 21 – (2+1) =18; 22 – (2+2) = 18; …; and finally, 29 – (2+9) = 18, etc.

It is interesting that all numbers on the shadowed diagonal have sum of their digits equals to 9! It makes, probably, even more intriguing form of the question after your friend got the final two-digit number: "Now, add both digits of the obtained number. You got…" And you again show the same piece of paper with written in advance number 9. It seems that such trick is even more intriguing than initial one!

Your skeptical friend may notice: "However, you considered only numbers less than hundred…" And he will be right! For instance if you take number 101 and perform with it all described above operations, you get:

101 – (1+1) =99 and 9+9 =18, not 9. Though, as you notice the obtained number is divided by 9 without reminder.

O.K., now let us explain why all it happens.

Take an arbitrary two-digital number consisting of X tens and Y units. The number can be written in the form X×10 + Y. Now extract from this number the sum of X and Y:

$$X \times 10 + Y - (X+Y) = [X \times 10 - X] + [Y - Y] = X \times (10 - 1) = X \times 9.$$

The last number is always can be divided by 9 without reminder! And moreover, this statement is correct for any initial numbers. For instance, take some huge number, say, 9376. The sum of digits gives 9+3+7+6 = 25. Take the initial number and extract 25: 9376 – 25 = 9351. Now, take sum of digits of the obtained number: 9+3+5+1 = 18! Again we have a number that can be divided by 9 without reminder.

Thus, you can declare that for arbitrary numbers the sum of a new number (obtained after completing the described above operations) can be always divided by 9 without reminder.

2.4. Twelve

You remember that number 12 played a special role in Sumerian numeration system. Probably, this number were connected with first observations of Nature: there are 12 full Moon nights during a year.

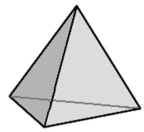
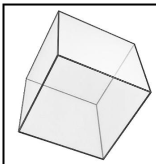
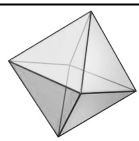
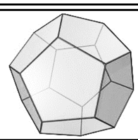
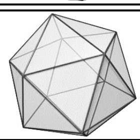
This number is absolutely unique in sense of appearance in mythology, religion, literature…

However, let us begin with more abstract things: consider the so-called Plato[29] solids. They are regular convex polyhedron. Their faces are congruent[30] regular polygons, with the same number of faces meeting at each vertex. There are five Platonic solids; the name of each of them is derived from the numbers of its faces. The symmetry and beauty of the Platonic solids have made them a favorite subject of geometers for millenniums.

Number 12 is very essential for these solids.

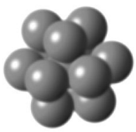
[29] They are named for the ancient Greek philosopher Plato (424-348 BC).

[30] In geometry, two figures are congruent if they have the same shape and size, though can be arbitrary oriented.

Polyhedron		Vertices	Edges	Faces
tetrahedron		4	6	4
cube, or hexahedron		8	12	6
octahedron		6	12	8
dodecahedron		20	30	12
icosahedron		12	30	20

So, among five Platonic solids only tetrahedron (pyramid) does not relate to number 12! Notice that the sum of number of vertices, facec and edges deliver in total number 13. However number 13 is a subject the following section.

In addition notice that there is one more interesting fact: one can located exactly 12 identical spheres touching each other around a sphere of the same radius.



Number 12 frequently appeared in Greek mythology.

12 Olympic Gods formed Pantheon[31]…

12 labors of Hercules,

12 fallen Trojans,

[31] Pantheon is formed by Greek words “Πάν” and “θειον” that means “all Gods”.

12 sacrificial bulls,
12 Penelope's bridegrooms,
12 foots of Scylla, etc., etc. …

Number 12 one can find in many places in the Bible (in the Old Testament as well as in the New Testament).

12 Jacob's sons: Reuben, Simeon, Levi, Judah, Dan, Naphtali, Gad, Asher, Issachar, Zebulun, Joseph, and Benjamin (Pay attention that Jacob had four wives "in parallel" ☺)…

12 Israeli tribes (from each of Jacob's sons)….
12 prophets in the Old Testament…
12 gate of Jerusalem…

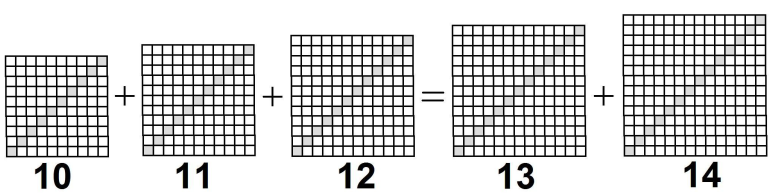
Christianity inherited this tradition to use number 12. First of all, Jesus had 12 followers-apostles. In Christianity number 3 is considered as divine and number 4 as human. So, guess yourself, what gives the number equals 3×4? Of course, it is the number of divine harmony!

Number twelve is called a dozen. This word is originated from French "douzaine", or Italian "dozzina", which, in turn, are originated from Latin "duodecim" that means "twelve".

"Idolaters" of number 12 noticed that the only quintuple of numbers in a row, namely, 10, 11, 12, 13 and 14 leads to a fascinating equality:

$$10^2+11^2+12^2=13^2+14^2.$$

And number 12 is in the center of this enigmatic equality. This equality can be illustrated by the following picture.



Let us finish this story about dozen by demonstration of way of writing this number in different numeration systems.

١٢	**Arabic**	ԺԲ	**Armenian**
𒌋𒐖	**Babylonian**	ιβ´	**Ionic (Old Greek)**
ΔII	**Attic (Old Greek)**	יב	**Jewish**
∩||	**Old Egyptian**	१२	**Indian**
В҃І	**Old Cyrillic**	十二	**Chinкитайская и японская**
••=	**Maya**	க௨	**Tamil**
XII	**Roman**	สิบ๒	**Thai**

2.5. Thirteen

After discussing about number 12, one needs to continue with 13, though a lot of people do not like this number… In Russian this number even is called “the Devil’s Dozen”.

In Middle East and China even nowadays the number 13 is believed to be connected somehow with a death. However, especially strong this kind of superstition in Christian countries. It is told that Jude was the 13th Jesus’ disciple.

In medicine there is a specific disease – triskaidekaphobia (from Greek “tris” meaning "3", “kai” meaning "and", “deka” meaning "10" and “phobia” meaning "fear"). This is superstitious fear of the number 13. Some “super superstitious” people are especially afraid Friday the 13th. This specific kind of superstition called *paraskevidekatriaphobia* or *friggatriskaidekaphobia*.

Austrian composer Arnold Shoenberg Schoenberg experienced triskaidekaphobia. He was born September 13th 1874 and etire his life believed that it was a bad omen. He wrote opera *Moses und Aron* that was originally spelled *Moses und Aaron*, but when he realized that this title contained 13 letters, he changed it. His superstitious nature may have triggered his death. In 1950, on his seventy-sixth birthday, he was deadly scared by his acquaintance astrologer ‘s letter with warning that 76 is a dangerous number for him, since 7 + 6 = 13. He died on Friday, 13 (!!!) July 1951 at 11:47 p.m., i.e. 13 minutes before midnight…

Why namely Friday? The Holy Bible says that Jesus Christ was crucified on Friday…

Taking care of “triskaidekaphobists”, some office buildings and hotels have no 13th floor: corresponding floor is either 12-A or even just 14th. Some avoid use 13 for numbering offices or ship’s cabins.

In England the number 13 is often called “the baker’s dozen”. Some explain this idiom by the fact that medieval English bakers were severe penalized for short weight (up to a hand severance!). So, bakers preferred to give to avoid accusation in trickery.

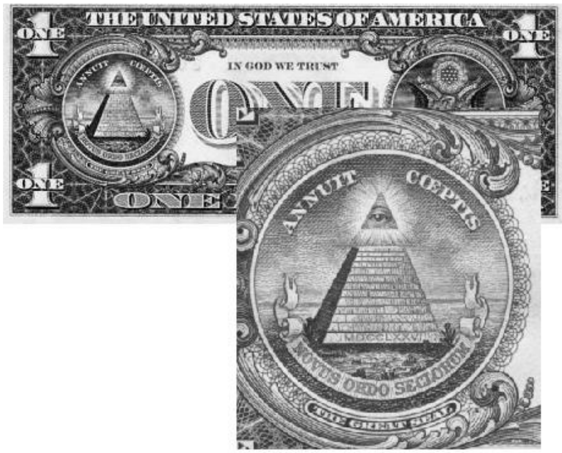
In the United States, however, the number 13 is considered rather happy than unhappy. One can find this number in many American symbols.

On the Great Seal of the United States, the eagle holds in one claw an olive branch with 13 leaves and 13 olives and in the other claw – a bundle of 13 arrows. The shield located in front of the eagle’s breast has 13 red and white stripes, and above his head there are 13 stars. In its beak it holds a scroll with the inscription: «E Pluribus Unum», which translates from Latin as "Out of many – one".

Of course, this multiple repetition of the number 13 symbolizes that the United States of America consisted of thirteen states. Thus, the very history of the United States makes the number 13 symbolic for the Americans.

Take a look on American dollar. You can see on the back side again the number 13: thirteen layers of pyramid’s stones, again 13-letter inscription…

… And you can see above the pyramid a triangle with an eye inside it. This is a standard Masonic[32] symbol that is called "Radiant Delta"!



Probably, there was a track of influence of one of the Founding Fathers of the United States of America George Washington was a mason.

However, for the United States, the number 13 was not always lucky. Apollo 13 was the seventh manned mission in the American Apollo space program and the third intended to land on the Moon. The craft was launched on April 11, 1970, at 13:13 from the Kennedy Space Center in Florida. The lunar landing was aborted after an oxygen tank exploded two days later, on April 13… Fortunately, the crew returned safely to Earth on April 17.

By the way, in native America the number 13 was always "in honor". Maya and Aztec consider the number 13 sacred, In their mythology the Heaven was divided into 13 layers, each of which

[32] **Freemasonry** is a fraternal organization that arose from obscure origins in the late 16th to early 17th century. The name origins frm French "franc-maçon" that literally means "freemason".

having its own god. In ancient Aztec calendar there were 13-day weeks.

2.6. Enigma of the Primes

These simple primes are not so simple
as they seem at the first glance!
Unknown author

Prime numbers intrigue mathematicians. First, let's define a prime? It is an integer number that can be divided only by 1 and itself (for instance, 2, 3, 5, 7, 11, 13, and so on). It is clear that even numbers except 2 are not prime. A list of the first primes under 500 is given below for those who are curious:

2	**3**	**5**	**7**	**11**	**13**
17	**19**	**23**	**29**	**31**	**37**
41	**43**	**47**	**53**	**59**	**61**
67	**71**	**73**	**79**	**83**	**89**
97	**101**	**103**	**107**	**109**	**113**
127	**131**	**137**	**139**	**149**	**151**
157	**163**	**167**	**173**	**179**	**181**
191	**193**	**197**	**199**	**211**	**223**
227	**229**	**233**	**239**	**241**	**251**
257	**263**	**269**	**271**	**277**	**281**
283	**293**	**307**	**311**	**313**	**317**
331	**337**	**347**	**349**	**353**	**359**
367	**373**	**379**	**383**	**389**	**397**
401	**409**	**419**	**421**	**431**	**433**
439	**443**	**449**	**457**	**461**	**463**
467	**479**	**487**	**491**	**499**	

The occurrence/distribution of primes is rather irregular: due to this property primes have even been called "bewitched." A sample of the distribution of primes is given below.

Interval	**Qty of Primes**
From 0 to 99	**25**
From 100 to 199	**21**
From 200 to 299	**16**
From 300 to 399	**16**
From 400 to 499	**17**

....	....
From 1000 to 1099	**16**
....	....
From 2000 to 2099	**14**
....	....
From 3000 to 3099	**12**
....	....
From 4000 to 4099	**15**
....	....
From 5000 to 5099	**12**

A mathematician, a physicist and an engineer prove the same statement: all odd numbers are primes.

The mathematician thinks and says:" 3 is prime, 5 is prime, 7 is prime, 9 is not prime... Thus, the statement is wrong".

The physicist with a paper and pencil: "3,5 and 7 are prime, 9 – is an error of the experiment, 11 and 13 and so on are prime, so it is very possible that all odd numbers are simple".

The engineer, looking on the ceiling, says: "So, well… 3 is prime, 5 is prime, 7 is prime, 9 is prime, 11 is prime… O.K., it is obvious that all odd numbers are prime!"

Among primes, there are the so-called "numbers-twins" that separated by a single even number, for instance, 11 and 13, 41 and 43. "Twins" appear not too often and with strange periodicity, though the larger the numbers, the more rare the appearance. For instance, among the first 100 numbers, there are the following pairs: (3, 5); (5, 7); (11, 13); (17, 19); (29, 31); (41, 43) and (59, 61). Numbers (3, 5, 7) could be called "triplets;" however, such triplets are evidently unique, probably, and no more such triplets exist. At the same tine, "twins," such as (9767, 9769) and (9857, 9859), are found in the area of 10000 in a distance about only one hundred!

A similar situation is observed with ordinary primes: around quintillion (10^{18}) only one of 28 numbers, on average, is prime. In further areas, one can find arbitrary long intervals without primes at all. The question: "Is the number of twins finite or infinite?" is still not answered.

By now the largest twins found are $33218925 \cdot 2^{169690} \pm 1$.

How many primes exist? Already Euclid knew that there are infinitely many such numbers. His elegant and simple proof given in "*The Elements*" (volume IX, statement 20) can be briefly formulated in the following manner: "Assume that the quantity of the primes is finite. Multiply all of them and add one. The new number cannot be divided by any of the primes because after dividing, one has residual equals 1. Thus this new number can be divided only by itself, i.e. it is prime". It is ingeniously simple and elegant, isn't it?

Today there are many algorithms for finding primes and testing whether a number is prime. However, this problem remains challenging. Mathematicians and programmers are trying to compile more reliable and more effective methods of generating primes.

Eratosthenes[33] found the very first (and simple) method. This method is called "*Eratosthenes' Sieve,*" and the principle is as follows. Eratosthenes wrote all the numbers from 1 to 1000 and made holes in the place of the numbers that had dividers different from the chosen number itself (i.e. removed non-prime numbers). So, his papyrus became like a sieve. Let us consider the procedure in more details. The first prime is 2. On the first step of the procedure, Eratosthenes made holes on places of all even numbers. (See the first row of the table below.)

Numbers	2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 ...
Remove every 2nd number	2 3 • 5 • 7 • 9 • 11 • 13 • 15 • 17 • 19 ...
Remove every 3rd number	2 3 • 5 • 7 • • • 11 • 13 • • • 17 • 19 ...
Remove every 5th number	2 3 • 5 • 7 • • • 11 • 13 • • • 17 • 19 ...
And so on	

Then he started again from the very beginning and made holes on places of the numbers that can be divided by 3. (See the second row.) The next prime is 5. The procedure continues: among remained numbers, one finds those that had divider equals 5, and so

[33] **Eratosthenes of Cyrene** (275 - 194 BC), ancient Greek poet and universal scientist, dealt with mathematics, astronomy, geography, philology, and chronology.

on. The process continues until all primes within the chosen interval are found.

Pierre Fermat[34] made a number of intriguing notes on the margins of Diophant's[35] book, the same book that is so famous due to the Fermat's note about the theorem that was named later after him. Fermat formulated a theorem about the primes. He found that all primes could be divided into two groups: one of them can be represented in the form $4n+1$, and other in the form $4n-1$, where n is some integer number. For instance, the number 13 belongs to the first group ($13 = 4\cdot3 + 1$), and 19 belongs to the second one ($19 = 4\cdot5 - 1$). Fermat's Theorem states that all numbers of the first group can be presented in the form of the sum of two squares (for instance, $13 = 2^2 + 3^2$), and elements of the second group cannot be presented in the form of the sum of two squares (for instance, number 19). Thus, he identified a property of the primes in a simple and elegant form, though all attempts of Fermat's contemporaries to prove the theorem were unsuccessful, and Fermat himself – as usual! – did not leave the proof. Leonhard Euler proved the theorem almost 100 years later!

In 1859 Georg Riemann introduced a concept of analytical zeta-function and stated a hypothesis (Riemann's Hypothesis) that the number of primes which value is less than X can be expressed through the distribution of the non-trivial roots of that function. This hypothesis still neither has been proved nor rejected.

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34 **Pierre Fermat** (1601-1665), French mathematician, one of the creators of analytical geometry and the number theory, author of many works in probability theory, optics, and calculus.

35 **Diofant of Alexandria**(the 3rd century BC), ancient Greek mathematician. In his main tractate "*Arithmetic*" (only 6 of 13 books are saved), he gave solutions of some problem in therm of so-called Diofant's equation (with only integer numbers). He also was the first who introduced letter symbol into algebra.

Georg Friedrich Bernhard Riemann (1826-1866)

German mathematician, pupil of Karl Gauss. He founded new directions in geometry, developing non-Euclidian (Riemann's) geometry, introduced Riemann's integral. The theory of quadratic differential forms developed by him is used in the theory of relativity.

His last work contained the famous Riemann's Hypothesis.

He was elected as the member of London Royal Society and French Academy of Sciences.

Most mathematicians believe that Riemann's Hypothesis is correct: at least the first 1,500,000,000 roots have been checked and confirm Riemann's guess...Riemann's hypothesis is one of the most important open problems of contemporary mathematics; the Clay Mathematics Institute has offered a $1,000,000 prize for a proof. So, if you need money – go ahead!

Among the primes, the Mersenne's[36] primes take a special place. These numbers have the form $M_p = 2^p - 1$, where p is a prime. Of course, not all numbers of kind 2^p-1 are prime. For instance, $M_2=2^2-1= 3$, $M_3=2^3-1=7$, $M_5=2^5-1=31$, $M_7=2^7-1=127$ are prime Mersenne's numbers, though $M_{11}=2^{11}-1= 2047=23\times 89$, i.e. is not prime.

By 1750 there were only 8 prime Mersenne numbers found: M_2, M_3, M_5, M_7, M_{13}, M_{17}, M_{19}, M_{31}. The latter number, M_{31}, was found by Leonhard Euler. In 1876 Edward Lukas[37] found that the following number is a prime one:

$$M_{127}=170141183460469231731687303715884105727.$$

It is interesting that he claimed that $2^{127} - 1$ when he was only 15 years old! He finished his proof in 14 years. &5 years this

[36] **Marin Mersenne** (1588-1648), French priest educated in Sorbonne. He was a friend of Fermat and Pascal, who visited him in his cell. He is one of the founders of the Paris Academy of Sciences.

[37] **Francois Edouard Anatole Lucas** (1842-1891), French mathematician.

number was the largest one among Mersenne numbers. This may stand forever as the largest prime number proven by hand.

Probably, the most astonishing case occurred in Russia: in 1883 a priest of a village in the Northern Russia Ivan Pervoushin[38] with no special calculation tools proved that $M_{61}=2^{305843009213693951}$ is prime. Later, it was proved that M_{89} and M_{107} are prime. Using computers allowed in 1952-1964 to prove that M_{521}, M_{607}, M_{1279}, M_{2203}, M_{2281}, M_{3217}, M_{4253}, M_{4423}, M_{2689}, M_{9941}, M_{11213} are prime. By 1964 23 prime Mersenne's numbers were known.

In 2003 chemistry student from the University of Michigan (USA) found the 40th prime Mersenne's number; it is $2^{20\ 996\ 001}-1$ (this number contains 6320430 digits). The largest prime Mersenne's number known as of this writing is $M_{24036583}=2^{24036583}-1$; it contains 7235733 decimal digits. This 41st number was found in 2004 by Josh Findley in the frame of project GIMPS (Great Internet Mersenne Prime Search).

In February of 2005 Martin Nowak, surgeon-ophthalmologist from Michelfeld (Germany), who was also working in the frame of GIMPS, has found $M_{42}=2^{25964951}$-1, which can be written with 7816230 digitals.

At last, the most recent (by the moment of the book preparation) Mersenne's number, M_{43} has been found in December of 2005. It has been found by the team led by Professors of University of Missouri Curtis Cooper and Steven Boone. This number is equal to $2^{30402457}$-1 and has 9 152 052 digits. Though this number is so huge, it is not enough to get the prize of $100,000 that Electronic Frontier Foundation awards for a prime with more than 10 million digits.

Of course, to find all primes with the Mersenne procedure is not possible: in interval from 1 to maximum known prime Mersenne's number, there are other primes. However, Mersenne's numbers are so popular due to the effective deterministic primality Lucas-Lehmer[39] test. We won't give any explanation of the test so

[38] **Ivan Mikheevich Pervushin** (1827 - 1900), Russian priest and mathematician, speciaslist in the Number Theory. For his mathematical achievements he was appointed correspondence member of Russian Saint Petersbur, Paris and Napolitano Academies of Sciences.

[39] This test was initially suggested by Lucas in 1878, and in 1930 **Derrick Norman Lehmer** (1867-1938) essentially improved it.

as not to put off even brave readers… Due to this test, prime Mersenne's numbers became the largest of all known primes. By the way, for finding a prime with over 10^7 decimal digits you can get a prize of $100,000... Less than for proving the Riemann Hypothesis, but still worthy of note! So, hurry up!

There are still many open questions about primes, for instance:

- Goldbach conjecture [40]: Is it true that any number larger than 5 can be presented as a sum of three distinct primes?
- Is it correct or not that there exists an infinite quantity of "twins"?

Goldbach's conjecture is one of the oldest open problems in mathematics. In a letter to Euler in 1742, he suggested that every integer $n > 5$ is the sum of three primes. Euler replied that this is equivalent to every even $n > 2$ is the sum of two primes. Now this slightly corrected conjecture is now known as Goldbach's conjecture.

Later it was shown that Goldbach's conjecture is equivalent to every integer $n > 17$ is the sum of three distinct primes.

It interesting to get a feeling for ourselves of Goldbach's conjecture by looking at a few numerical examples:

4=3+1
6=5+1
8=5+3=7+1
10=7+3
12=7+5=11+1
14=11+3=13+1
16=11+5=13+3
18=11+7=13+5=17+1
20=13+7= 17+3= 19+1
22=13+11=17+5=19+3, and so on.

So, such a simple statement still has not been proved, though a lot of attempts have been undertaken. By 2004 Goldbach's conjecture had been tested for all even numbers under $2{\cdot}10^{17}$.

---

[40] **Christian Goldbach** (1690-1764), Russian mathematician born in Prussia, professor of astronomy, one of the first academicians of the St. Petersburg Academy of Sciences.

It is time to say that for practical purposes (for instance, in cryptology) it is important not only to construct a sequence of primes: substantially more important is confirmation that the number is prime. The proof of – as one can say, abstract – problem might principally change the entire concept of modern cryptographic systems, which are playing such essential role in the modern society from security of state secrets to reliability of bank accounts.

2.7. What about fractions?

Integer numbers are very interesting… However, fractions are no less interesting. In everyday life most often we deal with rational fractions whose nominators and denominators are integers. If one deals with algebraic equations, irrational fractions could appear. Finally, there are transcendental fractions, which are irrational but do not present solutions of any algebraic equation.

Rational number can be presented in the form *m*/*n*, where *m* and *n* are integers. Rational number also can be presented by a finite decimal fraction, for instance, 2/5=0.4 or infinite periodical fraction: 1/7=0.1428571428571… =0.(142857) or 119/300 = 0.37666666... = 0.37(6).

If take all rational numbers and add to them all possible infinite periodical and non-periodical (irrational) fractions, then we will have all real numbers.

Algebraic numbers are those numbers which can be obtained as the solution of algebraic equations. These numbers can be rational (including integers) or irrational. Simple examples are presented by square roots, for instance, $\sqrt{4} = 2$ or $\sqrt{2}$ =1.4142135623731… Irrational algebraic numbers represents only a part of all irrational numbers. The other numbers are called transcendental numbers.

In connection with irrational numbers, an interesting mathematical object appears: the so-called *chain fraction*. This fraction has the form:

$$\frac{a}{b} = q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \cfrac{1}{\cdots \cfrac{\cdots}{q_n + \cfrac{1}{q_{n+1}}}}}}$$

Chain fractions can be finite or infinite. Numerical values of finite fractions can be calculated "from the bottom to the top." Consider as example the following chain fraction:

$$2 + \cfrac{1}{5 + \cfrac{1}{4 + \cfrac{1}{8}}}.$$

First, calculate $4 + \frac{1}{8} = \frac{33}{8}$, then $5 + \frac{8}{33} = \frac{173}{33}$, and, finally,

$$2 + \frac{33}{173}.$$

Let us write a rational fraction in the form of chain fraction using Euclid's algorithm. For example, take the fraction 50/13. For the first step we get:

$$\frac{50}{13} = 3 + \frac{11}{13} = 3 + \frac{1}{\Delta_1}.$$

From here it is clear that Δ_1 = 13/11. Let us continue the procedure as long as possible:

$$\Delta_1 = \frac{13}{11} = 1 + \frac{2}{11} = 1 + \frac{1}{\Delta_2},$$

$$\Delta_2 = \frac{11}{2} = 5 + \frac{1}{2},$$

and, finally,

$$\Delta_3 = \frac{2}{1} = 2.$$

Now let us substitute all obtained Δ_k and the final result is:

$$\frac{50}{13}=3+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{2}}}.$$

Example of an infinite chain fraction is presentation of $\sqrt{2}$:

$$\sqrt{2}=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ldots}}}}$$

The value of an infinite chain fraction is equal to the limit of the infinite sequence of finite chain fractions with increasing "lengths."

2.8. Irrational numbers in this rational world

In Latin the word "*irrational*" means "*illogical*", "*unreasonable*" or even "*imprudent*." Any number that cannot be presented as a fraction m/n of two natural numbers m and n is called irrational. Thus, you understand now what kind of numbers we are going to discuss…

Pythagoras' scientific school based its philosophy on the harmony of numbers. They believed that everything in the Nature could be explained using integers and their ratios. Rejection of that idea was considered to be treason. However, Pythagoreans themselves discovered "incommensurable segments" while analyzing the ratio of the diagonal of a square to its side. They proved that this ratio couldn't be expressed as the ratio of any two integers.

Pythagoras himself tried to find the universal law of World harmony based on numbers and geometrical schemes. He believed that only natural numbers and rational fractions formed the basis of Nature. Any attempt to reject this idea was considered as an apostasy.

It is said that Pythagoras himself rejected the discovery of irrational numbers because it would destroy all the beauty of his ideals. He attempted up to the last days of his life to find the numbers *m* and *n* which would give a ratio m/n equal to the diagonal of a square with unit sides.

Pythagoreans trying to conceal the discovery of irrational numbers created a myth about the death of Hippasus, Pythagoras' pupil, who disclosed the fact of irrational numbers. (Who knows: may be it is not a myth but a real event?)

According to the legend, Hippasus tried to find a fraction, which would be equal to the ratio between the side of a square and its diagonal. At last, he came to the conclusion that there is no such fraction at all.

Hippasus' discovery "could" destroy the entire Pythagorean mathematical concept of Nature. For this "scientific sin" the Pythagorean Brotherhood condemned him to death through drowning…

Two centuries would pass before Euclid would prove that $\sqrt{2}$ cannot be a rational number and, in doing so, gave us with the method of proof "by contradiction."

Euclid's proof is so simple and so elegant that we present it here.

Euclid of Alexandria
(365 - 300 BC)

Great ancient Greek mathematician, one of the founders of the Alexandrian scientific school. He was one of the first mentors at the Alexandria Museum and Library.

In his fundamental work "*The Elements*" he summarized all known mathematical approaches of that time and founded the base for the future developments. It was a geometry textbook for more than two millennia, up to the 20th century.

For more details see Chapter "Pantheon".

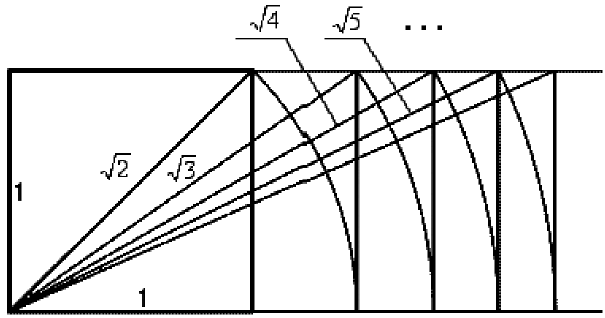
Assume that statement that $\sqrt{2}$ can be presented as some fraction is correct. Let us write this fraction in the form p/q where p and q are integers. Raising to the second power both sides of the equality $\sqrt{2} = p/q$, we get $2 = p^2/q^2$, or after rewriting, $2q^2 = p^2$. For any number q^2, after multiplying on 2, it became even, that is p^2 is also even, and consequently, p is also even. However, if p is even, then it can be presented in the form $2m$ where m is some other integer. Substituting $p = 2m$ in the equality for p^2, we got $2q^2 = (2m)^2 = 4m^2$. Now let us reduce both sides of the equality by 2: $q^2 = 2m^2$.

Using the same arguments, we conclude that q^2 should be even, i.e. q is even as well. Now, returning to the initial formulation, we can write: $\sqrt{2} = p/q = 2m/2n$.

Evidently, the fraction $2m/2n$ can be reduced by 2: $\sqrt{2} = m/n$. The new fraction is simpler than the previous one, though we are again at the same position! From the proof, it is clear that the process of reducing can be continued infinitely. However, the fraction with any initially large numerator and denominator cannot be reduced infinitely but our fraction can be reduced infinitely.

So, we came to the controversy, and so our initial assumption was wrong, that is $\sqrt{2}$ cannot be presented in the form of fraction, i.e. it is irrational.

Thus, the $\sqrt{2}$ was the first number proven to be irrational. A final impact of this proof/discovery was the seed for the development of irrational number theory and of "continuous mathematics." One can get infinitely many numbers of types $\sqrt{n}$, using, for instance, the following geometrical method.



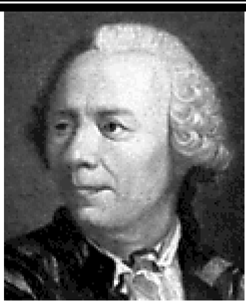
A diagonal of a square with a unit side is equal to $\sqrt{2}$. Using dividers, make a horizontal side of the next rectangular equal to that diagonal. So, we have a right triangle with sides 1 and $\sqrt{2}$. The hypotenuse of this new triangle by the Pythagorean Theorem is equal to $\sqrt{3}$. Continuing this construction, one can get $\sqrt{n}$ for any natural *n*. All these numbers are irrational, except those *n* which are perfect squares.

However, Euclid while accepting irrational intervals considered only rational numbers. After Euclid, ancient Greeks and Romans did not accept irrational numbers. Arabs translated their names as "*asamm*" that means "*deaf, invalid*", though Omar Hayam[41] wrote that irrational numbers possess properties close to the

41 **Omar Hayam**, or (full name) **Ghiyath al-Din Abu'l-Fath Umar ibn Ibrahim Al-Nisaburi Al-Hayami** (1048 – 1122), Persian poet, philosopher, mathematician and astronomer, follower of Aristotle and Ibn Sina (Avicenna) (980-1037), who was a famous Uzbek philosopher and physician. Created theory of solving square and cubical equations, first expose a hypothesis that parallel lines can cross each other. He led the famous Samarkand observatory having been built by Ulug Beg, compiled a new calendar. He is mostly known for his rubai – a new form in poetry.

rational ones. In works of Al-Kashi[42] and Simon Stevin[43], one can find an idea of approximate presentation of irrational numbers by decimal fractions.

At last, Descartes put all numbers on the same numerical axes showing that there is no principal difference between rational and irrational numbers. And finally, Euler proved that any irrational number could be represented by non-periodic decimal fractions.



Leonard Euler
(1707–1783)

One of the greatest mathematicians ever. He was born in Switzerland, then was a member of Russian Academy of sciences, afterwards quarter of century worked in Germany and returned back to Russia where he worked 17 years more. Being sick and blind, he dictated his works to his son and an assistant.

He made a great input in many spheres: in calculus, theory of numbers, combinatory, probability theory, mechanics, astronomy, physics and even in theory of music.
For more details see Chapter "Pantheon".

Now is the time to say several words about transcendental numbers. These numbers are irrational numbers that are not solutions of algebraic equations with real coefficients. All of us know of or at least have heard of the well known numbers "e" and "pi". Another transcendental number is the somewhat less well known "golden proportion."

So, transcendental numbers existed but nobody had a clear definition of their properties. The first transcendental number was

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[42]**Ghiyath al-Din Jamshid Mas'ud al-Kashi** (1380-1436), Uzbek mathematician and astronomer. In his main work "The Key to Arithmetic" he introduced decimal fractions, and suggested rules for the extraction of roots.

[43] **Simon Stevin** (1548-1620), Dutch mathematician and engineer was the first to introduce decimal fractions in Europe (the Al-Kashi works were not known in Europe by the time). He introduced negative roots of equations and gave approximate methods of root calculation.

mentioned by Josef Liouville[44] in 1844. Then Georg Kantor[45] in 1874 proved the existence of transcendental numbers by showing that the set of all algebraic numbers is countable while the total set of all numbers is a continuum.

In 1873 Charles Hermite[46] proved that the number "e", playing an exclusive role in calculus is transcendental. And in 1882 Ferdinand Lindeman[47], developing Hermite's method, proved that the number "pi" also is transcendental.

3. ABSOLUTELY SPECIFIC NUMBERS

3.1. Perfect Numbers

Olympic Games… World Championships… Miss Universe Competitions… Who does not like to be perfect (or at least to get such a title)?

Can you imagine: there are perfect numbers! A real number *P* is called perfect if it is equal to the sum of its divisors.

Ancient Greeks knew of only four perfect numbers. Numbers 6 and 28 were known to Pythagoreans (divisors of 6 are 1, 2 and 3 that gives 1+2+3=6, and 28 has divisors 1, 2, 4, 7 and 14 that gives 1+2+4+7+14=28.

Pythagoras studied perfect numbers very carefully. One of his discoveries was the observation that perfection of the number is tied to "duality". Let us explain it using modern notations: numbers 4, 8, 16 and so on can be written in the form 2^n, where n is the number of times two is multiplied by itself. Pythagoras found that all such numbers are "almost perfect": the sum of all dividers is less than the initial number by exactly 1! Indeed:

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[44] **Josef Liouville** (1809-1882), French mathematician, member of Paris Academy of Sciences.

[45] **Georg Kantor** (1845-1918), German mathematician, founder of the Theory of Sets.

[46] **Charles Hermite** (1822–1901), French mathematician, member of Paris Academy of Sciences and London Royal Society.

[47] **Ferdinand von Lindeman** (1852-1939), German mathematician.

- $2^2 = 2 \cdot 2 = 4$, dividers of this number are: 1 and 2, and their sum equals 3,
- $2^3 = 2 \cdot 2 \cdot 2 = 8$, dividers of this number are: 1,2 and 4, and their sum equals 7,
- $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$, dividers of this number are: 1, 3, 4 and 8, and their sum equals 15,

and so on...

Two centuries after Pythagoras, Euclid developed this idea further found that perfect numbers consist of two multipliers: one of them is a power of 2, and another is next power of 2 minus 1, i.e., for instance:

$$\mathbf{6 = 2^1 \cdot (2^2 - 1),}$$
$$\mathbf{28 = 2^2 \cdot (2^3 - 1),}$$
$$\mathbf{496 = 2^4 \cdot (2^5 - 1),}$$
$$\mathbf{8128 = 2^6 \cdot (2^7 - 1).}$$

Euler rigorously proved that if both numbers p and 2^p-1 are primes, then $P_p=2^{p-1}\times(2^p-1)$ is perfect. Let us demonstrate it on a couple numerical examples.

Let $p=3$, then 2^p-1=7, i.e. both numbers are prime. The result, $2^2\times7=28$ is perfect number. Now let p=5, then $2^5-1=31$, and 24×31=496 is again a perfect number.

With computers the hunting for perfect numbers accelerated. Now a "numerical monster" is known: $2^{216090}\cdot(2^{216091} - 1)$ is a perfect number, which contains more than 130,000 digits.

Pythagoras found that perfect numbers are equal to the sum of consequent natural numbers; for instance, 6=1+2+3, 28= 1+2+3+4+5+6+7 and so on.

Christian religions held perfect numbers to be special. In the 12th century the church taught that he who found the fifth "Divine number" would have eternal blessing. Actually this number, P_{13}, was found in the 15th century, though the name of the person who won the lottery ticket to Heaven remains unknown…

Perfect numbers were objects of special interests among early Christian priests. In particular, St. Augustine (354-430) in his composition "City of God" discussing the Creation of the World, wrote: "These works are recorded to have been completed in six days, because six is a perfect number -- not because God required a protracted time, as if He could not at once create all things, but because the perfection of the works was signified by the number six" (Volume XI, Chapter 30 "Of the perfection of the number six").

We won't argue with one of the fathers of Christian church – he knew best…

From a purely mathematical viewpoint perfect numbers also are unique, indeed:

- The sum of all inverse values of dividers (including the initial number itself) always equals 2.
- Residual of dividing a perfect number by 9 (except 6, of course!) equals 1.
- In the binary system, the perfect number P_p begins with p units and they are followed by p-1 zeros, for instance: P_2=110, P_3=11100, P_5 =111110000, P_7 =1111111000000, and so on.
- The last digit of a perfect number equals either 6 or 8, and if it is 8 then it is preceded by 2.

Perfect numbers still keep many enigmas. Is the total number of them limited? Do odd perfect numbers exist? (There is a hypothesis that if such a number exists, it should have no fewer than 36 digits.)

Perfect numbers led to the so-called "friendly numbers". These are pairs of numbers such that the sum of the divisors of the first number equals the second number and vise versa. Of course, Pythagoreans were the first to discover a set of "friendly numbers:" they found 220 and 284. Divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55 and 110, and their sum equals 284. From the other hand, divisors of 284 are 1, 2, 4, 71 and 142, and their sum equals 220.

Besides these "Pythagorean" friendly numbers there were no other such numbers until Pierre Fermat, who in 1636 found 17296 and 18416. Of course, this is only a curiosity, though it shows that Fermat knew natural numbers very well. The third pair of friendly numbers was found by Descartes (9363 584 and 9437 056), and then Euler added to the list 59 new pairs! It is interesting that Descartes and Euler overlooked a very small pair of friendly numbers; in 1866, 16-year old Nicollo Paganini (no, no! he was not a relative of famous violinist[48]) found 1184 and 1210.

However, curious mathematicians continued to find new strange numbers. In the 20th century "related numbers" were introduced. They represent closed cycles of three or more numbers. For instance, in the triplet

(1 945 330 728 960; 2 324 196 638 720; 2 615 631 953 920)

The sum of the dividers of the first number gives the second number, the sum of the dividers of the second number gives the third number, and – as you can guess – the sum of the dividers of the third number gives the first number. The longest of the known cycles contains 28 "related" numbers, and the first of them equals 14 316...

The question arises: SO WHAT?

If you are tired of perfect and friendly numbers, look at these "funny" numbers...

- If the number 111 111 111 is multiply by itself, then a "funny" number appears 12345678987654321.
- If the number 21978 is multiplied by 4, it will reverse the order of the digits: 21978 × 4 = 87912.
- Year of I96I… What was interesting in this number? Stand up side down and read: you again see 1961. The next such "funny" year will be 6009. Afraid that it is too long to wait until then…

[48] **Nicollo Paganini** (1782 – 1840), Italian violinist and composer, one of the greatest virtuosos in the World musical performance.

3.2. Euler's Number

The number "*e*" often is called "Euler's number," because it was Euler who introduced this number into mathematical practice. This number has many interesting properties, especially in calculus. Logarithms to the base *e* are called natural logarithms.

Leonard Euler mentioned that *e* is a transcendental number, and he gave two presentations of it in the form of chain fractions:

$$\frac{e-1}{2}=\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cfrac{1}{18+\dots}}}}}$$

and

$$e-1=1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+\dots}}}}}}}}.$$

Of what use is the exponential function $y=e^x$? This function is the only solution of the differential[49] equation $\frac{dy(x)}{dx}=y(x)$. This function possesses the interesting property

$$\int e^x dx = e^x \text{ and } \frac{d}{dx}e^x = e^x.$$

[49] Authors ask for forgiveness from readers who have no knowledge of beginning calculus.

The natural logarithm of number *a* is denoted by ln *a*. From this definition follows that

$$e^{\ln x} = x = \ln(e^x).$$

The number *e* is equal to 2.7182818284...

To calculate the value of *e*, one can use the following formulas:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

or

$$e = \sum_{0\le n\le\infty}\frac{1}{n!}.$$

In the last formula *n*! is called the *factorial* of *n*. Factorial is defined as

$$n! = 1\cdot 2\cdot 3\cdot\ldots\cdot n = \prod_{k=1}^{n} k,$$

and it is assumed/defined that 0!=1.

Of course, it is easier to write any formula than to calculate with its help (especially, if you have operations with the infinity!). We demonstrate the process of calculation by these two methods. One can see that first method converges not too fast.

n	**$(1+1/n)^n$**
10	**2.593742460**
20	**2.653297705**
30	**2.674318776**
40	**2.685063838**
50	**2.691588029**
100	**2.704813829**
200	**2.711517123**
300	**2.713765158**
400	**2.714891744**
500	**2.715568521**
1000	**2.716923932**
10000	**2.718145927**

The second method, for instance, gives the value of 2.718… already on the sixth step while the first method need about 10000 steps.

0	**1**
1	**2**
2	**2.5**
3	**2.666666667**
4	**2.708333333**
5	**2.716666667**
6	**2.718055556**
7	**2.718253968**
8	**2.71827877**
9	**2.718281526**
10	**2.718281801**
11	**2.718281826**
12	**2.718281828**
13	**2.718281828**
14	**2.718281828**
15	**2.718281828**

It was in 1873 when Charles Hermite[50] finally proved that the number *e* is transcendental.

3.3. Number π

The symbol π («pi») is used to note the ratio of the length of a circle to its diameter.

π=3.14159265358979…

Would you like to know the value of number pi with more accuracy? Go to the end of this section; there is "Guinness' record book" for this number. Though you'll never need such accuracy…

The notation π for this number was introduced by the English mathematician William Jones in 1707 who chose the first

50 **Charles Hermite** (1822-1901) was a French mathematician, made many works in calculus, number theory and algebra.

letter of Greek word περιφερια, which means "circle". However, it became common in mathematics only after Leonhard Euler used it in his works.

Concerning this number itself, people have known of it since time immemorial. The earliest reference on the value of number "pi" is found in an Egyptian papyrus dated approximately 2000 BC. In ancient Egypt the area of circle was calculated as $\left(d-\frac{d}{9}\right)^2$, i.e. $d^2\left(\frac{8}{9}\right)^2$, where d is the diameter of the circle. We know that the exact area of the circle is $\pi\frac{d^2}{4}$, so it is easy to find that number "pi" in ancient Egypt was taken equal to $\pi = 4\cdot\frac{64}{81}=\frac{256}{81} \approx 3.1605...$

The number pi can be found in some proportions of the Great Khufu (Cheops) Pyramid in Giza, the last of the Seven Wonders in the Ancient World that survives. The ratio of the length of the side of the pyramid to its height divided by two gives us the value of number "pi" with 6 digitals accuracy! This value is much more accurate that that calculated by the great mathematician Archimedes who lived about two millennia later…In addition references to the value of number "pi" are found in the well-known Rhind's papyrus.

Ancient Babylonians gave a simpler though less accurate value of pi: $\pi = 3\frac{1}{8} = 3.125$.

In ancient India, several centuries BC, the value of pi was calculated to be $\sqrt{10} = 3.1622...$ (Notice that it is equal to ratio of the diagonal to the small side of the rectangle with sides 1 and 3.)

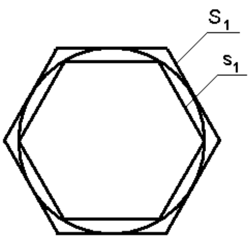
The ancient Chinese used the approximation $\frac{22}{7} \approx 3.1429$. Then, Chinese mathematician Zu Chongzhi[51] prior to 500 A.D.

51 **Zu Chongzhi,** or **Tsu Chung-Chi** (430-501), Chinese mathematician.

found the value: 3 + 1/(7 + 1/(15 + 1/1)) = 3 + 1/(7 + 1/16) = 3 + 1/(113/16) = 3 + 16/113 = 355/113.

This is an excellent approximation that was known to Europeans, who would not make mention of this fraction for another 1000 years!

Ancient Greeks tried to build a segment equal to the perimeter of the circle using only a divider. Naturally, their attempts were unsuccessful. The problem of the "*circle squaring*" is left in the history of mathematics as one of three classical problems of Antiquity[52]. Attempts to solve this non-solvable problem gave a number of interesting by-products, and one of them was the beginning of the usage of geometrical methods for this purpose.



Archimedes suggested a method, which gives the possibility to get the value of interest with any given accuracy. Moreover, "incidentally" he invented first iterative algorithm ever!

The Archimedes idea was as follows: he inscribed and circumscribed hexagons in a circle and measured their perimeters s_1 and S_1, measuring them in length of radius of the circle. If we denote the perimeter of the circle by X, then it is clear that $s_1 < X < S_1$.

Then he doubled the number of sides of the polygon and got new perimeters of inscribed 12-sided polygon S_2 and circumscribed s_2. It might be clear from the diagram that $s_2 > s_1$, $S_2 < S_1$. (By the way, ancient Greeks often used such kind of proving: "See the diagram!")

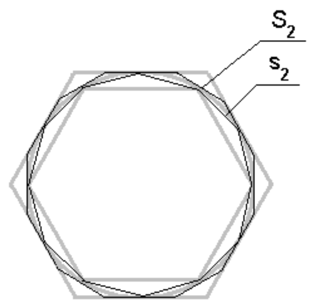
At the same time, the condition $s_2 < X < S_2$ is preserved, i.e. the upper bound decreased and the lower one increased.

52 Three problems of Antiquity: to build a cube with a double volume, divide an angle into three equal parts and circle squaring.

Continuing the procedure of increasing of numbers of polygon sides, the perimeter of the inscribe polygon will increase and converge to the perimeter of the circle from below. In the same manner the perimeter of the circumscribed polygon will decrease and converge to the perimeter of the circle from above. Archimedes in his work "*measurement of the circle*" constructed the polygon with 96 sides and found that the ratio of the circle perimeter to its diameter lies between $3\frac{10}{71}$ and $3\frac{1}{7}$, or in modern notations

$$3.140845... < \pi < 3.142875...$$

Everything is so easy, isn't it? No, it is not only simple, it is brilliant! Archimedes created the method of approximate calculations. Concerning calculations of the value of "pi", all ancient mathematicians used this method to get more precise results. On the basis of his own calculations, Archimedes defined approximate value of "pi" equal to 3.1419...



In Medieval times, calculating the value of "pi" became almost sportive competition; however, even such a great mathematician like Leonardo Fibonacci (we well tell about him in one of the next issues of the book) in 1220 defined 'pi" with only three decimal digitals.

Leonardo da Vinci wrote in one of his works: "The wheel of the vehicle turns around on the length of 10 elbows (an old unit of length) because the diameter of the wheel is equal to $3\frac{2}{11}$ elbows. It can be proven by multiplying the diameter by $3\frac{1}{7}$, because the product will be exactly 10". From this citation follows that Leonardo assumed that ratio

$$\pi = \frac{length\ of\ the\ circle}{radius\ of\ the\ circle} = 3\frac{1}{7}.$$

Scientist and poet Omar Khayyam also calculated the value of pi. Then in the beginning of 15^{th} century, astronomer and mathematician al-Kashi working at the Ulugbek[53] observatory near Samarkand, calculated the value of pi to 16 decimal figures. (He took a circle with an inscribed triangle and then made 27 iterations doubling the polygon sides.) It was a real mathematical record, which kept the first place almost two and a half centuries!)

In a century and a half after al-Kashi, in Europe Francois Viette[54] found pi to 9 places, using the same method which Archimedes invented! However, Viette made an important discovery: he noted that pi could be calculated with the help of the summation of some special infinite sets. It was very important because it led the way to calculations that were much more effective and convenient than the Archimedean one.

And after Viette's approach, among mathematicians there began a real "pi-searching fever"! John von Neumann[55] got 2037 places, and with appearance of modern computers the science became a "sportive enterprise:" now pi is calculated up to a billion decimal figures! And programmers go further and further because a computer is "iron" – it can bear anything!

Notice again that it was only at the end of 18^{th} century when pi was proved to be an irrational number, and it was at the end of 19^{th} century when it was proved to be transcendental.

While calculating pi to many places may be of interest only for some "Guinness World Records," the methods themselves are extremely interesting. Below we give some methods that were used

[53] **Muhammad Taragay Ulugbek** (1394 –1449), one of the most prominent astronomers and mathematicians of the fifteenth century. He had constructed a magnificent three-level observatory in Samarkand. By the way, Ulugbek was a grandson of Great Temur's (Tamerlane), legendary conqueror of the medieval time.

[54] **Francois Viette** (1540-1603), French mathematician who practically developed entire modern algebra. Viette's formulas give the relations between roots and coefficients of quadratic equation.

[55] **John (Janos) von Neumann** (1903-1957), Hungarian mathematician who immigrated in 1930 to the USA. He made a huge contribution to the development of the first computers and their implementation.

for finding value of pi. Though the methods are quite different, the result is the same!

Thus, Viette proved that

$$\frac{\pi}{4}=\left(\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}}\cdot\ldots\right)^{-1}$$

.One of the greatest mathematicians of all times – Leibniz was the first to suggest calculating pi with the following infinite sum:

$$\frac{\pi}{4} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots+(-1)^n\frac{1}{2n+1}+\ldots$$

Gottfried Wilhelm von Leibniz
(1646–1716)

German philosopher, physicist, and mathematician. He wrote on philosophy, science, mathematics, theology, history, and comparative philology, even writing verse. He was self-taught in mathematics, but nonetheless developed calculus independently of Newton. Although he published his results slightly after Newton, his notation was by far superior (including the integral sign and derivative), and is in use today.

Later Euler calculated number *pi* using another infinite set:

$$\frac{\pi}{4}=\left(\frac{1}{2}-\frac{1}{3\cdot 2^3}+\frac{1}{5\cdot 2^5}-\frac{1}{7\cdot 2^7}+\ldots\right)+\left(\frac{1}{3}-\frac{1}{3\cdot 3^3}+\frac{1}{5\cdot 3^5}-\frac{1}{7\cdot 3^7}+\ldots\right)$$

Wallis[56] found an interesting infinite product for calculation of pi:

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot\ldots}{1\cdot 1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot\ldots}.$$

Looking at all these different approaches to calculate pi, we come to understand how ancient mathematicians came to believe that pi was a magic number!

---

[56] **John Wallis** (1616-1703), English mathematician.

In conclusion, we give a reference in the Bible. A little known verse of the Bible, where a list of specifications for the great temple of Solomon is given, reads: "And he made a molten sea, ten cubits from the one brim to the other: it was round all about, and his height was five cubits: and a line of thirty cubits did compass it about" (*Kings*, III, Ch. 7, verse 23). It gives the value of *pi*=3.

* * * * *

Jokes about "pi" . Mathematicians invented a number of "around-mathematical" jokes about "pi". Why not enjoy them?

π π π π π

An engineer meets a mathematician and asks him:

- I don't understand, why when riding on train we hear those rhythmic knocking sounds: the wheels are round… Why is there this sound?

- I think, - the mathematician responds, - that the formula of a circle is "pi" by r in square. Probably that square is the knocking sound.

π π π π π

A mathematician, a physicist and an engineer gathered together. They are asked: "What is the number pi?"

The mathematician:

- "Pi" is the number equal to the ratio of the perimeter of a circle to the diameter of that circle.

The physicist:

- "Pi" is 3.142± 0.0005.

The engineer:

- "In my belief, "pi" is something close to three…

π π π π π

A mathematician is sitting at his working table. His son comes and asks:

-Dad, how I should write the figure 8?

- It is so simple, son: write the infinity sign and rotate it by pi divided by two.

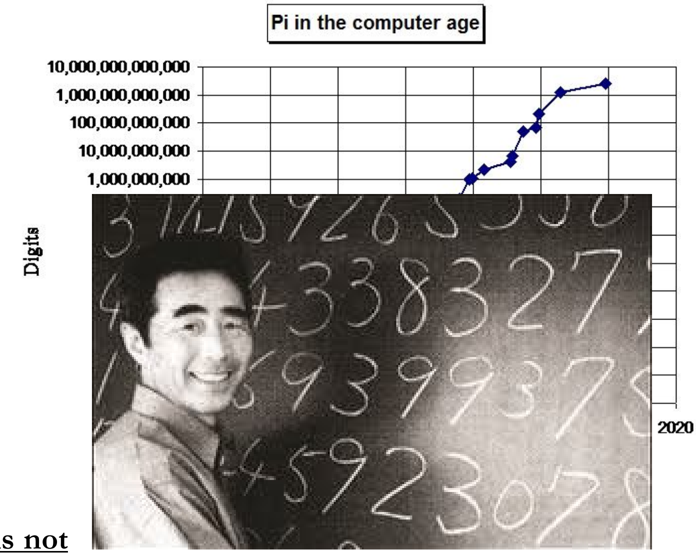
π π π π π

Records for "π-Guinness Book"

Below in the table, one can find facts about who, where and when calculated number π. You can se that p to the 16th century scientists tried to find what was useful, though later (especially from the middle of the last century) they just satisfied their curiosity. Nevertheless, we found this table rather interesting.

When	Who and where	Value or Number of places	Tool
XX BC	Egypt	3.1605	Hand
XX BC.	Babylon	3.125	Hand
X BC.	India	3.1622	Hand
X BC	China	3.1429	Hand
III BC	Archimedes, Greece	3.1419	Hand
I AD	Ptolemy, Greece	3.1417	Hand
III	Chung Huing, China	3.1416	Hand
V	Zu Chongzi, China	3.141593	Hand
VI	Aryabhata, India	3.1416	Hand
VII	Brahmagupta, India	3.1623	Hand
IX	Al-Khorezmi, Persia	3.142857	Hand
XIII	Fibonacci, Italy	3.141818	Hand
XV	Al-Kashi, Persia	16 places	Hand
XV	Madhava, India	3.14159265359	Hand
XVI	Leonardo da Vinci, Italy	3.142857	Hand
XVI	Viette, France	9 places	Hand
XVI	Roomen, Belgium	16 places	Hand
XVII	Ceulen, Holland	35 places	Hand
XVIII	Machin, England	100 places	Hand
XVIII	Vega, Slovenia	140 places	Hand

XIX	Sharp, England	72 places	Hand
XIX	Richter, Germany	330 places	Hand
XIX	Shenks, England	527 places	Hand
1949	Neumann, USA	2037 places	Computer
1986	Bailey, USA	19 million places	Computer
1987	Kanada, Japan	134 million places	Computer
1989	Brothers Chudnovsky, USA	1 billion places	Computer
1991	Brothers Chudnovsky, USA	2.3 billion places	Computer
1994	Brothers Chudnovskys, USA	4 billion places	Computer
1995	Kanada, Japan	4.3 billion places	Computer
1997	Kanada, Japan	51 billion places	Computer
1999	Project HINTS, USA	206 billion places	Super computer
2009, December	Brothers Chudnovsky, USA	2.7 trillion places	Computer
2000, August	Brothers Chudnovsky, USA	5 trillion places	Computer
2009, August	Takahashi, Japan	2.6 trillion places	PC (!!!)
2009, December	Bellard, France	2.7 trillion places	PC (!!!)
2010, August	Kondo & Yee, Japan-USA	6.4 trillion places	Computer
2011, October	Brothers Chudnovsky, USA	Over 10 trillion places	Computer



<u>**What follows is not a joke…**</u>

Reuters Agency, July 2, 2005, 9:53 A.M. : ***Japanese breaks "pi" memory record.***

A Japanese mental health counselor has broken the world record for reciting "*pi*" from memory. Akira Haraguchi, 59, managed to recite the number's first 83,431 decimal places, almost doubling the previous record held by another Japanese. He had to stop three hours into his recital after losing his place, and had to start from the beginning.

Mr. Haraguchi took several hours reciting the numbers, finishing in the early hours of Saturday. He hopes to be listed in the Guinness Book of World Records to replace his fellow countryman Hiroyuki Goto, who managed to recite 42,195 numbers as a 21-year-old student in 1995.

This is not the end of the story: in August of 2006 Akira Haraguchi recited number *pi* with 100 thousand decimal places. It took for him 16 hours to pronounce this number. No limits for a human e-eh… I even don't know what to say…

By the way, to recite a new World record in about 5 trillion digits, one needs over 150 years to pronounce it without stopping!.. Good luck!

Do we need the value of pi with such accuracy? Hardly…

For practical purposes on the Earth we need no more than 11 places for pi: the Earth radius is 6400 km, or 6,4·1012 mm, so it gives an error not more than several mm if one forgets about 12th place… The average Earth orbit around the Sun is 150 million km, or $1.5 \cdot 10^{14}$ mm, so it is enough to know 14 places of pi to measure it with accuracy of several millimeters!…The maximum distance from the Sun to the farthest planet Pluto only in 40 times larger than that to the Earth, so 16 places is enough for the entire Solar system.

OK, let us be generous: consider our Galaxy with its diameter equal to 100 000 light years, i.e. 10^{18} km, or 10^{30} mm (one light year is about 10^{13} km)... Thus, even for measurement the Galaxy diameter with accuracy of 1 mm, one needs the value of pi with only 34 places! This value had been computed in the 17th century!

So, you see: truly, science has no [numerical] limits!

Nevertheless, maybe you want to have a look at ***pi*** to an accuracy of 100 decimal places? Here it is:

3.14159 26535 89793 23846 26433 83279 50288 41971
69399 37510 58209 74944 59230 78164 06286 20899
86280 34825 34211 70679...

In conclusion, let us notice that “pi” is the only number in the World that has its own celebration day. Naturally, it is March 14th (3.14). It is a day for math teachers to be creative and use 21st century resources to engage a new generation of math learners.

4. VERY HUGE NUMBERS

4.1. Chess and very huge numbers

The true origin of the game of chess is not known with certainty. By some legends, its invention is attributed to the Biblical King Solomon, by another, to the Greek god Hermes... But most probably, chess originated in India sometime around the 6th or 7th century AD. From there, the game crossed into Persia, and afterwards to Europe. The word Chess perhaps was derived from the Persian word for king ("*shah*"), and the word *checkmate* from "*shah mat*", meaning "*the king is dead*".

One interesting story is as follows: A mighty Indian rajah, having conquered a number of neighboring countries, claimed that he was almighty. However, a wise Brahmin objected and said that anybody without an army was not powerful. The Rajah in anger asked the wise man to prove his statement or he would be beheaded. The next morning the Brahmin brought to the Rajah a game that he had invented during the night – chess. The game proved in parable style that the Brahmin was right: a king with no soldiers was nothing!

The Rajah liked the game so much that offered the Brahmin any reward. The man did not ask for money or privileges but made a modest request: "Ask your vizier, the greatest of the greats, to put a single wheat grain on the first cell of the chess board, then twice more grains on the second cell, then again double the number up to four, then up to 8, and so on… It will be enough when all cells will be covered by the needed number of grains".

The Rajah was amazed that the Brahmin gave his game for such a miserable price! He ordered to his vizier to satisfy the man. However, the next day court mathematicians informed their ruler that nobody was able to gather such a large amount of wheat: there was not that number of grains in all the kingdoms of the Earth!

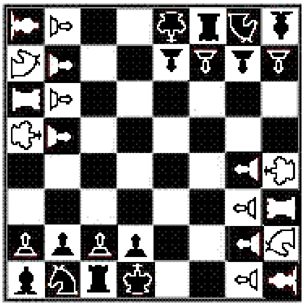
The modest Brahmin asked only for $1+2+2^2+ ...+2^{63}$ grains... Is it really so many? Yes, it is *unimaginably* many! If that number is written in a compact form, it is $2^{64}-1$, which in our decimal system is:

18 446 744 073 709 551 615

and is pronounced as 18 quintillion 446 quadrillion 744 trillion 73 billion 709 million 551 thousand and 615. To store that amount of wheat, one needs a barn of 5 meters height, 16 meters wide and as long as from the Earth to the Sun!

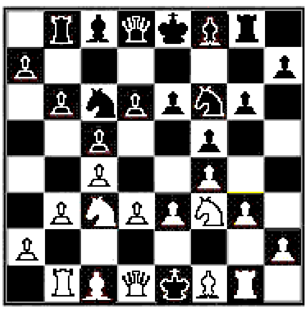
No wonder such numbers are called "astronomical"… We will discuss astronomical numbers a bit later. Let us in brief see about the history of the development of chess: usually those who like math like chess as well (though the converse statement is not always correct☺).

4.2. Chess History



Chess in its original form was a game for four with four sets of pieces. It was called *chaturanga*, or *chatrang*. The Sanskrit name "chaturanga" means "*quadripartite*" (divided into four parts) and was also used to describe the ancient Indian army, in which a platoon had four parts: one elephant, one chariot, three soldiers on horseback, and five foot-soldiers.

The game symbolized a simultaneous war of four sides, which initially was located in the corners of the chessboard. All pieces were of different colors: black, red, yellow and green.



Each move was done after dice rolling: if one had 2, then he played a soldier, if 3, then an elephant (now a bishop), and so on. The victory was defined as destroying entire opponent's army (not mating the king).
With time, the game became a game for two, and in such form it appeared in Europe.

At the beginning the rules were different from those of today: there was no castling, a pawn moved only one step forward, a bishop moved only 3 cells but it could jump over other pieces, etc.

The arrival of chess in the Arab World led to the changing of the pieces because Islam forbids using animal images. (A horse was "revived" only when the game arrived in Europe.)

Because the pace of the game was too slow for some, there was a development called the "tabias," in which the game starts out with an already developed position. One example of such a position was called "*almudjannah*", in which 12 moves already had been "played," is presented below.

In Europe the game changed: a mate became the only way to win, castling was introduced, and the first move of a pawn could be through two cells. In addition a two-color board was introduced.

4.3. Googol and "Its Friends"

Everybody knows of the Internet search engine called Google. However, probably not everybody knows about a mathematical monster – Googol[57]. What is this "monster"? What does it mean?

American mathematician Edward Kasner[58] coined the term "googol" in 1938. This number equals

10^{100} = 1 0000000000 0000000000 0000000000 0000000000 0000000000 0000000000 0000000000 0000000000 0000000000 0000000000.

(There are exactly one hundred zeros following unity ☺. Believe it.)

Kasner told that his 9-year old nephew Milton Sirotta invented the name to this number.

How large is this number? Probably, a little bit smaller than infinity☺. Indeed, the following values are less than googol:

(a)The quantity of nanoseconds from the "Big Bang" (an assumed moment of creation of the Universe) up to our days.

(b)The maximum diameter of the Universe measured in angstroms is less than one googol. Remember that an angstrom = 10^{-8} cm;

(c) The number of elementary particles within the entire Universe is less than one googol… A googol is approximately equal to "70 factorial"

However, Kasner did not stop at a googol: he invented the "googolplex". That number has the form:

$$Googolplex = 10^{googol} = 10^{10^{100}} = 1\ 000\ldots \text{(Googol zeros)} \ldots 000.$$

Of course, googolplex has no real sense at all! Could you imagine that it is even impossible to write down this number: there isn't enough time, paper or ink in the world.

Doubtlessly, anybody can invent ugly monstrous numbers. Why not invent "super googolplex", which equal 1 followed by googolplex zeros? Or even more absurd number, for instance,

57 By the way, Google was named as a play on the number googol.

58 **Edward Kasner** (1878-1965), American mathematician.

$(googol)^{googol}$ and name it G☺☺G☺L… Of course, the next step is the number

$$(\text{G☺☺G☺L})^{\text{G☺☺G☺L}}\ldots$$

Wow!!! However, the question arises: S-O W-H-A-T?

However, it is quite different matter if the huge number appears during the solving of some physical or mathematical problems. In this case, "non-system" numbers appear in physics and combinatorial numbers in mathematics. (Notice that googol$\approx$70! We will discuss properties of factorials later.)

So, let us forgive Kasner for his inventions – he got what he deserved: he and his nephew can be found in any modern encyclopedia on mathematics.

Large numbers attracted mathematicians from time immemorial. Probably, the first to show interest in large numbers was Archimedes. He decided to count what always was considered as uncountable: the number of sand grains. Moreover, he decided to count all grains of sand that could fill the Universe. (At that time, ancient Greeks assumed that the Universe was a sphere with the radius equal to distance from the Earth to the Sun.) He found that the number of grains of sand contained in such a sphere would be not more than 10^{51}.

Archimedes in his calculations took as a basis "myriad of myriads". Myriad was known at that time as 10000, i.e. myriad of myriads equals $(10^4)^2 = 10^8$. It is interesting that in almost all European languages the word "myriad" lost its numerical value and denotes just "countless quantity" of everything. It is believed that this word came into the Greek language from ancient Egypt.

Myriad of myriads was called by Archimedes as "*di-myriad*" ("double myriad"). He also introduced further names for large numbers: tri-myriad for $(10^8)^2 = 10^{16}$, tetra-myriad for $(10^{16})^2 = 10^{32}$, and penta-myriad for $(10^{32})^2 = 10^{64}$, and so on. So, within each new level the number of zeros is doubled, i.e. any number is a square of the previous one.

Archimedes scheme has a "last" number, which in modern notation can be written as $10^{8\cdot 10^{15}}$. "The largest number" appeared only because Archimedes had a lack of appropriate notations! He did not know how to write numbers in the form n^{n^n}, and so on.

Interestingly, the names of modern large numbers are rooted in Roman numbers, even though the Roman numbers in their "natural" form do not exceed a thousand. Names of large numbers are formed in a simple way: in the beginning there is Roman name of the number and the suffix "illion" is added at the end.

There are two different system of naming large numbers: one is The so-called "short scale" (US and modern British) and "long scale" (Europe and older British). Both scales coincides… only for 1,000 and 1,000,000 – thousands and millions. After this short scale has "jumps" in 1000 times and long scale has jumps in million times. So, the numbers with the same name are quite different!

There are following names of numbers:

Name	**Short scale**	**Long scale**
Million	10^6	10^6
Milliard	-	10^9
Billion	10^9	10^{12}
Trillion	10^{12}	10^{18}
Quadrillion	10^{15}	10^{24}
Quintillion	10^{18}	10^{30}
Sextillion	10^{21}	10^{36}
Septillion	10^{24}	10^{42}
Octillion	10^{27}	10^{48}
Nonillion	10^{30}	10^{54}
Decillion	10^{33}	10^{60}
…	…	…
Sexdecillion	10^{51}	10^{96}
Septendecillion	10^{54}	10^{102}
Octodecillion	10^{57}	10^{108}
…	…	…

Probably that's enough…

Actually, the names of numbers larger than a quadrillion are almost never used; everybody prefers to say "10 in power such-and-such…"

Let us give now a "real example" of the occurrence of a large number. After being transported to heaven on a white stallion, Mohammed said: "I saw there an angel, the most gigantic of all created beings. It had 70,000 heads, each head had 70,000 faces, and each face had 70,000 mouths, each mouth had 70,000 tongues, and each tongue spoke 70,000 languages; all were employed in singing God's praises". So, Mohammed met an angel who spoke almost two septillion languages. (Can you imagine where the angel could find such a number of different languages? Maybe it was from inhabitants of other Galaxies?)

Notice that in the ancient Buddhist tractate of the 1st century BC the number "*asankhyeya*" [59] was introduced. This number can be represented as one followed by 140 zeros! It was believed that was the number of cosmic cycles needed to get to nirvana. You see that this number has a real application☺.

4.4. Non-System Numbers

At last, we come to the very large numbers (sometimes called "non-system" numbers) that have some actual sense. These numbers appear in the mathematical theory of numbers during attempts to prove Riemann's hypothesis related to frequency of prime numbers.

The existence of this number was proved in 1912 by John Littlewood[60] , and then in 1933 his pupil Stanley Skewes[61] found the upper bound of this number

$$\mathrm{Sk}_1 = e^{e^{e^{79}}} \approx 10^{10^{10^{34}}}$$

that was named "*First Skewes' number*" after its author.

G. H. Hardy[62] once described the First Skewes' Number as "the largest number which has ever served any definite purpose in mathematics," though soon enough even much larger number of such kind appeared.

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59 From Chinese word "*acentsi*" – "*uncountable*".

60 **John Edensor Littlewood** (1885 –1977) was a British mathematician,

61 **Stanley Skewes** (1899–1988) was a South African mathematician, best known for his discovery of Skewes number in 1933.

62 **Godfrey Harold "G. H." Hardy** (1877-1947) was a prominent English mathematician, known for his achievements in number theory and mathematical analysis.

The First Skewes number has since been reduced to 1.165×10^{1165} in 1966, to $e^{27/4} \approx 8.185 \times 10^{370}$ in 1987, to 1.39822×10^{316} in 2000, and to $1.397162914 \times 10^{316}$ in 2004.

The Second Skewes number Sk_2 introduced by Skewes in 1955 was related to the Riemann hypothesis is much larger than the Skewes number Sk_1, $\mathrm{Sk}_2 = 10^{10^{10^{10^3}}}$.

The largest number used in mathematical proofs ever is the Graham's[63] number, introduced in 1977. This number relates with such a mathematical problem that even its simplified formulation is beyond the scope of this book.

However, large numbers having some sense exist not only in mathematics. In some physical theories one might find numbers of the order of 10^{39} -10^{44}, and their squares and cubes, i.e. they are much larger than googol. However, let us agree that googolplex is nothing more than an extravagancy…

* * * * *

Jokes about mathematicians

***

Sherlock Holmes and Dr. Watson several hours flew on the balloon through a huge thick cloud and were completely lost. At last at some hole in the clouds they saw a house with a lawn where a man with a book was sitting on the bench. Watson shouted: "Hey, on the ground! Let us know where we are?" The man on the ground thought a bit and then shouted in response: "You are in the basket of the balloon!"

Sherlock Holmes commented: "Watson, I/m sure that this man is a mathematician…" Amazed Watson exclaimed: "How you guess about it?!" – "Because his answer was absolutely correct and in the same time absolutely useless…"

***

About a math exam: A professor is trying to calm a very nervous student and with a friendly smile asks him:

– It seems to me that we have already met sometime? –asks the professor encouragingly.

---

63 **Ronald Graham** (b. 1935) is a contemporary American mathematician.

– Yes, I took an exam last year… Unfortunately, I failed…
– Don't panic, I'm sure that this time everything will be O.K. Did you remember what question I asked you at the previous exam?
– You asked me: "It seems to me that we have already met sometime?"

***

A mathematician decided to hang a picture on the wall. He took a ladder, climbed on it taking a hammer but forgetting to bring a nail. He shouted loudly: "Sweetheart, please give me a nail!" His wife brought him a nail. The mathematician took the nail and put the nail head to the wall. He called to his wife: "Darling, you have brought me a nail from the opposite wall!.."

***

A corporal lined up soldiers for a digging job in the yard. "So, today we will dig up this plot of land. Who of you is into math?" – "I am, sir!" – "Smith? O.K. Take a spade and go to extract roots!"

***

An assistant informs his math professor: "Just while you were away for a moment somebody called from the maternity ward to let you know that you have a new-born daughter!" – "Well, well… Please call my wife and inform her about it…"

5. THE MAGIC OF NUMBERS

The World is created on the strength of numbers.

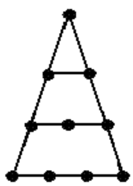
Pythagoras

5.1. Pythagoreans

Probably, the very first time in human history that numbers began to play mystical and important roles was when the great Greek scientist Pythagoras established his scientific school, which led to the "Pythagoreans." Pythagoras paid enormous attention to numbers, trying to explain almost everything with the help of numbers. He saw in them some hidden sense of the World. Pythagoras' school was the first religious-philosophical association having the characteristic of a closed aristocratic brotherhood with a strong influence on the contemporary society.

The existence of the Pythagorean brotherhood was veiled with mystery and secrecy, because they believed that fundamental knowledge of Nature had to be kept secret. Only those who were able to understand the truth and evaluate its significance were accepted into the brotherhood: they held that science is a matter for the priests not for a mob.

The basic concepts of Pythagorean teaching were transmigration of souls and harmony of the World. They believed that music and intellectual labor purified a human soul, so they tried to reach perfection in "four arts": arithmetic, music, geometry and astronomy.



Actually, the Pythagorean School might be considered the founder of mathematics. The basis of that school was teaching about the "decade": 1+2+3+4=10. These four numbers, by Pythagorean thought, describe all processes in the World. Pythagoras' tetraktis (pyramid of ten dots), was a symbol of great importance, because it opened, as Pythagoreans believed, the very secrets of the World.

In particular, the decade reflects the laws of musical harmony: it expressed the musical intervals – octave (2:1), quint (3:2), quarta (4:3).

The most important musical instrument was a tetrachord, or a four-string lyre. A lyre was tuned by ear until a musician heard a harmonic sound – a process which Plato called the "torture of tuning pins".

There is an interesting legend about how Pythagoras discovered the laws of musical harmony. Once he was going near the smithy and heard the beating of hammers. All simultaneous hits were composed mostly of harmonic sounds, but some hits were no harmonic. Pythagoras entered the smithy and examined the hammers. He found that those hammers which weights formed simple relations like 1:2, 2:3 and so on produced pleasant harmonic sounds!

Pythagoreans found more magic properties of numbers. The square of a number was for them a symbol of justice and equality. Nine was a symbol of constancy, because all numbers multiplied by 9 give a result which digits sum to nine: 9×2=18, and 1+8=9; 9×3=27, and 2+7=9; 9×4=36, and 3+6=9; 9×5=46, and 4+5=9; 9×6=54, and 5+4=9; 9×9=81, and 8+1=9! Something magic is in this property, isn't it?

Moreover, if one multiplies any two numbers, each of which is a product of some number and 9, the sum of the digits of the result also will be 9! For instance, (3×9) ×(4×9)=27×36=972 that gives 9+7+2=18 and on the next step of the procedure 1+8=9.

The number 8 was a symbol of death because all numbers multiplied by 8 have a decreasing sum of figures. Indeed, start with 8. Then take 8×2=16, for which 1+6=7. Then consider 8×3=24, for which 2+4=6; then 8×4= 32 with the sum of figures 3+2=5, and so on… Indeed, it is so easy to believe in magic (especially if you are an ancient Greek☺!).

Pythagoreans believed even numbers to be feminine and odd numbers to be masculine, and that odd numbers were the "fertilizing" ones: take an even number add an odd one and get again odd number. The symbol of marriage was 5 because it is the sum of 2 and 3. By the same reasoning, right triangles with sides 3, 4 and 5 was called by Pythagoreans the "*bridal figure*." The very first perfect number (6) was referred as a symbol of a soul.

Of course, the magic of numbers was only a mystique. The main purpose of the Pythagorean School was moral purification and the religious indoctrination of the members of the brotherhood. For them, moral principles were the most important. Their teachers taught: "In your words and your deeds be always just;" "Let the supreme judge be your conscience" and so on. At the end of each day everybody asked himself about his deeds during the waning day: "Before to falling into peaceful dreams, remember the day that just passed: What did I do wrong? What could I have done? What did I not do?"

Pythagoreans developed a teaching about music of spheres that reflected the harmony of the Solar system where each planet corresponds to a specific musical note. They introduced the so-called musical psychology: music was used for upbringing and healing of a body and a soul.

Pythagoras developed the theory of harmony working with a "*monochord*", a one-string musical instrument of his own invention. He viewed the Universe to be a huge monochord with the string attached to Absolute Spirit on one end and to Absolute Substance at the other end. In other words, this string tied together earthly and heavenly sides.

Pythagoreans developed astronomy and medicine. They created a grammar of the Greek language and wrote many allegoric comments to Homer. It may be said that they were founders of science in a broad sense.

5.2. Numerology

Numerology is an old esoteric[64] science about numbers. Numerology studies the occult meanings of numbers and their influence on human life.

Of course, to call these exercises with numbers a science is a great exaggeration: numerology is not far from astrology and palm or crystal ball reading☺. Nevertheless, because we are telling about numbers, we are forced to mention about numerology.

The following principle is the basis of numerology: any number with the help of a simple procedure can be reduced to the "root" numbers from 1 to 9. And afterwards everything is extremely simple: each "root" number has its own "face", its own "character" and its own occult influence on the object to which this number is attributed.

It is not an easy task to say when numerology arose; however, Pythagoras developed the main principles of modern numerology in the 6th century BC. He invented the method of finding the root number. For this purpose he took an arbitrary multi-place number (the number under study) and found the sum of all digits. If a new number was still multi-place, he repeated the procedure. For instance, for the number 27 the corresponding root

64 Esoteric (from Greek "*esoterikos*" – "*inner*") means – secret, hidden.

number is 9, because 2+7=9, and for the number 28 the root number is 1: 2+8=10, and then 1+0=1.

Notice that in principle one can number the letters of any alphabet and do the same with words (first or last names). In other words, if you don't like "your numbers", try "your letters". In addition, in this case you can switch to another language!

* * *

One. (Birthdays 1, 10, 19 or 28).

It is believed that "1" has a great power, because it is the first number used when counting. People with those birthdays are single-minded, strong in character, proud and stubborn. They are creative, ambitious, and they rely only on themselves. One can be happy, loving, romantic, dynamic and charismatic, but on the downside one can be egotistical, selfish and melodramatic. They are individualistic, obstinate and loner. They like to command and control, sometimes becoming tyrannical.

O-o-ops!.. Excuse us, it is not our fantasy – it is from the numerology books! ☺

> For the curious: Nicolaus Copernicus (February 19), Gotfrid Leibniz (June 1), Sergei Rakhmaninov (April 1), Jacqueline Kennedy (July, 28).

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Two. (Birthdays 2, 11, 20 or 29.)

The number "2" traditionally is considered feminine, intuitive, and corresponds with our protective/defensive instincts. People of this number are soft, tender, modest and obedient. They like to be led and controlled; they are good teammates. Such people often change their viewpoints and hesitate.

At the same time, two represents polarities such as good and evil, black and white, male and female, left and right, since one pole cannot exist without the other. Negatively, two can be grasping, overprotective and cranky.

So, you can see that you have a multiple choice here – it is so convenient for fortune-tellers!

For the curious: Nikolai Lobachevsky (November 20), Salvador Dali (May 11), John Kennedy (May 29),

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Three. (Birthdays 3, 12, 21 or 30.)

Three is seen as a very magical number. This can be seen in the divisions of a human: mind, body, and spirit. Some consider this number very special due to the Trinity. Pythagoreans knew nothing about the Trinity; nevertheless they called this number perfect because it symbolizes the beginning, middle and end (☺ 😐 ☹). Indeed, what can be perfect if has no beginning, middle or end?!

If your birthday is on the 3rd, you are successful; your life is easy; you get money and respect. (If it is your day, congratulations!) Those who possess this number are creative, sharp-tongued and artistic. Although "3" is characterized by wisdom, understanding and knowledge, negatively it can exhibit pessimism, foolhardiness and unnecessary risk taking. In some cases, the number three can take on some negative aspects and is seen as demonic or unnatural, as there are no creatures in our world that walk upon three legs.

For the curious: Friedrich Gauss (April 30), Vincent van Gogh (March 30), Fyodor Dosotevsky (October 30), Andrei Kolmogorov (April 12), Abraham Lincoln (February 12), Franklin Roosevelt (30 January), Mark Twain (November 30), Ernest Hemingway (July 21), Frank Sinatra (December 12),

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Four. (Birthdays 4, 13, 22 or 31.)

The number "4" derives its significance from various sources. It is the first "composite" number (it can be created from multiplying numbers other than itself and 1), in that $2\times 2 = 4$. The simplest solid object, a tetrahedron has four sides, hence – in numerology – it denotes solid matter in general and the Earth in particular. In addition, the Earth is bounded by four cardinal points (North, South, East and West). In Medieval astrology everything consists of 4 elements: fire, air, earth and water. Time is another concept strongly associated with "4": the year has four seasons and the month has (roughly) four weeks. Christian-oriented observers also note that the life story of Jesus is told in four gospels. In the Jewish religion, the number four is significant because of the Tetragrammaton, the four-letter name of God that is so holy it is never spoken. In Chinese numerology (as well as that of other Oriental languages), the word “4” is a homonym of the Chinese word for "death". As thus, some hospitals do not have a 4th floor ☹.

Those who have a number “4” birthday are honest, very practical and emotionless, sometimes even gloomy, suspicious and boring… They are usually very hard-workers; however, they cannot work with aspiration or inspire others. They are able to perform a colossal job for insignificant results.

Four is an unlucky number… By tradition, it is considered to be a number of poverty, unhappiness and defeat. Most of contemporary numerologists try to keep silent about those properties of “4”.

For the curious: Louis Armstrong (August 4), George Byron (January 22), Frederick Chopin (February 22), Thomas Jefferson (April 13), Henry Matisse (December 31), Isaac Newton (January 04), George Washington (February 22),

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Five. (Birthdays 5, 14 or 23.)

Five reflects 5 senses and is connected to sensual awareness. Those who possess this number are vigorous, smart, curious and emotional. Everything unusual and strange attracts them; they are open to new experiences as well as new ideas. These people often are adventurers.

It is also a number, which represents service to others. They like to change their life, so they usually never reach perfection. They hate responsibility and avoid it.

Some numerologists believe that “5” is a number of Lovelace or Don Juan – men who are irresponsible in their matrimonial behavior. (Notice: We are not responsible for saturnalia of numerological fantasies – everything is taken from the sources. By the way, there is no mention of women in this category. ☺)

<u>For the curious: </u>Albert Einstein (March 14), David Hilbert (January 23), Simon Laplace (March 23), Claude Monet (February 14), William Shakespeare (April 23)

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Six. (Birthdays 6, 15 or 24.)

Number "6" is perfect! Naturally, six relates to tact, beauty and harmony. Six possesses charm, grace and the ability to make small talk on any stratum, and is therefore much of a diplomat. Six is very nurturing, and is considered the mother/father number.

It is very much a relationship builder, which corresponds to one-to-one encounters. It deals with that which we are attracted to and those things we find great pleasure in.

<u>For the curious: </u>Galileo (February 15), Martin Luther King (January 15), Ronald Regan (February 6), Marc Shagal (June 24), Leonardo da Vinci (April 15)

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<u>Seven</u>. (Birthdays 7, 16 or 25)

Seven is a spiritual number because it is illusive and contains veils that must be uncovered, one after another. It is believed that seven is a sacred number: in the Bible text of Genesis it is read that the World was created in seven days, the ancient solar system consisted of seven luminaries, the human body consists of seven plexuses…While seven possesses qualities of dreaminess, spirituality and psychic awareness, negatively it can be dubious, deceptive and insincere.

People possessing "7" are loners, individualists, and are stubborn and pedantic. They are smart; they like science and might not like physical labor. They are not interested in money and personal comfort. They don't like to discuss and can hardly express their thoughts in an appropriate way… (Here we would like to ask for forgiveness from professors and teachers whose birthdays fall on 7, 16 or 25. However, what we can do with number "7"? ☺)

In general, "7" is the most enigmatic and superstitious number.

For the curious: Ludwig Beethoven (December 16 декабря), Pablo Picasso (October 25), Peter Tchaikovsky (May 7).

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Eight. (Birthdays 8, 17 or 26.)

A person with "8" has a harmonically developed mind and good leadership ability. More than any other number, eight seeks money and material success. Those with "8" are strong and practical, though their life path is not simple: they have to work hard and constantly fight for survival.

They could be unpleasant, pragmatic and egotistic though behind their cold and dark fa?ade an eccentric and unrestrained nature might be hidden.

Numerologists have come even further: they observed that man's body has 7 holes and woman's body has eight, and the eighth is those, which gives a new life, so –they decided – number "8" is the number of life.

For the curious: George Gershwin (September 26)

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Nine. (Birthdays 9, 18 or 27.)

Nine was considered to be a sacred number by the ancients. The strength of "9" is in the fact that it is a "double three" (9=3×3). It represents change, invention and growth, which springs forth from inspiration. Nine is the humanitarian. Nine has traditionally held esoteric significance, which is evidenced by the fact that it takes nine calendar months to bring a baby into the world from its initial conception.

Those who possess "9" are romantic, emotional and impulsive, who like to help others and serve humanitarian purposes. They are excellent teachers, scientists and artists. However, some of them do good things with arrogance and egocentrism.

For the curious: John Lennon (October 9), Henry Longfellow (February 27), Leo Tolstoi (September 9).

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5.3. Magic squares

At last, numbers that really have some magic properties! And we speak now not about specific "wonderful" numbers: those numbers were enigmatic but there was no explanations why they are enigmatic.

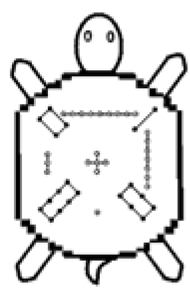
There are some mathematical puzzles that are magic for those who try to find something mystical, and at the same time, for those who try to construct something magic! We are going to tell you about magic squares.

The sense of these mathematical objects is as follows. A square with side equal n units is divided into n^2 smaller squares. Into each cell of this large square, one writes natural numbers 1, 2, 3, ... up to n^2. The problem is to put the numbers in such a way that the sum taken over each column or over each row or over any of the two diagonals yields identical results!

4	9	2
3	5	7
8	1	6

The problem is not as simple as it might seem at first glance.

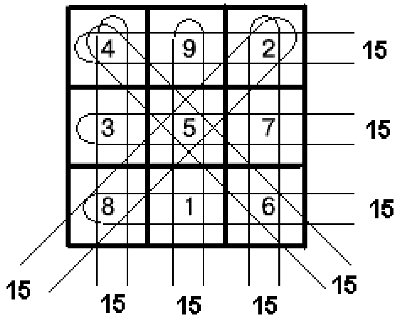
Ancient Chinese are credited for the first mention of a magic square. By legend, at the time of Emperor Yu (about 2nd millennium BC) a sacred turtle emerged from the yellow waters of the Yellow River (Huang He). It had mystical symbols on its shell.



Those signs known as "*Lo Shu*" formed a magic square. Indeed, let us decipher those symbols in the following simple way: count each point in the figure.

You can see that the sum of the digits in each column, or each row, or any diagonal equal 15!

Naturally, the Chinese ascribed to this magic square a supernatural property. It was a significant symbol in ancient China. The number 5 in the middle denoted the Earth and around it were Fire (2 and 7), Water (1 and 6), Wood (3 and 8) and Metal (4 and 9).



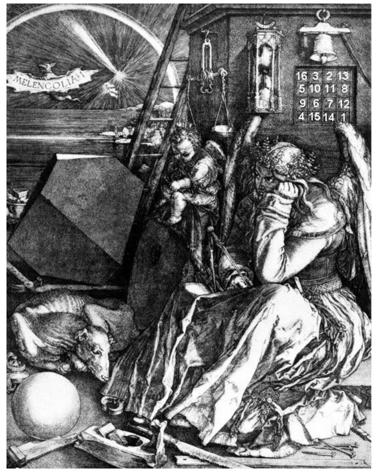
Let us try to construct a magic square ourselves. First, find the sum of all numbers in it: in the square of size 3 ×3 the sum is 1+2+3+4+5+6+7+8+9 = 45. So, in each group of three numbers (column, row or diagonal) the sum has to be 15. Triplet of natural numbers from 1 to 9, which has the total sum equal 15, can be decreasingly ordered in 8 following different ways:

9+5+1; 9+4+2; 8+6+2; 8+5+2; 8+4+3; 7+6+2; 7+5+3 and **6+5+4.**

In the magic square of 3 ×3 the sums of three numbers in all 8 directions (3 rows, 3 columns and 2 diagonals) have to be equal to a "magic constant" 15. Notice that the number in the center of the magic square belongs to one row, one column and two diagonals, i.e. it belongs to 4 of 8 triplets. As you can see from the presented above sums of triplets, there is the only such number, namely, 5. Consequently, the number in the center of the magic square has to be 5.

Now let us take 9. It appears in 2 of 4 triplets. This number cannot be located in the corner of the square, since then it would have been belonged to three triplets, therefore, 9 can be located only in the middle of one of the square's sides. Let us choose the cell above 5.

On the right and left of 9 only 2 and 4 can be placed (remember about "magic constant"?). The order of the numbers does not play a role: it could be "2, 9, 4" or "4, 9, 2" (they are mirror-like). Let us choose for simplicity the first order: "2, 9, and 4." In the column beginning with 9, after 5 the only possibility is 1. In the diagonal beginning with 2, we can place only 8 to get the sum equal 15. In the diagonal beginning with 4 the only appropriate number is 6 (so, 4+5+6=15). For remaining 3 and 7, there is no choice!



Thus, we have constructed the magic square 3×3, and moreover, proved its uniqueness. (Of course, don't be confused that by rotating the square around its symmetry axes – both diagonals and the central row or the central column – one can get different presentations of this magic square, however, all of them will be only kind of mirror reflections.)
The magic square came to India in the 6th century, and it got to Japan even later.

Europeans met that "mathematical wonder" only in the 15th century. The first European magic square is believed to have been constructed by Durer[65], who put it on his famous lithograph "Melancholy" just above the head of an angel.
You see that there is no central cell, so the problem seems if not more difficult then at least different. One can construct the magic square 4×4 in the following simple way. (We will give a rule without detailed explanations as we did in the previous case.)

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Let us take a square and divide it into 16 equal smaller squares. Then write down all numbers from 1 to 16 in order from the upper left corner to the lower right one. Then rotate each diagonal around its central point. In result one gets a new square that is magic!

[65] **Albrecht Durer** (1471-1528), great German artist and graphic who was called "the Northern Leonardo" for his universal knowledge and perfect painting.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

In addition, Durer transposed two middle columns (without loss of the "magic properties" of the magic square) and got the year of the creation of his lithograph – 1514.

Of course, in the same manner one can transpose rows and get a new magic square. Example of such transpositions is presented below.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

9	10	11	12
5	6	7	8
1	2	3	4
13	14	15	16

9	11	9	12
5	7	5	8
1	3	1	4
13	15	13	16

Magic squares in medieval times considered truly to be real magic: it was believed that a magic square engraved on a silver plate would offer protection even from the bubonic plague.

Muslims had a special regard for magic squares with 1 in the center, considering such squares to be reflections of the uniqueness of Allah.

5.4. Squares which entranced the World

If you are not addict of this mathematical game still, allow me to introduce it. This game name is Sudoku. It is a captivating mathematical brain-twister for which you should not have any mathematical knowledge!

The name of this game originated from two Japanese words "su" (number) and "doku" (standing along). What is a target of this game? There is a sheet of paper with a grinded square divided on 99= 81 equal square cells. Within some cells there are number from 1 to 9. A player has to fill other cells with numbers in such a way that all nine figures in each row and in each column are different. Probably the best way of explanations of the essence of the game is a demonstration of initial and final phases:

	2				4			
9			8					1
					2	8		
2		9	5		6	3		
		8				2		
		3	2		9	5		4
		4	1					
7					3			6
			6				7	

8	**2**	5	7	1	**4**	6	9	3
9	4	6	**8**	3	5	7	2	**1**
1	3	7	9	6	**2**	**8**	4	5
2	1	**9**	**5**	4	**6**	**3**	8	7
4	5	**8**	3	7	1	**2**	6	9
6	7	**3**	**2**	8	**9**	**5**	1	**4**
5	6	**4**	**1**	2	7	9	3	8
7	8	2	4	9	**3**	1	5	**6**
3	9	1	**6**	5	8	4	**7**	2

You can see that Sudoku possesses one more property: within smaller square 3×3 al figures are always different.

Today you can find special sites with Sudoku or buy a booklet with printed puzzles.

The puzzles differ by complexity of solution: there are simple Sudoku, average Sudiku, difficult Sudoku and very difficult Sudoku (the so-called "Devil's Sudoku"). They differ by the number of in advance filled cells. Examples of Sudoku of different complexity are given below.

Simple Sudoku

4					1			
				3		1		
9		8	5	7	6		4	
3	5			8				4
	7	1	4		3	6	5	
2							8	3
	6		1	4	2	5		8
		2					6	
			9					1

Average compplexity Sudoku

3		5	9	4				
1			8	7			2	
8					5	3		
	7	2	1	9				
		8				1		
				5	4	2	8	
		3	7					2
	1			8	2			5
				3	9	7		1

Difficult Sudoku

4					1			
	2			3		1		
9		8	5	7	6		4	
3	5			8				4
	7	1	4		3	6	5	
2				1			8	3
	6		1	4	2	5		8
		2		5			6	
			9					1

"Devil's Sudoku"

3		5	9	4				
1			8	7			2	
8					5	3		
	7	2	1	9				
		8				1		
				5	4	2	8	
		3	7					2
	1			8	2			5
				3	9	7		1

At http://www.websudoku.com/ you can find a countless number of Sudoku (about 3 billion simple ones, about 6 billion average ones, about 7 billion difficult ones and up to 10 billion of “Devil’s Sudoku”!).

Now you can find even Sudoku for children with squares size 4×4 cells and figures from 1 to 4.

Simple Sudoku

	3		1
1		4	
	4		2
2		3	

Difficult Sudoku

1			2
	3	4	

Many people think that Sudoku is Japanese puzzle. However, only name is Japanese, and the game itself originated in France: the first Sudoku was published in French newspaper ”Le Siecle” in 1892 year, though the mathematical puzzle did not become such popular as now. Then in 1979 an American pensioner Howard Garns published in a mathematical journal “Dell Magazines” his puzzle “Number Place”. At last, this puzzle appeared in a journal of Japanese publishing company “Nikoli” in 1986. However real “Sudoku fever” began in 2004 London newspaper “Time” began regular publication of this puzzle..

5.5. Latin and Graeco-Latin squares

Now let us return from fascinating puzzle to mathematics. In statistical testing one frequently uses the so-called Latin squares. These mathematical objects were introduced and studied by outstanding Swiss scientist Leonard Euler who entire his scientific career had been working at the Russian Academy of Sciences in Sanct Petersburg.

Euler defined a Latin square as a square table of *n* objects that contains all its row and columns containing different objects. He coined the name "Latin squares" because he considered Latin letters for forming the tables. For instance, Latin square for 4 objects has the form presented in Figure ***a***. Notice that if we substitute a= 1, *b* = 2, *c* = 3 and *d* = 4, we got Sudoku for children (see Figure ***b***). Of course, in principle, objects can be of arbitrary nature (see Figure ***c***).

a	***b***	***c***	***d***
b	***c***	***d***	***a***
c	***d***	***a***	***b***
d	***a***	***b***	***d***

(a)

1	**2**	**3**	**4**
2	**3**	**4**	**1**
3	**4**	**1**	**2**
4	**1**	**2**	**3**

(b)

(c)

If symbols of the Latin square are positioning in their natural order, i.e. has its first row and its first column coinciding, it is called reduced, or normalized, or standard. By permuting (reordering) Latin square in numerical form can be transformed into magic square.

In Euler's archives there were found manuscript (dated by 1776) in which construction of Latin squares of large dimension had been described, in particular, 9×9, 16×16, 25×25 и 36×36.

In his work "Investigation of a New Type of Magic Squares", Euler found a new class of mathematical objects- the so-called Graeko-Latin squares. Now this type of object is called Euler square. Such square of dimension N×N contains two sets of numbers from 1 to N in such a way that:

1) Each digit of each set has to be located in the first cell of a row once;
2) Different pairs of digits (including pair of the same digits) have to be located in each row and in each column only once.

One can see that these squares relate to Latin squares. For instance for four symbols *a*, *b*, *c*, *d*, and four symbols α, β, γ, δ this will be the two following Latin squares, superposition of which gives the Graeko-Latin square.

a	***b***	***c***	***d***
b	***a***	***d***	***c***
c	***d***	***a***	***b***
d	***c***	***b***	***a***

α	**β**	**γ**	**δ**
γ	**δ**	**α**	**β**
δ	**γ**	**β**	**α**
β	**α**	**δ**	**γ**

***a*α**	***b*β**	***c*γ**	***d*δ**
***b*γ**	***a*δ**	***d*α**	***c*β**
***c*δ**	***d*γ**	***a*β**	***b*α**
***d*β**	***c*α**	***b*δ**	***a*γ**

5.6. Numerical pyramids

Pyramids... People have been admiring their mighty, beauty and elegance...

However, there are other pyramids, consisting of numbers. These pyramids conquer our brain by its mathematical elegance. One of the simplest such pyramids is presented below:

1×1=	1
11×11=	121
111×111=	12321
1111×1111=	1234321
11111×11111=	123454321
111111×111111=	12345654321
1111111×1111111=	1234567654321

11111111×11111111=	123456787654321
111111111×111111111=	12345678987654321

The next pyramid consists of units only!

1×9+2=	11
12×9+3=	111
123×9+4=	1111
1234×9+5=	11111
12345×9+6=	111111
123456×9+7=	1111111
1234567×9+8=	11111111
12345678×9+9=	111111111
123456789×9+10=	1111111111

And how beautiful is this pyramid of eights!

9×9+7=	88
98×9+6=	888
987×9+5=	8888
9876×9+4=	88888
98765×9+3=	888888
987654×9+2=	8888888
9876543×9+1=	88888888
98765432×9+0=	888888888

Or look at this pyramid:

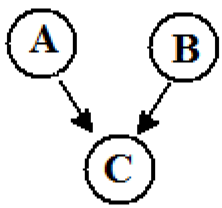
1×8+1=	9
12×8+2=	98
123×8+3=	987
1234×8+4=	9876
12345×8+5=	98765
123456×8+6=	987654
1234567×8+7=	9876543
12345678×8+8=	98765432
123456789×8+9=	987654321

I don't know who invented these numerical pyramids above. However, we should bow to the ground to the inventors of these beautiful mathematical "constructions".

And finally, let us consider the so-called Pascal triangle that will be discussed in detail in the fourth book of the series, which is named "Enigmatic Terra Al-Jabr". This triangle has the form:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
……………………………………

In this pyramid each number of the lower level is obtained as the sum of two neighbor numbers, standing just above the number of interest (see explanation in the figure below):



C = A + B

Jokes about numbers

A father is checking a notebook of his first grade son:

- Why you write these pothooks so ugly?
- Dad, they are not pothooks! They are integrals.

∞ 8 ∞ 8 ∞ 8 ∞

Father-Mathematician is working at his desk. His son comes and asks

- Dad, how to write number 8?
- It is very simple: write the infinity sign and rotate it on "pi" divided by 2.

∞ 8 ∞ 8 ∞ 8 ∞

Everybody knows that two by two equals four… However, sometimes I wish more!

∞ 8 ∞ 8 ∞ 8 ∞

Do not multiply numbers: there are too many of them already!

∞ 8 ∞ 8 ∞ 8 ∞

PANTHEON

Euclid
(365 - 300 BC)

One of the greatest ancient mathematicians, author of the first theoretical works on mathematics saved up to our time.

Euclid of Alexandria, the most prominent mathema-tician of antiquity, is best known for his treatise on mathematics "*The Elements*." The long lasting nature of "*The Elements*" must make Euclid the leading mathematics teacher of all time. For his work in the field, he is known as the father of geometry and is considered one of the great Greek mathematicians.

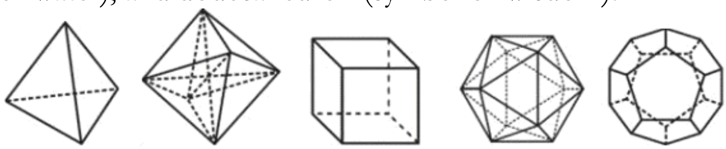
Very little is known about Euclid's life, though he was one of the greatest mathematicians of all times. Both the dates and places of his birth and death are unknown. According to some sources he was born in Greece, in a town near Athens; according to another source, he was born in Alexandria, the then capital of Egypt.

It is believed that he was educated at Plato's Academy in Athens because he was follower of Socrates[66] and Plato[67]. He even

[66] **Socrates** (470 - 399 BC), ancient Greek philosopher, founder of dialectics, great teacher who proclaimed honesty and openness in politics. He was condemned to death for "introducing new gods and corruption of the youth in a new spirit". When at the appointed hour a slave brought him a cup with poison, he said good buy to his friends and with cold blood drank the poison.

his famous "Elements" concluded with consideration of the so-called "Plato's bodies" – regular polyhedral.

Here they are: tetrahedron (symbol of fire), pentahedron (symbol of the Earth), hexahedron (symbol of air), octahedron (symbol of water), and dodecahedron (symbol of wisdom).



The most important period of Euclid's life was the so-called Alexandrian one. The King of Egypt, Ptolemy[68], established the "Museion", i.e. a shrine dedicated to the Muses. Actually it was a scientific center with the famous Alexandria Library, observatory, and Zoo. As a result, Alexandria in several years had become a cultural center of that time. When the King invited Euclid, Euclid left Plato's Academy and joined the Museion. Here Euclid founded the school of mathematics and remained there for the rest of his life. He probably mentored Archimedes.

Some historians of science assumed that Euclid was not really a person, but the name of a group of mathematicians, in the manner of the well known group of the 20th century named

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[67] **Plato of Athens** (427 – 347 BC), Socrates' pupil. After his tutor, whom he called the most justice man in the World, had been executed went for 20-year long journey, and afterwards founded Academy in the Athens where he had been teaching up to his death. The Plato's Academy remained the main philosophical school ancient Greece and Rome about a millennium. Plato's works were written in a form of dialogues between the author and other figurants, one of them was Socrates. Those works influenced very much on philosophy development, and frequently published up to our days on many languages.

[68] **Ptolemy I Soter** (360 - 283 BC), a general of Alexander the Great who became an Egyptian satrap when Alexander's Empire fell apart after his death. The last representative of the Ptolemy's kin was Cleopatra, Anthony's mistress. Do not confuse this Ptolemy with Claudius Ptolemy, the greatest scientist of ancient time.

Burbaki[69]. An Arab manuscript of the 12th century has been found that mentions "Euclid, son of Naucrates, known by his nickname "Geometrist," a Greek by origin lived in Syria." However, who cares about such evidence written thousand years after the event?

Adding to this identity issue, Euclid of Athens has been mistakenly taken for Euclid from Megara, who actually was a Greek philosopher who lived about 100 years before Euclid the Geometrist.

Nevertheless, it is said that around 325 BC Euclid completed his great work "Elements." This work became a textbook for many centuries.

The objective of that manuscript was the systematic presentation of mathematical knowledge collected from Euclid's great predecessors: Thales, Pythagoras, Eudox[70], Theaetetus[71], Aristotle[72] and others.

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69 **Nicolas Burbaki** – a collective pseudonym for a group of mathematicians (mostly French) who began in 1937 to write a book on mathematics attempting to base it on axiomatic methods in the manner that Euclid presented mathematics of his time in his "*Elements*." They wrote 30-volume work "*El?ments de math?matiques*", but the group fell apart in 1968 without completing the book.

70 **Eudox** (408-355 BC), ancient Greek astronomer, geometrist, geographer, physician and lawyer. In his youth, he went to Athens to study sciences. Being poor, he lived outside of Athens and everyday walked 7 miles to the city to listen to the lectures of philosophers. Returning home, he founded his mathematical school. He was one of the outstanding mathematicians of ancient time: he developed the theory of proportions and a method of working with infinitesimally small values (a predecessor of modern integral calculus).

71 **Theaetetus of Athens** (417-369 BC), mathematician who had a significant influence on the development of Greek geometry.

72 **Aristotle of Stagirus** (384 - 322 BC), born in Stagirus, Macedonia; he taught for 20 years at Plato's Academy. Leaving the Academy after the teacher's death, he became a tutor of Alexander the Great. In 335 he returned to Athens where founded his own school – the Lyceum. He died on the Island of Euboea where he had fled after being accused of dishonoring of the gods after the death of his patron Alexander the Great. The Lyceum continued to exist many centuries after the founder's death. Aristotle's works on philosophy, mathematics, astronomy, logic, physics,

Arithmetic and algebraic parts of "*Elements*" have played a big role in the development of mathematics; however, geometry made his name immortal. After him for almost two millennia nobody could supersede his results. In so-called Euclidian geometry, he introduced the axiomatic method that became the dominant approach in modern mathematics. Euclid in his "*Elements*" gave definitions of primary geometrical concepts in the form of axioms and postulates, and then, based on them, formed a single logical chain of 465 theorems.

There is a legend that King Ptolemy asked the mathematician if there was some easier way to learn geometry than by learning all the theorems. Euclid replied: "There is no royal road to geometry".

Euclid distinguished axioms from postulates (now they have similar senses): axioms concerned some basic properties, for instance, "Two objects equal to the same third are equal themselves" (1^{st} axiom) or "The whole is larger than its part" (8^{th} axiom), and postulates declared a possibility of some geometric constructions, for instance, "From any point one can make a line to another point" (1^{st} postulate). By Euclid's understanding, axioms as well as postulates cannot be proved and only could be explained, so that they might be accepted on "faith."

However, mathematicians are not "believers": even Euclid's contemporaries tried to check the consistency of his axioms. A specific interest inspired his famous 5th postulate (about non-intersection of parallel lines). It was only by the 19^{th} century that those attempts led to the formation of non-Euclidean geometry.

Euclid completed all his theorems with the phrase "*Quod erat demonstrandum*" ("that which was to be demonstrated") that began to be used by mathematicians, especially in school.

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biology, ethics, sociology, history and poetry have played a significant role in the development of the European civilization. There are many translations of his works in all modern languages.

Since advent of printed books, "*Elements*" has been published thousands times. This book was a school and university textbook up until the beginning of the 20th century.

Unfortunately, no ancient original copy of Euclid's books has survived. The oldest copy is dated in the 9th century; however, many references and citations of his original books leave hope that the saved text is close enough to the original one.

Someone who had begun to learn geometry with Euclid, when he had learned the first theorem, asked Euclid: "What shall I get by learning these things?" Euclid called his slave and said: "Give him three drahmas because he must make gain out of what he learns".

***Archimedes of Syracuse* (287 – 212 BC)**

There was more imagination in the head of Archimedes than in that of Homer.

Voltaire

One of the greatest ancient mathematician, physicist and inventor.

Archimedes was born in Syracuse, a Greek seaport colony in Sicily. His father was court astronomer and mathematician and a relative to the King of Syracuse, Hiero. Archimedes received a good primary education, being taught with the king's son. It is said that "Archimedes" is a nickname rather than the first name because it means "*bright mind*", and later he was called "a wise man" or "the great geometer".

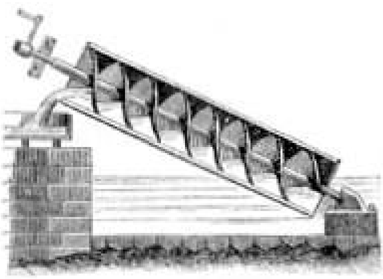
Archimedes spent almost entire his life in Syracuse, though he may have visited Alexandria in Egypt. At least, historians mentioned about his brief correspondence with Egyptian mathematicians, in particular with Eratosthenes[73] who was the Head of the Alexandrian Library.

In the preface to "*On spirals*" Archimedes relates an amusing story regarding his friends in Alexandria. He tells us that he was in the habit of sending them statements of his latest theorems, but without giving proofs. Apparently some of the mathematicians there had claimed some of the results as their own. So, Archimedes says that on the last occasion when he sent them theorems, he

[73] **Eratosthenes** (276-194 BC), outstanding ancient scientist. ***For more see book 1 "Who are We? Where from? Where are we going?"***

included two which were false: "... so that those who claim to discover everything, but produce no proofs of the same, may be deceived into pretending to discover the impossible".

By some sources, Archimedes invented the so-called "*Archimedes's screw*" during his visit to Egypt, and his invention was almost immediately used to raise water from the river to the field. This is a pump, of the kind still used in many parts of the world.



By the way, Archimedes' screw might be considered as a prototype of screw propellers and airscrews.

Archimedes' scientific merits cannot be overestimated. In particular, he was the first in history to solve physical problems with mathematics, thus affording him the title of "father of mathematical physics".

He made many pioneering discoveries: the concept of the center of gravity and methods of its determination, the law of the lever, development of hydrostatic theory (Archimedes' *Law*).

He made a significant contribution in optics. In his book "*Catoptrics*", which we know only by the citations of other ancient authors, he investigated the properties of images in concave and convex mirrors, and he described his experiments on light diffraction.

Archimedes' works on astronomy were not saved, though references to them can be fund in the works of great ancient astronomers like Hipparchus[74] and Ptolemy[75], when they found the length of a year. Tit Livius[76] named Archimedes "a unique observer of the sky and stars."

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[74] **Hipparchus of Nicea** (190-125 BC), ancient Greek scientist, one of the founders of astronomy.

[75] **Claudius Ptolemy** (85-165), great astronomer, mathematician and geographer working in Alexandria. ***For more see book 1 "Who are We? Where from? Where are we going?"***

[76] **Tit Livius** (59 г. BC - 17 AC), one of the greatest Roman historians.

In one of his latest works "*Psammiet*"[77] ("*Sandreckoner*") Archimedes tried to determine the size of the Universe (as believed to be in that time, i.e. restricted to the known Solar system). In this work, he introduced "octads" – the system he invented to name and denote large ("astronomical") numbers. This system reminds our system of using powers of 10. It was a genius approach for that time.

In his book the "*Measurement of the Circle*" Archimedes calculated the number "pi" with a good accuracy: $3{,}1408 < \pi < 3{,}1428$. He needed it for his researches in astronomy.

For approximating the value of "*pi*", Archimedes measured the perimeters of polygons inscribed and circumscribed about a given circle. Rather than trying to measure the polygons one at a time, Archimedes uses a theorem of Euclid to develop a numerical procedure for calculating the perimeter of a circumscribing polygon of 2n sides, once the perimeter of the polygon of n sides is known. Then, beginning with a circumscribing hexagon, he uses his formula to calculate the perimeters of circumscribing polygons of 12, 24, 48, and finally 96 sides. He then developed the corresponding formula to repeat the process using inscribing polygons. In this way he formed the boundaries for a circle and a square. This anticipated the results of the calculus of Newton[78] and Leibniz[79] by almost 2000 years!

In his "*On balance of plane bodies*" Archimedes gave a mathematical formulation of the center of gravity. He explained that if a body is attached at its center of gravity, it would keep its initial position in the space when suspended by the attachment. He also found the location of the center of gravity of many various geometrical bodies.

The theory of a lever is associated with the name of Archimedes. Of course, the lever as a tool was already known many-many years before Archimedes; however, the methods of

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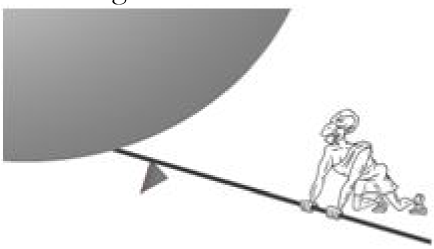
[77] "Psammit" comes from the name of Greek goddess Psamithe that means: "*psammos*"="sand" and "*theid*"= "goddess".

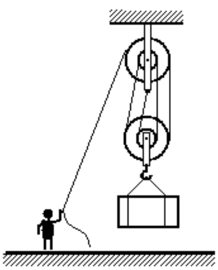
[78] **Isaac Newton** (1643-1727), one of the greatest mathematician, philosopher, physicist and astronomer of all times.

[79] **Gottfried Wilhelm Leibniz** (1646 - 1716), great German mathematician.

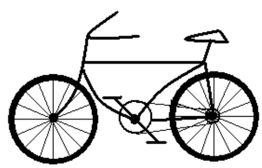
calculating leverage was first made known only from Archimedes' work "On Scales". The text of this work did not come to us directly, but from citations by other ancient authors, allowing us to reproduce the formulations of some of Archimedes' theorems. For example, "Weights on a scale balance at distance from the fulcrum that is inversely proportional to their weight."

Once, soon after he discovered the laws of levers and pulleys, Archimedes exclaimed in presence of the king: "Give me a place to stand on and I will move the earth".





Pulley, or polyspast is a kind of lever and probably needs a short explanation. (Notice that word "*polyspast*" comes from two Greek words "*poly*"="*many*" and "*spao*"="*pull*".) In a sense, pulley is the same idea of a lever realized with the use of wheels, where the wheels radii play the role of lever arms. Using a small force, one can slowly raise a heavy load.



Probably, a simplest mechanism using this idea is a bicycle transmission.

In the words of the historian Plutarch[80], to support this statement, Archimedes constructed a compound pulley system and moved to shore a heavy loaded ship using a single hand to control the device. According to the account, such a ship could not be moved even by all the men of Syracuse.

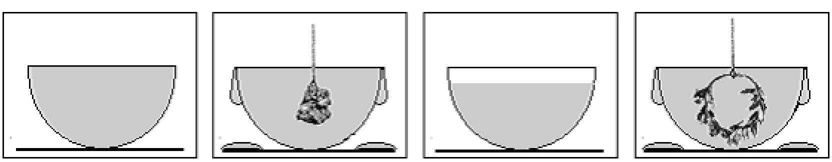
[80] **Plutarch** (45-127), Greek writer and historian. He saved for us unique information about biographies of outstanding people of ancient Greece and Rome.

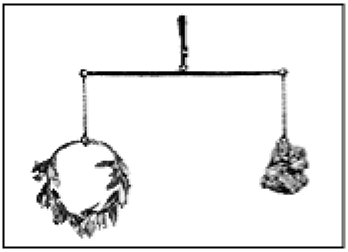
By another legend, the king of Syracuse suspected that a goldsmith, making a golden crown by the king's request, stole some gold. The crown was of the same weight as the golden bar given to the goldsmith, and Archimedes had to find a way of determining if the crown was pure gold. The crown was made in a form of laurel wreath like Roman Caesars bore and was used for putting on the gods' heads during some religious rituals. Because the crown in a sense was sacred checking it should not cause any damage.

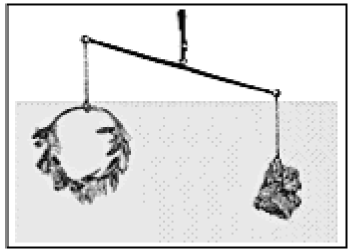
One day while considering the question, he went to the public bathhouse and entering a bathtub recognized that the amount of water that overflowed the tub was proportional the amount of his body that was submerged. This observation, now known as *Archimedes' Principle*, was just the insight he needed to solve the problem asked by the king. He was so excited by the idea that he ran naked through the streets of Syracuse to the king's palace shouting "*Eureka! Eureka!*" ("*I have found it!*"). And that is where the common expression of success came from.

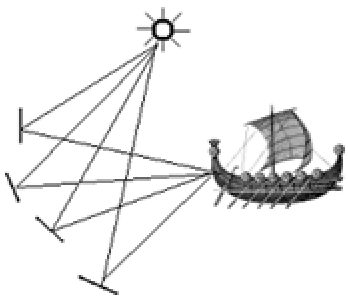
Subsequent generations of scientists tried to duplicate Archimedes' experiment, minus the streaking☺. They tried one of two approaches shown below:

The first approach is described approximately in the following way: Archimedes took a bar of gold of the same weight as the crown. If the crown was made of an alloy of gold and silver, it would occupy a larger volume than the bar of pure gold, simply because gold is denser than silver. Then the bar of gold was put into a cup already filled with liquid right up to the rim, about to overflow. When the bar was placed into the cup, liquid equal to the volume of the bar poured out. Then the bar was taken out and replaced by the crown. Because the crown actually had a larger volume than the bar, additional water poured out of the cup. Thus, the fraudulent goldsmith was brought to justice.









Thus a practical problem solved by Archimedes allowed its creator to leave to future generations the Principle he espoused, which now bears his name: "*If a solid body is submerged in a liquid, it will displace a volume of liquid equal to the volume of the submerged part of the body.*"

Archimedes' engineering talent was most evident during the defense of Syracuse. The Romans and Carthaginians renewed their antagonisms in 218 BC, the beginning of the Second Punic War. At that time, Archimedes had been King Hiero's military adviser for many years and had well prepared Syracuse for any attack. He invented a number of various military devices which were new and efficient and totally unknown to the enemy.

When in 214 BC Roman general Marcellus laid siege to the city of Syracuse, catapults began to throw huge logs of wood, which rolled through the Roman ranks killing soldiers. Stone shrapnel thrown by other catapults killed more soldiers.

Marcellus was forced to change tactics: he tried to capture the city from the sea. However, here the Roman fleet was met with an even more efficient defense tool, giant catapults which threw huge stones of up to 500 pounds onto the Roman triremes breaking their sides and decks.[81].

During another failed assault, it is said that Archimedes had the soldiers of Syracuse use specially shaped and polished shields to

[81] **Trireme** was a military galley of ancient time.

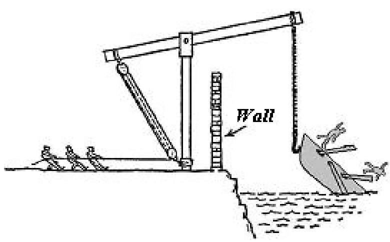
focus sunlight onto the sails of the Roman ships; thus several tens of soldiers equipped with mirror shields set Roman ships on fire from a distance.

The ships which survived the catapults and the "burning mirrors" were met with large grappling hooks attached to levers, which lifted the ships out of the water and then dropped them or struck them against the rocks.

This was more than the terrified sailors could stand, and the fleet withdrew. Marcellus' fleet was defeated by Archimedes' mechanical inventions.

That wasn't the end of it. Marcellus began a long siege of Syracuse. Unfortunately, the defenders' successes led to a loss of their vigilance: once when citizens celebrated a holyday dedicated to the goddess of hunt, Artemis, Romans broke the city wall and penetrated into the city. A number of Greeks were killed, Archimedes among them. His death at the hands of a Roman soldier is described by the outstanding Greek historians of later times, Livius[82] and Plutarch.

The traditional story is that Archimedes was unaware of the taking of the city. While he was drawing figures in the dust, as he did usually, a Roman soldier stepped on them and demanded he come with him. Archimedes responded, "Don't disturb my circles!" The soldier was so enraged that he pulled out his sword and slew the great geometer.



In the 17th century French naturalist and mathematician Rene Descartes (1596-1650) exposed the legend about one of Archimedes' scientific/military invention, stating that it would be absolutely impossible. However, about a century later, another French naturalist and mathematician, George Buffon (1707-1788), wrote: "The history about Archimedes' burning mirrors is famous and well known. Ancient writers told us that he fired enemy's ships at a distance. However, the story, which was trusted for 15 or 16

82 **Tit Livius** (59-17 BC), one of the most well known Roman historians.

centuries, now is called into question, and it is even claimed to be impossible and fantastic. Descartes has rejected such a possibility, and his opinion overshadows the opinion of ancient writers. And his opinion is shared by modern physicists".

To appeal against "Descartes pronouncement", Buffon needed facts, not words. So, he designed a mirror, which was able light afire a piece of wood at a distance of about 50 meters. Later he sarcastically noticed: "Nothing is more poisonous than delusion supported by a sound name".

Moreover, in the second part of last century a Greek engineer literally imitated Archimedes trial: he ranked a hundred soldiers armed with polished bronze mirrors of about one meter by a half meter and asked them to direct the sun reflection to a single spot on the side of a copy of an ancient Roman trireme. The ship soon caught fire.

Plutarch wrote that Archimedes long before the war, left a last will requesting that his tombstone be decorated with a sphere contained in the smallest possible cylinder, with the ratio of the cylinder's volume to that of the sphere inscribed on the sphere.

In his work "*On the Sphere and Cylinder*" Archimedes proved that the ratio of the volume of a sphere to the volume of a cylinder equals to 2:3. Indeed,

$$\frac{(4/3)R^3}{4\pi R^2 + 2\pi R^2} = \frac{2}{3}.$$

In the same work, he got the same ratio of the surface of a sphere to the surface of a cylinder:

$$\frac{4\pi R^2}{4\pi R^2 + 2\pi R^2} = \frac{2}{3}.$$

Because expressions for the volume and surface area of a cylinder were known before his time, Archimedes' results established the first exact expressions for the volume and surface area of a sphere. Archimedes himself considered the discovery of this ratio the greatest of all his accomplishments.

The results of Archimedes works are all the more remarkable when one considers the times in which he lived. Archimedes made fundamental discoveries in several fields, and he then advanced them so far that his results were not improved upon for many centuries. Archimedes certainly ranks as one of the greatest minds in recorded history.

Leonardo Fibonacci
(Leonardo Pisano)
(1175-1250)

Leonardo Fibonacci was one of the greatest mathematicians of medieval times.

He was probably the greatest genius in number theory in the 2000 years between Diophantus and Fermat.

Leonardo's full name is Leonardo of Pisa, or Leonardo Pisano in Italian, the name, as was typical of earlier times, indicating that he was born in Pisa (Italy). Leonardo of Pisa also is now known by his nick-name Fibonacci, which is a shortening of the Latin "*filius Bonacci*," which means "the son of Bonacci." In turn, his father's name, "*Bonacci,*" means "a man of good nature."

Not surprisingly, little is known about Fibonacci's life. Even the only known portrait of Fibonacci was painted well after his death, so it is assumed that the painting reflects an image created on the basis of a verbal description... A hundred years or so later a statue of Leonardo was erected in his birthplace.

Nobody knows his exact date of birth; the general assumption is that he was born in 1175 in Pisa, Italy. His father, Guilielmo Bonacci, was a republic of Pisa diplomat who directed trade between Pisa and Bugia, Algeria (now it is Algerian port Bejaia). Around 1192 he brought Fibonacci to Bugia for schooling. He wanted Fibonacci to become a merchant so he sent his son to a school to major in mathematics. Fibonacci himself wrote, "When my father served as public notary in the customs office at Bugia I was still a child, and he summoned me to Bugia, because, with an eye on a useful future skill for me, he wanted me to stay there with him to attend accounting school. It was there that I was introduced to the art of the Indians' nine symbols through. This knowledge pleased me above all else and I came to understand it."

Fibonacci traveled extensively until about the year 1200, at which time he returned to Pisa and began to serve his birth-place city with is talents.

His most well known chef d'oeuvre, "*Liber abaci,*" (meaning "*Book of the Abacus*" or "*Book of Calculating*") was finished and published in 1202 and was then re-published in 1228. The book contains 15 chapters: About new Indian signs and their use for notation as numbers (Chapter 1); about addition, subtraction, multiplication and division of numbers (Chapters from 2 to 5); about arithmetical operations with fractions (Chapters from 6 to 7); about pricing of goods and the principles of exchange (Chapters from 8 to13); about finding square and cube roots (Chapter 14); and, at last, some knowledge on geometry and algebra (Chapter 15). By the way, this book contains the famous Fibonacci problem about rabbits that was considered in an earlier section in this book.

Fibonacci introduced the Hindu-Arabic positional decimal system and the use of Arabic numerals into Europe. His book persuaded many European mathematicians of his day to use the new system.

Fibonacci wrote several books which played an important role in reviving ancient mathematical skills as well as presenting significant contributions of his own. Fibonacci lived in the days before printing, so his books were hand-written and the only way to get a copy of one of his books was to have another hand-written copy made. Of his books we still have copies of "*Book of Calculating*" ("*Liber abaci*"), "*Practical Geometry*" ("*Practica geometriae*"), "*Flowers*" ("*Flos*") and "*Book on Square Numbers*" ("*Liber quadratorum*").

In 1220 Frederick II, an enlightened monarch who had been crowned Holy Roman emperor, became aware, through the scholars at his court, of Fibonacci's work. One member of Frederick's court, presented a number of problems as challenges to the great mathematician Fibonacci. Three of these problems were solved by Fibonacci, who wrote the answers in his book "*Flos,*" which he sent to Frederick II.

By the way, in "*Flos*" he solved the equation $10x + 2x^2 + x^3 = 20$ with a remarkable degree of accuracy. Without explaining his methods, Fibonacci gave the approximate solution in Babylonian notation as 1.22.7.42.33.4.40 (this is written to base 60, so it is $1+\frac{22}{60}+\frac{7}{60^2}+\frac{42}{60^3}+\ldots$). This converts to the decimal 1.3688081075 which is correct to nine decimal places. A remarkable achievement!

"*Liber quadratorum*", written in 1225, is Fibonacci's most impressive piece of work, although not the work for which he is most famous. The book was dedicated to number theory, in which, among other things, he examined methods for finding Pythagorean triples. Fibonacci first notes that square numbers can be constructed as sums of odd numbers, essentially describing an inductive construction using the formula $n^2 + (2n+1) = (n+1)^2$. In "*Liber quadratorum*" Fibonacci wrote, "I thought about the origin of all square numbers and discovered that they arose from the regular ascent of odd numbers. For unity is a square and from it is produced the first square, namely 1; adding 3 to this makes the second square, namely 4, which root is 2; if to this sum is added a third odd number, namely 5, the third square will be produced, namely 9, which root is 3; and so a sequence of square numbers is generated through the regular addition of odd numbers."

Constructing Pythagorean triples, Fibonacci proceeded as follows: "Thus when I wish to find two square numbers which addition produces a square number, I take any odd square number as one of the two square numbers and I find the other square number by the addition of all the odd numbers from unity up to but excluding the odd square number. For example, I take 9 as one of the two squares mentioned; the remaining square will be obtained by the addition of all the odd numbers below 9, namely 1, 3, 5, 7, which sum is 16, a square number, which when added to 9 gives 25, a square number". Fibonacci also proves many interesting number theory results such as: (1) there is no x, y such that $x^2 + y^2$ and $x^2 - y^2$ are both squares; (2) $x^4 - y^4$ cannot be a square, and others.

Fibonacci's work in number theory was almost wholly ignored and virtually unknown during the Middle Ages.

There is the only existing document which refers to Fibonacci. That document is a decree issued by the Republic of Pisa in 1240 that awards a salary to "... the serious and learned Master Leonardo Bigollo ..." (Bigollo means "traveler"; it was yet another nickname for Fibonacci.) This salary was given to Fibonacci in recognition for the services that he had given to the city as advisor on matters of accounting and instruction of the citizens.

Nothing is known about his death. A coined version that he was killed during the 5th Crusade while accompanying Frederick the 2nd in 1228 is evidently in conflict with the above quoted document.

Leonhard Euler
(1707–1783)

Euler made such a huge service to mathematics that his works have been publishing in Switzerland for over a hundred years. He was one of the most significant mathematicians of all times.

Leonhard Euler was born in Basel, Switzerland, to a Calvinist pastor. His childhood was spent in a tiny town, Riehen, not far from Basel, where his father served as a priest. Leonhard's father had studied theology at the University of Basel, and while doing so, he attended lectures on mathematics by Jacob Bernoulli[83]. This allowed him was able to teach his son elementary mathematics.

Leonhard was sent to school in Basel, where he lived with his grandmother. Euler learned no mathematics at all from the school, but due the mathematical knowledge he had gained from his father, he was able to read mathematics texts on his own and took some private lessons. Euler's father wanted his son to follow him into the priesthood, and, to that end, he sent his son to the University of Basel to prepare for the ministry. Leonhard entered the University in 1720, at the age of 14, first to obtain a general education before going on to more advanced studies. He studied

[83] **Jacob Bernoulli** (1654 – 1705), outstanding Swiss mathematician, one of the founders of probability theory. He was a member of famous Bernoulli mathematical klan: Bernoullis in 18th – 19th centuries were professors at Basel University.

theology and Hebrew, and at the same time he was privately tutored in mathematics by Johann Bernoulli[84]. At the home of his teacher, Leonhard met his sons – Nicolas and Daniel who also took a great interest in mathematics. Johann Bernulli was so impressed by his pupil's ability that he convinced Euler's father to allow Leonhard to become a mathematician.

Euler graduated from the University of Basel in 1723 with Master's degree in philosophy presenting his thesis on the comparison of the philosophical ideas of Descartes and Newton. At this period he wrote a paper on acoustics, which was so professional that his candidacy for the open position of Professor of Physics was presented. However, the University administration rejected him, finding him too young (he was only 17 years old).

About this time, Leibniz at the request of Peter the Great[85] had compiled a Charter of the Russian Academy of Sciences, which was founded in 1724 by Catherine I, the wife of Peter the Great. Because there were no native Russian scientists, she invited foreigners. Amongst the first 22 guests were Nicolas and Daniel Bernoulli. In a year, by their recommendations, Euler was invited to St. Petersburg as an adjunct professor in physiology. Having no appropriate mathematical career in Switzerland, Euler accepted the invitation and moved to Russia in 1727. Through the requests of Daniel Bernoulli and Jakob Hermann[86], Euler was appointed to the mathematical-physical division of the Academy rather than to the physiology post he had originally been offered. Soon he obtained a professorship in mathematics and in 1733 he became a member of St. Petersburg Academy. He was 26-year old at the moment.

When Daniel Bernoulli, who held the senior chair in mathematics at the Academy, returned to Basel in 1733, Euler was appointed on his place. The financial improvement which came from this appointment allowed Euler to marry in 1734. His wife,

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[84] **Johann Bernoulli** (1667-1748), Swiss mathematician, younger brother of Jacob Bernoulli.

[85] **Peter the Great** (1672–1725), Russian Tsar, a patron of Russian science.

[86] **Jakob Hermann** (1678-1733), Swiss mathematician, Euler's distant relative.

Katharina Gsell, also had Swiss roots. They had 13 children altogether although only five survived their infancy.

Euler wrote that he made some of his greatest mathematical discoveries while holding a baby in his arms with other children playing round his feet.

After 1730 he carried out several important state projects on science education, fire engines, machines, and ship building. Simultaneously he led many pure mathematical projects (in number theory; calculus, differential equations and the calculus of variations) and rational mechanics. He viewed these fields as closely interconnected: number theory was vital to the foundations of calculus, and special functions and differential equations were essential to rational mechanics.

During this period, he published a number of articles and a book "*Mechanica*". In this book for the first time Newtonian dynamics was presented in the form of mathematical analysis.

In 1735 the Academy needed to make some complex cartographic work. The plan had this work scheduled for completion in three months. Euler laid a bet that he could complete this task by himself in three days! And he did do it! However, he paid an enormous price for this: due to that strong neurotic stress, he developed a severe fever and almost lost his life. As a result he lost eyesight… He took this with calmness if not bravado: "Well, now I can concentrate completely on mathematics!"

By 1740 Euler had won two Grand Prizes of the Paris Academy of Sciences for his brilliant mathematical achievements.

Meanwhile, the political situation in Russia worsened: Czarina Anna died and the country was in hands of regents because the heir was a child. Life for foreigners became unbearable and even dangerous. Euler accepted an offer from Frederick the Great[87] who had established Berlin Academy of Science and in 1741 invited Euler to join it.

Even while in Berlin Euler continued to cooperate with the St. Petersburg Academy and even received some salary from Russia for writing reports and educating young Russians.

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[87] **Frederick the Great** (1712 – 1786), the king of Prussia.

During the twenty-five years spent in Berlin, Euler wrote almost 400 papers and several books (on the calculus of variations; on the calculation of planetary orbits; on artillery and ballistics; on analysis; on shipbuilding and navigation; on the motion of the moon).

However, relations between Euler and the king were somehow not the best. When the previous president of the Berlin Academy died, Euler should have been the obvious successor, but the monarch offered the position to the French mathematician D'Alembert[88]. D'Alembert refused to accept the position on the basis that none could be placed above Euler. However, Euler understood the message: it was the time for him to find a new home.

Meanwhile, Russia had come under the rule of the more liberal Catherine the Great. Hearing about some friction between Euler and the Prussian king, the Czarina in 1766 offered Euler one of the highest posts in the Academy and tripled his. In her message to the ambassador, she wrote: "If he is not satisfied with my offer, let him state his conditions; don't let him delay his return to St. Petersburg".

Euler wrote a retirement letter to Frederick but no answer… He wrote the second time. Frederick was greatly angered at Euler's decision, though gave him a permission to leave.

In 1766 Euler, who was already almost 60 years old, returned to St. Petersburg. The next day he was invited to the Czarina's reception where she gifted him a beautiful house in the center of St. Petersburg. Finding that Euler was too skinny, she sent to his service one of her cooks.

Soon after his return to Russia, Euler became almost entirely blind after an illness. However, he worked very hard up to

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[88]**Jean-le-Rond d'Alembert** (1717-1783), French mathematician, physicist and philosopher. He was also one of the editors of the "*Encyclop?die*", the famous early28-volume French encyclopedia, for which he wrote most of the mathematical and scientific articles. He was abandoned as a baby on the steps of the Saint-Jean-le-Rond de Paris church. According to custom, he is named after the protecting saint of the church.

the last day of his life: during his last 15 years he wrote almost 300 new scientific works.

In 1771 his home was destroyed by a fire which destroyed almost the entire center of the city. During the fire, Euler refused to leave the burning house until boxes with his mathematical manuscripts were saved.

Shortly after the fire, a cataract operation restored his sight for a few days but Euler in spite of the surgeon recommendations removed a bandage and started to work too soon. It led to his total blindness forever.

Euler's powers of memory and concentration were legendary. He could recite the entire Aeneid[89] word-for-word. Because of this ability, he was able to continue with his work on optics, algebra, and lunar motion.

Of course, this remarkable level of output was possible due to help of his sons, Johann and Christoph, and Nicolaus Fuss. The latter, also Daniel Bernoulli's pupil, became Euler's secretary. Later Fuss married Euler's granddaughter; he continued to perform his secretarial duties, even becoming an academician of the St. Petersburg Academy. Fuss helped Euler prepare over 250 articles for publication, including an important work on insurance, which was published in 1776.

On the 18th of September, 1783 Euler spent the first half of the day as usual. He gave a mathematics lesson to one of his grandchildren, did some calculations with chalk on two boards on the motion of balloons; then discussed with Fuss the recently discovered planet Uranus. At about five o'clock in the afternoon he suffered a brain hemorrhage and uttered only "I am dying" before he lost consciousness. He died about eleven o'clock in the evening.

One of the obituaries said: "Euler ended his live and his calculations simultaneously".

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89 The "*Aeneid*" is a Latin epic written by **Virgil** (70-19 BC) that tells the legendary story of Aeneas, a Trojan who traveled to Italy where he became the ancestor of the Romans.

Euler's work in mathematics is so vast that it cannot be overestimated. He was the most prolific writer of mathematics of all time. He made large leaps forward in the study of modern analytic geometry and trigonometry, calculus and number theory. He integrated Leibniz's differential calculus and Newton's method of fluxions into mathematical analysis. He introduced beta and gamma functions, and integrating factors for differential equations. He studied continuum mechanics, lunar theory, elasticity, acoustics, the wave theory of light, hydraulics, and music. He laid the foundation of analytical mechanics.

Possibly his most impressive work was his approximation of the three-body problem of the Sun, Earth and Moon, which he solved while completely blind, performing all the computations in his head.

We owe to Euler the notation $f(x)$ for a function, "*e*" for the base of natural logarithms, "*i* "for imaginary 1, i.e. $\sqrt{-1}$, π for "pi", Σ for summation…

Euler's important contributions were so numerous that terms like "Euler's formula" or "Euler's theorem" can mean many different things depending on context. For instance, one has Euler angles in mechanics, Euler's theorem for rotation, Euler's equations for motion of fluids, and the Euler-Lagrange equation in calculus of variations. The "Euler's formula" defines the exponentials of imaginary numbers in terms of trigonometric functions. There are both Euler numbers and Eulerian numbers, and they aren't the same thing. Euler's study of the bridges of K?nigsberg can be seen as the beginning of combinatorial topology (the Euler characteristic bears his name).

Euler, who is best remembered for his contributions to mathematics, was involved in philosophy as well. While in Berlin, he would constantly get philosophical debates, especially with Voltaire, though not always successfully.

Well known is a story about his meeting with the famous French philosopher Diderot, whom Catherine the Great had invited to her court. One time Diderot annoyed the Czarina with attempts to convert her subjects to atheism. She asked Euler to quiet him. Euler claimed that he had a mathematical proof of the existence of God and Diderot asked to hear it. Euler then stepped forward and stated: "Sir, $(a+b^n)/n=x$, hence God exists; reply!" Diderot had no idea what Euler was talking about, though he did understand the chorus of laughter. Soon after this he returned to France…

Leonhard Euler was the greatest mathematician of the eighteenth century and one of great mathematicians of all times. His publication list has 886 titles. Euler's complete works fill about 90 volumes. Switzerland began to publish Euler's works in 1911 and the publication is still not finished…

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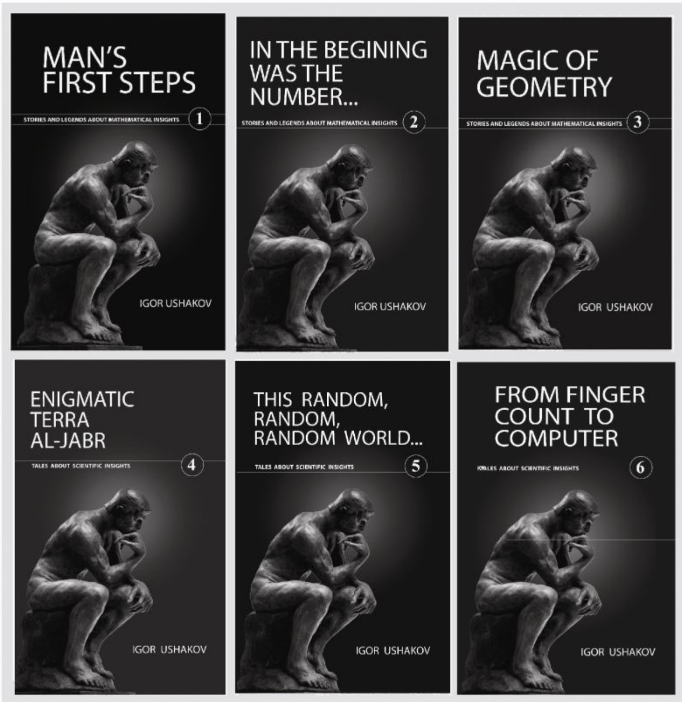

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The author has published roughly 30 scientific monographs in English, Russian, Bulgarian, Czechoslovakian, and German.

In addition to scientific writings, the author has published several book of prose, poems and lyrics (in Russian).